ON BUTSON-TYPE HADAMARD MATRICES $H(17, 17)$

Kyoung-Tark Kim
poisonn00@hanmail.net
(with Mitsugu Hirasaka and Yoshihiro Mizoguchi)

Pusan National University

January 26, 2013
Hakata Workshop 2013

Butson-type Hadamard Matrices

An $n \times n$ matrix $H = (H_{ij})$ is called an $H(p, n)$-MATRIX if

$H_{ij}^p = 1$ and $HH^{CT} = nI$ where I is the $n \times n$ identity matrix.

$\Longrightarrow$ An $H(p, n)$-matrix is called a Butson-type Hadamard matrix.

Basic Properties for $H(p, n)$-matrix H

1. A permutation of the rows (columns) of H is permissible.
2. A multiplication of the elements of a row (column) of H by a fixed pth root of unity is permissible.

DIFFERENCE MATRICES

H : an $H(p,n)$-matrix

$\Longrightarrow$ A DIFFERENCE MATRIX $\mathrm{Diff}(H) \in \mathrm{Mat}_{n\times n}(\mathbb{Z}_p)$ of H :

$$H = (H_{ij}) = \left(\exp\left(\frac{2\pi\sqrt{-1}E_{ij}}{p}\right)\right) \overset{\text{def.}}{\Longleftrightarrow} \mathrm{Diff}(H) = (E_{ij})$$

EXAMPLE : H is an $H(3,3)$-matrix

$$H = \begin{pmatrix} 1 & 1 & 1 \\ 1 & e^{2\pi i/3} & e^{4\pi i/3} \\ 1 & e^{4\pi i/3} & e^{2\pi i/3} \end{pmatrix} \Longleftrightarrow \mathrm{Diff}(H) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix}$$

FOURIER MATRIX

The Fourier matrix F of order p is a $p \times p$ matrix such that

$$\mathrm{Diff}(F) = (ij)_{i,j=0}^{p-1} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 2 & \cdots & p-1 \\ 0 & 2 & 4 & \cdots & 2(p-1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & p-1 & 2(p-1) & \cdots & (p-1)^2 \end{pmatrix}$$

$\Longrightarrow$ F is indeed an $H(p,p)$-matrix.

Orthogonality Correspondences

We will consider ONLY $H(p,p)$-matrix H where p is prime.

For two distinct rows of H

$$(\omega_1, \cdots, \omega_p) \perp (\eta_1, \cdots, \eta_p)$$

$$\Updownarrow$$

$$\omega_1 \cdot \overline{\eta_1} + \cdots + \omega_p \cdot \overline{\eta_p} = 0$$

For two distinct rows of $\mathrm{Diff}(H)$

$$(E_{i1}, \cdots, E_{ip}) \perp (E_{j1}, \cdots, E_{jp})$$

$$\Updownarrow$$

$$\{E_{i1} - E_{j1}, \cdots, E_{ip} - E_{jp}\} = \mathbb{Z}_p$$

Calculation of Candidates for Entry E_{ij}

Fix i and j. For $0 \leq a < i$ and $0 \leq b < j$,

$$\mathrm{Diff}(H) = \begin{pmatrix} & \vdots & & \vdots & \\ \cdots & E_{ab} & \cdots & E_{aj} & \cdots \\ & \vdots & \ddots & \vdots & \\ \cdots & E_{ib} & \cdots & E_{ij} & \cdots \\ & \vdots & & \vdots & \ddots \end{pmatrix}$$

$\Longrightarrow E_{ib} - E_{ab} \neq E_{ij} - E_{aj}$.

Candidates for E_{ij}

E_{ij} satisfies $E_{ij} \neq E_{ib} + E_{aj} - E_{ab}$ for $0 \leq a < i$ and $0 \leq b < j$.

Our INITIAL PART for $\mathrm{Diff}(H)$ is symmetric :

$\Uparrow$

$$\mathrm{Diff}(H) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 2 & 3 & 4 & \cdots \\ 0 & 2 & E_{22} & E_{23} & E_{24} & \cdots \\ 0 & 3 & E_{32} & E_{33} & E_{34} & \ddots \\ 0 & 4 & E_{42} & E_{43} & E_{44} & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}$$

SYMMETRIC REDUNDANCY

We may assume that $E_{23} \leq E_{32}$.

CONCLUSION FOR $H(17, 17)$-MATRICES

For $p = 17$ we use a parallel algorithm on super computer with MPI (Massage Passing Interfaces) over TATARA FUJITSU PRIMERGY CX400 (in Kyushu Univ).

COMPUTATIONAL RESULT

There is a unique $H(17, 17)$-matrix, namely,

the Fourier matrix of order 17.

◦ This calculation is done by super computer with 64 threads over computational complexity = 68 hours.