Equivalence between the eigenvalue problem of non-commutative harmonic oscillators and existence of holomorphic solutions of Heun differential equations, eigenstates degeneration and the Rabi model

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Abstract

The initial aim of the present paper is to provide a complete description of the eigenvalue problem for the *non-commutative harmonic oscillator* (NcHO), which is defined by a (two-by-two) matrix-valued self-adjoint parity-preserving ordinary differential operator [28], in terms of Heun's ordinary differential equations, the second order Fuchsian differential equations with four regular singularities in a complex domain. This description has been achieved for odd eigenfunctions in Ochiai [25] nicely but missing up to now for the even parity, which is more important from the viewpoint of determination of the ground state of the NcHO. As a by-product of this study, examining the monodromy data (characteristic exponents, etc.) of the Heun equation, we prove that the multiplicity of the eigenvalue of the NcHO is at most two. Moreover, we give a condition for the existence of a finite-type eigenfunction (i.e. given by essentially a finite sum of Hermite functions) for the eigenvalue problem and an explicit example of such eigenvalues, from which one finds that doubly degenerate eigenstates of the NcHO actually exist even in the same parity. Also, we determine the possible shape of (so-called) *Heun polynomial solutions* of the Heun equations, which are obtained by the eigenvalue problem of the NCHO corresponding to finite-type eigenfunctions. Furthermore, as the second purpose of this paper, we discuss a connection between the quantum Rabi model [15, 2, 20, 41] and a certain element of the universal enveloping algebra $\mathcal{U}(\mathfrak{sl}_2)$ of the Lie algebra $\mathfrak{sl}_2$ naturally arising from the NcHO through the oscillator representation. Precisely, an equivalent picture of the *quantum Rabi model* drawn by a confluent Heun equation is obtained from the Heun operator defined by that element in $\mathcal{U}(\mathfrak{sl}_2)$ under a (flat picture of non-unitary) principal series representation of $\mathfrak{sl}_2$ through an appropriate confluent procedure.

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1 Introduction

In recent years, special attention has been paid to studying the spectrum of self-adjoint operators with non-commutative coefficients, in other words, interacting quantum systems, like the quantum Rabi model, the Jaynes-Cumming (JC) model [15], etc., not only in mathematics [7, 8] but also in theoretical physics [20, 4, 2, 38] and experimental physics ([24, 5], see also e.g. [6]). For instance, the quantum Rabi model [33] is known to be the simplest model used in quantum optics to describe interaction of light and matter and the JC model is the widely studied rotating-wave approximation of the Rabi model [20, 8]. Further, in [24, 5], the authors succeeded in reaching ultra strong coupling regime in the so-called circuit QED. It shows that their experimental results can be described by the quantum Rabi model, though it cannot be done by the JC model. The non-commutative harmonic oscillator (NcHO [28]) Q defined below has been expected to share/provide one of these Hamiltonians describing such quantum interacting systems.

The purposes of this paper are, in short, providing explicit descriptions of i) the eigenvalue problem of NcHO in terms of Heun's differential equations, ii) the degeneration of eigenvalues and concrete examples together with presenting their possible shape in the Heun picture (Heun polynomials), and iii) a connection between the NcHO and quantum Rabi model through the confluence process of Heun's ODE using representation theory of the three dimensional simple Lie algebra $\mathfrak{sl}_2$.

The normal form the Hamiltonian $Q_{(\alpha,\beta)}(x,D)$ of NcHO ([31, 32, 28]) is given by

$$Q_{(\alpha,\beta)}(x,D) = A\left(-\frac{1}{2}\frac{d^2}{dx^2}+\frac{1}{2}x^2\right)+J\left(x\frac{d}{dx}+\frac{1}{2}\right),$$

where the mutually non-commuting (in general) coefficients A and J are given by

$$A = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}, \quad J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

From the definition, $Q_{(\alpha,\beta)}(x,D)$ is obviously a parity-preserving differential operator. We assume that $\alpha,\beta>0$ and $\alpha\beta>1$ throughout the paper.

