Classification of maximal 2-distance sets by Magma

Hiroshi Nozaki

Aichi University of Education

Hakata Workshop 2013
Reference Eki Higashi Building
January 26, 2013

Definition and Purpose

Definition 1

- $X \subset \mathbb{R}^d$ is a 2-distance set $\Leftrightarrow$ $|\{\sqrt{\sum(x_i - y_i)^2} \mid x, y \in X\}| = 2$
- A 2-distance set in $\mathbb{R}^d$ is maximal $\Leftrightarrow$ $\nexists x \in \mathbb{R}^d \setminus X$ such that $X \cup \{x\}$ is still a 2-distance set in $\mathbb{R}^d$

Purpose

We classify maximal 2-distance set X in $\mathbb{R}^d$ with $|X| \geq d + 3$ for $d \leq 6$ by using Magma.

The algorithm is mainly based on Lisonek's (JCTA 1997):

Simple graph $\rightarrow$ 2-distance set

Theorems

Theorem 2 (Menger)

Let d be a positive integer. A metric space M of cardinality $|M| \geq d + 3$ is isometrically embeddable in $\mathbb{R}^d$ if and only if any subspace of M of cardinality exactly $d + 3$ is.

Corollary 3

Let G_i be the induced subgraph removing a vertex v_i of a simple graph G. Then G is isometrically embeddable in $\mathbb{R}^d$ if and only if $G_1, \ldots, G_{d+4}$ is isometrically embeddable in $\mathbb{R}^d$

Theorem 4 (Einhorn-Schoenberg(1966))

A simple graph G is isometrically embeddable in $\mathbb{R}^{n-2}$, where $|G| = n$.

Algorithm for classification of maximal 2-distance set in $\mathbb{R}^d$

Let L_n $(n \geq d+2)$ denote the set of graphs of size n which is isometrically embeddable in $\mathbb{R}^d$.
Rough sketch:

1. Obtain graphs of size $d+3$ from the database of Magma. Pick out all graphs which are isometrically embeddable in $\mathbb{R}^d$ from the graphs of size $d+3$. It is L_{d+3}.
2. Construct L_{j+1} from L_j as follows.
   Fix $G=(V,E) \in L_j$. Take a new vertex v_0 such that for $v_1 \in V$, we have $G_1 = ((V \setminus \{v_1\}) \cup \{v_0\}, E_1\}) \in L_j$. Let $G_i := (V_i = (V \setminus \{v_i\}) \cup \{v_0\}, (E \cup E_1)|_{V_i}\})$. By Corollary 3, $G_2, \ldots, G_{d+2} \in L_j$, if and only if $G' := (V \cup \{v_0\}, E \cup E_1) \in L_{j+1}$
   - If there does not exist such v_0, or $G' \notin L_{j+1}$, then the embedding of (V,E) is maximal.

Results

$d/\|X\|$	7	8	9	10	11	12	13	15	16	17	21	27
4	5	2	1	1								
5		26	2	13	1	1		1	1			
6			90	21	17	5	4	1	35	1	1	1

Number of maximal spherical two-distance sets

$d/\|X\|$	8	9	10	11	12	14	16	17
4	6	3						
5	4	1	1					
6		3	10	1	10	1	1	1

Number of maximal non-spherical two-distance sets

Refer the database of maximal 2-distance sets:
http://auemath.aichi-edu.ac.jp/~hnozaki/data_English.html