The former requirement $\alpha,\beta>0$ comes from the formal self-adjointness of the operator $Q_{(\alpha,\beta)}(x,D)$ relative to the natural inner product on $L^2(\mathbb{R},\mathbb{C}^2)(=\mathbb{C}^2\otimes L^2(\mathbb{R}))$. To be more precise, one realizes $Q_{(\alpha,\beta)}(x,D)$ as an unbounded operator $Q=Q_{\alpha,\beta}$ in $L^2(\mathbb{R},\mathbb{C}^2)$ with the dense domain

$$\mathcal{D}=\mathcal{D}_{\alpha,\beta}:=\{u\in L^2(\mathbb{R},\mathbb{C}^2)\mid Q_{(\alpha,\beta)}(x,D)u\in L^2(\mathbb{R},\mathbb{C}^2)\}$$

such that $Qu=Q_{(\alpha,\beta)}(x,D)u$ $(u\in\mathcal{D})$, where $Q_{(\alpha,\beta)}(x,D)u$ is understood in the distribution sense. Hence Q is the so-called maximal operator (associated with $Q_{(\alpha,\beta)}(x,D)$). By the global pseudo differential calculus due to L. Hörmander, one sees that $\overline{Q|_{\mathcal{S}(\mathbb{R},\mathbb{C}^2)}}=Q$, that is, Q is the closure in the graph-norm of its restriction to the Schwartz space $\mathcal{S}(\mathbb{R},\mathbb{C}^2)$ (see [28, 29]). The latter condition $\alpha\beta>1$ guarantees the operator $Q_{(\alpha,\beta)}(x,D)$ is globally elliptic so that Q is self adjoint in $\mathcal{D}$. In the globally elliptic case, one has $\mathcal{D}=B^2(\mathbb{R},\mathbb{C}^2)$, where for $s\in\mathbb{R}$, $B^s(\mathbb{R}^n,\mathbb{C}^N)=B^s(\mathbb{R}^n)\otimes\mathbb{C}^N$, the spaces $B^s(\mathbb{R}^n)$ being the global spaces tailored to the s-th powers of the harmonic oscillator $P(x,D):=\frac{1}{2}(-\frac{d^2}{dx^2}+x^2)$ introduced in [36] (see §3 in [28] in details). Since the resolvent is compact, due to the compact embedding of the space $B^2(\mathbb{R},\mathbb{C}^2)$, the spectrum is made of a sequence of eigenvalues diverging to $+\infty$ with finite multiplicities, with eigenfunctions belonging to the Schwartz class, whence in particular those space are finite dimensional. Thus, from the assumptions, one concludes that the eigenvalues of the eigenvalue problem $Q\varphi=\lambda\varphi$ $(\varphi\in L^2(\mathbb{R},\mathbb{C}^2))$ are all positive and form a discrete set with finite multiplicity. One remarks also that the operators considered here are globally elliptic, and no essential spectrum can therefore be present when they are realized as self-adjoint operators on the maximal domain. In other words, to have essential spectrum one has to destroy global ellipticity, and an example is given when $\alpha=\beta=1$ and generalized also recently to the cases $\alpha\beta=1$ [30].

It should be first noted that, Q is unitarily equivalent to a couple of quantum harmonic oscillators when $[A,J]=0$, i.e. $\alpha=\beta$ holds, whence the eigenvalues are explicitly calculated as $\{\sqrt{\alpha^2-1}(n+\frac{1}{2})\mid n\in\mathbb{Z}_{\geq 0}\}$ having multiplicity 2 ([32]). Actually, when $\alpha=\beta$, there exists a structure behind Q corresponding to the tensor product of the two dimensional trivial representation and the oscillator representation [11] of the Lie algebra $\mathfrak{sl}_2$ ([32]). However, when $\alpha\neq\beta$, representation theoretically, the apparent lack of an operator which commute with Q (second conserved quantity) besides the Casimir operator, the image of generator of the center $\mathcal{ZU}(\mathfrak{sl}_2)$ of the universal enveloping algebra $\mathcal{U}(\mathfrak{sl}_2)$ of the Lie algebra $\mathfrak{sl}_2$. (Moreover, it has been shown that there is no annihilation/creation operators associated to NcHO [29] when $\alpha\neq\beta$.) Therefore, the clarification of the spectrum in the general case where $\alpha\neq\beta$ is considered to be highly non-trivial (see [13, 27, 9], also references in [28]). It is, nevertheless, worth noticing that the spectral zeta function of Q [12] (which is essentially given by the Riemann zeta function if $\alpha=\beta$) yields a new number theoretic study including the subjects such as Apéry-like numbers, elliptic curves, modular forms, Eichler integrals, Eichler cohomology groups and their natural generalization (see [16, 17, 18] and references therein, [19], and the recent study [22]).

We have constructed the eigenfunctions and eigenvalues [32] in terms of continued fractions determined by a certain three terms recurrence relation, which can be derived from the expansion of eigenfunctions relative to a basis constructed by suitably twisting the classical Hermite functions.

We say that the eigenfunction $\varphi(x)$ in $L^2(\mathbb{R}, \mathbb{C}^2)$ is of a finite-type if $\varphi(x)$ can be expanded by a finite number of this Hermite basis. The eigenvalue corresponding to the finite-type eigenfunction is said to be of finite-type. Otherwise, we say that the eigenvalues/eigenfunctions are of infinite-type. We denote Σ_0 (reps. Σ_∞) the set of eigenvalues corresponding to eigenfunctions of finite (reps. infinite)-type. Since the operator Q preserves the parity, we define $\Sigma^{\pm}$ to be the set of eigenvalues whose eigenfunctions are even/odd, that is, those satisfying $\varphi(-x) = \pm\varphi(x)$. Then there is a classification of eigenvalues: $\Sigma_0^{\pm} = \Sigma_0 \cap \Sigma^{\pm}$ (resp. $\Sigma_\infty^{\pm} = \Sigma_\infty \cap \Sigma^{\pm}$) corresponding to even/odd eigenfunctions of finite (resp. infinite) -type. In [32], it is shown that $\Sigma_0^{\pm} \subset \Sigma_\infty^{\pm}$ and the multiplicity of each $\lambda \in \Sigma_\infty^{+}$ (resp. Σ_∞^{-}) is at most 2. This means that once the eigenvalue degenerates in the same parity, one of the eigenfunction is of the form $p(x) \times e^{-ax^2}$, $p(x)$ being a polynomial and a a positive constant depending on the value $\sqrt{\alpha\beta - 1}$, and this resembles the situation of the (doubly) degenerating eigenvalues case for the quantum Rabi model [20] (see also [2, 37, 41]). The spectral analysis of the quantum Rabi model seems to be much simpler than the one of NcHO, while the latter seems to share certain interesting properties the former has. Actually, one finds that the NcHO gives essentially the quantum Rabi model through the confluence limit procedure at the stage of Heun equations' picture (see §5).

It is known [23] that the eigenvalues of NcHO build a continuous curve with arguments α and β. It comes as an important problem to analyze the behavior of eigenvalue curves, in particular, a main issue of present day research, especially in mathematical physics, addresses the characterization of crossing/avoided crossing of eigenvalue curves (see e.g. [35, 7, 8]). From the observation in [23], since the eigenvalue curves are continuous, one can observe that $\Sigma_\infty^{+} \cap \Sigma_\infty^{-} \neq \emptyset$ (see Figure 1 in [23], p.648; the graph of eigenvalue curves is drawn with respect to the variable $s = \beta/\alpha$ with a fixed α; $\alpha = 3.0$). However, one does not know whether $\Sigma_0^{+} \cap \Sigma_\infty^{-}$ (resp. $\Sigma_0^{-} \cap \Sigma_\infty^{+}$) is empty or not, while it is shown in [32] that $\Sigma_0^{+} \cap \Sigma_0^{-} = \emptyset$. Therefore, the multiplicity of eigenvalue is at most 3 and may a priori reach 3. In this paper, we will solve one of the longstanding problems for the eigenstate degeneration of NcHO, in particular, prove that $\Sigma_0^{+} \cap \Sigma_\infty^{-} = \Sigma_0^{-} \cap \Sigma_\infty^{+} = \emptyset$ (Theorem 1.2).

Generally, in harmonic analysis on the real line, even/odd eigenspaces have completely analogues structures. Also, in view of the description of the lowest eigenvalue, the study on even eigenstates is more important [9]. Moreover, we could not see any difference between the even/odd eigenspaces in the papers [31, 32]. However, in the complex domain picture drawn in [25], the odd part Σ^- corresponds to the second order equation given by Heun's ordinary differential equation whereas the even part Σ^+ corresponds to the third-order equation (constructed by the same Heun operator). Therefore, working out a solution to this asymmetry has been desirable for a long time. In this paper, we prove that there exists a completely parallel structure of even eigenfunctions to that of the odd eigenfunctions. For readers' convenience we state the results for the odd case obtained in [25] in a parallel way. In fact, one of the main techniques to derive this correspondence is based on a brilliant idea developed in [25], but employing a modified Laplace transform different from that in [25] which provides an (exact) intertwiner for the oscillator representation corresponding to the even parity (see §2.2). The reason why one could not obtain in [25] the proper Heun operator (only the third order operator) in the even case is that the restriction of the modified Laplace transform in [25] to the even parity has only quasi-intertwing property, that is, there is an extra term which breaks the $\mathcal{U}(\mathfrak{sl}_2)$-equivariant actions (see Lemma 2.2).

In conclusion, we are able to show that there exists a second-order Fuchsian differential operator $H(w, \partial_w)$ so that the eigenvalue problem $Q\varphi = \lambda\varphi$ ($\varphi \in L^2(\mathbb{R}, \mathbb{C}^2)$) of the NcHO is equivalent to the existence of holomorphic solutions $f(w)$ of the differential equation $H(w, \partial_w)f(w) = 0$ on a suitably chosen domain. Technically speaking, since the eigenspaces of Q are finite dimensional, they coincide with their topological closure and the same should apply the counterpart of the Heun operators via this equivalence. With this understanding one can state the first main result as follows.

Theorem 1.1. *There exist linear bijections:*

$$\text{Even}: \quad \{\varphi \in L^2(\mathbb{R}, \mathbb{C}^2) \mid Q\varphi = \lambda\varphi,\ \varphi(-x) = \varphi(x)\} \xrightarrow{\sim} \{f \in \mathcal{O}(\Omega) \mid H_\lambda^{+} f = 0\},$$

$$\text{Odd}: \quad \{\varphi \in L^2(\mathbb{R}, \mathbb{C}^2) \mid Q\varphi = \lambda\varphi,\ \varphi(-x) = -\varphi(x)\} \xrightarrow{\sim} \{f \in \mathcal{O}(\Omega) \mid H_\lambda^{-} f = 0\},$$

where Ω is a simply-connected domain in $\mathbb{C}$ (w-space) such that $0, 1 \in \Omega$ while $\alpha\beta \notin \Omega$, $\mathcal{O}(\Omega)$ denotes the set of holomorphic functions on Ω, and $H_\lambda^{\pm} = H_\lambda^{\pm}(w, \partial_w)$ are the Heun ordinary *differential operators given respectively by*

$$H_\lambda^+(w,\partial_w) := \frac{d^2}{dw^2} + \left(\frac{\frac{1}{2}-p}{w} + \frac{-\frac{1}{2}-p}{w-1} + \frac{p+1}{w-\alpha\beta}\right)\frac{d}{dw} + \frac{-\frac{1}{2}(p+\frac{1}{2})w - q^+}{w(w-1)(w-\alpha\beta)} \tag{1.1}$$

and

$$H_\lambda^-(w,\partial_w) := \frac{d^2}{dw^2} + \left(\frac{1-p}{w} + \frac{-p}{w-1} + \frac{p+\frac{3}{2}}{w-\alpha\beta}\right)\frac{d}{dw} + \frac{-\frac{3}{2}pw - q^-}{w(w-1)(w-\alpha\beta)}. \tag{1.2}$$

Here the numbers $p = p(\lambda)$ and $\nu = \nu(\lambda)$ are defined thorough the following relations:

$$p = \frac{2\nu - 3}{4}, \quad \lambda = 2\nu\frac{\sqrt{\alpha\beta(\alpha\beta - 1)}}{\alpha+\beta}. \tag{1.3}$$

The accessory parameters $q^\pm = q^\pm(\lambda)$ in these Heun's operators are expressed by the parameters α, β and eigenvalue λ as

$$q^+ = \left\{\left(p+\frac{1}{2}\right)^2 - \left(p+\frac{3}{4}\right)^2\left(\frac{\beta-\alpha}{\beta+\alpha}\right)^2\right\}(\alpha\beta-1) - \frac{1}{2}\left(p+\frac{1}{2}\right), \tag{1.4}$$

$$q^- = \left\{p^2 - \left(p+\frac{3}{4}\right)^2\left(\frac{\beta-\alpha}{\beta+\alpha}\right)^2\right\}(\alpha\beta-1) - \frac{3}{2}p. \qquad \square \tag{1.5}$$

Remark 1.1. Since the Fuchsian operators $H_\lambda^\pm(w,\partial_w)$ have regular singulars (only) at the points $0, 1$ and $\alpha\beta$, the general theory (see e.g. [34]) indicates that the connected simply-connected open subset $\Omega(\in \mathbb{C})$ in Theorem 1.1 can be chosen arbitrary provided it satisfies $0, 1 \in \Omega$, $\alpha\beta \notin \Omega$. For instance, Ω can be taken to be an open disk of radius $(1+\alpha\beta)/2$ centered at the origin. See also the discussion in §3.2.

Remark 1.2. The bijections in the theorem are essentially constructed by the extension of the map $T_C : L^2(\mathbb{R})_{\text{fin}} \to \mathbb{C}[y]$ defined in §2.1 (see (2.1)) to their completion $L^2(\mathbb{R}) \to \overline{\mathbb{C}[y]}$ (see Remark 2.3) and the modified Laplace operator $\mathcal{L}_1$ (Even case) and $\mathcal{L}_2$ (Odd case), respectively in §2.2. Concerning the L^2-condition and the holomorphic condition for the solutions of the Fuchsian differential equation with 6 regular singularities, which is equivalent to our Heun's ODE obtained by the variable change $w = (\text{constant}) \times z^2$, see §3.2. Summarizing those discussions done in advance, we shall put the proof of Theorem 1.1 in §3.3.

In this paper, the canonical form of Heun's equations are taken as

$$\frac{d^2}{dw^2} + \left(\frac{\gamma}{w} + \frac{\delta}{w-1} + \frac{\epsilon}{w-t}\right)\frac{d}{dw} + \frac{\rho\mu w - q}{w(w-1)(w-t)}. \tag{1.6}$$

The parameters $\gamma, \delta, \epsilon, \rho, \mu$ are generally complex and arbitrary, except that $t \neq 0, 1$. The first five parameters are linked by the equation

$$\gamma + \delta + \epsilon = \rho + \mu + 1. \tag{1.7}$$

The equation (1.6) is of Fuchsian type with regular singularities at $w = 0, 1, t, \infty$ and the characteristic exponents (or Frobenius exponents) at these regular singularities are given by $\{0, 1-\gamma\}$, $\{0, 1-\delta\}$, $\{0, 1-\epsilon\}$ and $\{\rho, \mu\}$ respectively. (The characteristic exponents are the roots of the indicial equation at the regular singularity [34].) As shown in (1.7), the sum of these exponents must take the value 2 according to the general theory of Fuchsian equations. The Riemann schema puts the equation (1.6) in the form (of a Riemann P-symbol)

$$\begin{pmatrix} 0 & 1 & t & \infty & ; w \;\; q \\ 0 & 0 & 0 & \rho & \\ 1-\gamma & 1-\delta & 1-\epsilon & \mu & \end{pmatrix}. \tag{1.8}$$

The number q represents the accessory parameter. It sometimes plays the part of an eigenparameter. As to the general theory of Heun ODEs and Riemann's scheme, see §1.1 and Appendix 7 in [34] (also [14, 35]).

We note that the Heun operators $H_\lambda^{\pm}(w,\partial_w)$ in the theorem above have four regular singular points, $w = 0, 1, \alpha\beta$ and ∞. Hence, the respective Riemann's scheme of the operators $H_\lambda^{\pm}(w,\partial_w)$ are expressed as

$$\mathrm{H}_\lambda^{+} : \begin{pmatrix} 0 & 1 & \alpha\beta & \infty & ; w \;\; q^{+} \\ 0 & 0 & 0 & \frac{1}{2} & \\ p+\frac{1}{2} & p+\frac{3}{2} & -p & -(p+\frac{1}{2}) & \end{pmatrix} \tag{1.9}$$

and

$$\mathrm{H}_\lambda^{-} : \begin{pmatrix} 0 & 1 & \alpha\beta & \infty & ; w \;\; q^{-} \\ 0 & 0 & 0 & \frac{3}{2} & \\ p & p+1 & -p-\frac{1}{2} & -p & \end{pmatrix}. \tag{1.10}$$

One notes that each element of the first row indicates a regular singular point of $H_\lambda^{\pm}$ and those in the second and third rows are expressing the corresponding exponents.

As a corollary of this theorem we may actually provide examples of finite-type eigenvalues (and corresponding solutions). In other words, we have an even (resp. odd) polynomial solution when the parameter $p + \frac{1}{2} \in \mathbb{N}$ (resp. $p \in \mathbb{N}$) satisfies a certain algebraic equation obtained by the determinant of Jacobi's (i.e. tri-diagonal) matrix (Proposition 4.1). It is also worth noticing that the ground state of the NcHO consists of only even functions [10], whence its simplicity follows from the result in [39]. The criterion for this simplicity (Theorem 1.1 in [39]) can be proved also by Theorem 1.1 above together with an upper bound estimate of the lowest eigenvalue given in [28] (Theorem 8.2.1) (see [39]). Furthermore, combining the results in Theorem 1.1 for even and odd eigenfunctions, and examining the monodromy data (characteristic exponents, etc.) of the corresponding Heun differential equations, we will show the following.

Theorem 1.2. *Suppose $\alpha\beta > 1$. The multiplicity m_λ of the eigenvalue λ for the non-commutative harmonic oscillators Q is at most 2. Moreover, when $\alpha \neq \beta$, $m_\lambda = 2$ holds if and only if either of the following two cases holds:*

1. *$\lambda \in \Sigma_0^{+}$ (resp. Σ_0^{-}); in this case one has a unique (up to scalar multiples) finite-type solution, i.e., a finite linear combination of even (resp. odd) twisted Hermite functions. Moreover, λ is of the form $\lambda = 2\frac{\sqrt{\alpha\beta(\alpha\beta-1)}}{\alpha+\beta}(2L+\frac{1}{2})$ (resp. $2\frac{\sqrt{\alpha\beta(\alpha\beta-1)}}{\alpha+\beta}(2L+\frac{3}{2})$) for $L \in \mathbb{N}$,*
2. *$\lambda \in \Sigma_\infty^{+} \cap \Sigma_\infty^{-}$.* □

Remark 1.3. As for the precise definition of twisted Hermite functions in Theorem 1.2, see [32].

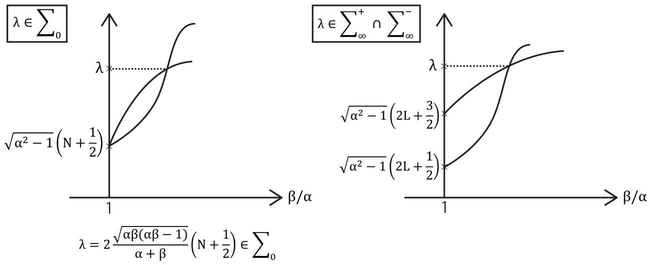

Figure 1: Examples of configuration for doubly degenerations of the spectrum of NcHO

Furthermore, employing the analogous discussion developed in [26], that is, using monodromy representation, in §4.3, we will determine the shape of the Heun polynomial solutions, which are the solutions of the Heun equations $H_\lambda^{\pm} f = 0$ derived from the eigenvalue problems of the NCHO (by Theorem 1.1) corresponding to finite-type eigenfunctions.

In the final section, we will discuss a connection between the operator $\mathcal{R}$ (a degree 2 element of the universal enveloping algebra $\mathcal{U}(\mathfrak{sl}_2)$) obtained naturally from NcHO through the oscillator representation of $\mathfrak{sl}_2$ (see Lemma 2.1) and the quantum Rabi model. Although the quantum Rabi model has had an impressive impact on many fields of physics [6], only recently (in 2011) could this model be declared solved by D. Braak [2]. The Hamiltonian is given as

$$H_{\mathrm{Rabi}}/\hbar = \omega a^\dagger a + \Delta\sigma_z + g(\sigma^+ + \sigma^-)(a^\dagger + a),$$

with $\sigma^\pm = (\sigma_x \pm i\sigma_y)/2$ (see §5), by taking the confluence procedure of Heun's equation (see [35, 34]). Actually, employing a (flat picture of) principal series representation ϖ_a of $\mathfrak{sl}_2$ (see Lemma 2.3), which is inequivalent to the oscillator representation when $a \neq 1, 2$ that gives the NcHO, one constructs the confluent Heun differential operator corresponding to the quantum Rabi model from $\mathcal{R}$. The result in §5 amounts to saying that the Rabi model is considered to be a sort of confluent version of the NcHO. We shall suppose throughout that $\alpha\beta > 1$.

2 Representation theoretic setting

2.1 Oscillator representation of $\mathfrak{sl}_2$

Although there is no exact (continuous) symmetry on Q ($\alpha \neq \beta$) described by the Lie algebra $\mathfrak{sl}_2$, as it is the case for the quantum harmonic oscillator, there seems to exist still a vague hidden (or modified) $\mathfrak{sl}_2$-symmetry behind it, beside the parity $\mathbb{Z}_2$. Thus a formulation of the problem by the language of $\mathfrak{sl}_2$ is useful, as we have observed in [31, 32, 25]. Moreover, as we will see in §5, in order to observe the relation between the NcHO and the Rabi model, a viewpoint employing Lie algebra representation (e.g. [21, 11]) of $\mathfrak{sl}_2$ is important.

Let H, E and F be the standard generators of the Lie algebra $\mathfrak{sl}_2$ defined by

$$H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

They satisfy the commutation relations

$$[H,\, E] = 2E,\ [H,\, F] = -2F,\ [E,\, F] = H.$$

For the triplet $(\kappa, \varepsilon, \nu) \in \mathbb{R}^3_{>0}$, define a second order element $\mathcal{R}$ of the universal enveloping algebra $\mathcal{U}(\mathfrak{sl}_2)$ of $\mathfrak{sl}_2$ by

$$\mathcal{R} := \frac{2}{\sinh 2\kappa}\Big\{\Big[(\sinh 2\kappa)(E-F) - (\cosh 2\kappa)H + \nu\Big](H-\nu) + (\varepsilon\nu)^2\Big\} \in \mathcal{U}(\mathfrak{sl}_2).$$

Define also an element $\tilde{\mathcal{R}}$ by

$$\tilde{\mathcal{R}} = \frac{2}{\sinh 2\kappa}\Big\{(H-\nu)\Big[(\sinh 2\kappa)(E-F) - (\cosh 2\kappa)H + \nu\Big] + (\varepsilon\nu)^2\Big\} \in \mathcal{U}(\mathfrak{sl}_2).$$

We define the oscillator representation π of $\mathfrak{sl}_2$ by

$$\pi(H) = x\partial_x + 1/2,\ \pi(E) = x^2/2,\ \pi(F) = -\partial_x^2/2,$$

where $\partial_x = d/dx$. We will also denote the algebra homomorphism from the universal enveloping algebra $\mathcal{U}(\mathfrak{sl}_2)$ to the ring $\mathbb{C}[x, \partial_x]$ of differential operators by the same letter π. By this realization, the eigenvalue problem $Q\varphi(x) = \lambda\varphi(x)$ ($\varphi \in L^2(\mathbb{R}, \mathbb{C}^2)$) turns to be solving the following equation.

$$[A\pi(E+F) + J\pi(H) - \lambda I]\varphi(x) = 0.$$

As we have shown in [32] (see also [25]) this equation can be rewritten as

$$[\pi(E+F) + \frac{1}{\sqrt{\alpha\beta}}J\pi(H) - \lambda A^{-1}]\tilde{\varphi}(x) = 0,$$

where $\tilde{\varphi}(x) = A^{\frac{1}{2}}\varphi(x)$.

Now, as usual, let us realize the oscillator representation on the polynomial ring $\mathbb{C}[y]$ in place of $L^2(\mathbb{R})$ using the Cayley transform $C := \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$.

Define the annihilation operator $\psi = (x + \partial_x)/\sqrt{2}$ and creation operator $\psi^\dagger = (x - \partial_x)/\sqrt{2}$. Then one has $[\psi, \psi^\dagger] = 1$. Put $\varphi_0(x) := e^{-x^2/2} \in L^2(\mathbb{R})$. Then φ_0 is the vacuum vector, that is, $\psi\varphi_0 = 0$, and φ_0 gives the ground state of the quantum harmonic oscillator $\mathcal{H} := \psi^\dagger\psi + \frac{1}{2}$. We define in general $\varphi_n := (\psi^\dagger)^n \varphi_0$, the Hermite functions. Then the set $\{\varphi_n \,|\, n = 0, 1, 2, \ldots\}$ forms an orthogonal basis with $(\varphi_n, \varphi_n) = \sqrt{\pi} n!$, $(\, , \,)$ being the standard inner product of $L^2(\mathbb{R})$ (see e.g. [11]). We denote the set of all finite linear combinations of the Hermite functions φ_n by $L^2(\mathbb{R})_{\text{fin}}$.

Let

$$T_C : L^2(\mathbb{R})_{\text{fin}} \to \mathbb{C}[y] \tag{2.1}$$

be the linear map defined by the property $T_C(\varphi_n) = y^n$. Then one immediately sees that $T_C(\psi^\dagger\varphi) = yT_C(\varphi)$ and $T_C(\psi\varphi) = \partial_y T_C(\varphi)$. Then, if we define the representation $(\pi', \mathbb{C}[y])$ of $\mathfrak{sl}_2$ by

$$\pi'(H) = y\partial_y + 1/2, \ \pi'(E) = y^2/2, \ \pi'(F) = -\partial_y^2/2,$$

one may easily show that $\pi'(CXC^{-1})T_C = T_C\pi(X)$ $(X \in \mathfrak{sl}_2)$. Moreover, if we define a Fisher inner product on $\mathbb{C}[y]$ by $(f, g)_F = \sqrt{\pi}(f(\partial_y)\bar{g}(y))|_{y=0}$ $(f, g \in \mathbb{C}[y])$, one finds that $(y^m, y^n)_F = \delta_{m,n}\sqrt{\pi} n!$, whence T_C gives an isometry. If we denote the completion of $\mathbb{C}[y]$ with respect to this inner product by $\overline{\mathbb{C}[y]}$, then it follows that the map T_C can be extended to an isometry between the Hilbert spaces $L^2(\mathbb{R})$ and $\overline{\mathbb{C}[y]}$.

The first claim of the following lemma follows immediately from [25] (Corollary 9 with Lemma 8), which translates the eigenvalue problem of Q into a single differential equation. The second follows in a similar way.

Lemma 2.1. *Assume $\alpha \neq \beta$. Determine the triplet $(\kappa, \varepsilon, \nu) \in \mathbb{R}^3_{>0}$ by the formulas*

$$\cosh\kappa = \sqrt{\frac{\alpha\beta}{\alpha\beta - 1}}, \ \ \sinh\kappa = \frac{1}{\sqrt{\alpha\beta - 1}}, \ \ \varepsilon = \left|\frac{\alpha - \beta}{\alpha + \beta}\right|, \ \ \nu = \frac{\alpha + \beta}{2\sqrt{\alpha\beta(\alpha\beta - 1)}}\lambda.$$

Then the eigenvalue problem $Q\varphi = \lambda\varphi$ $(\varphi \in L^2(\mathbb{R}, \mathbb{C}^2))$ is equivalent to the equation $\pi'(\mathcal{R})u = 0$ $(u \in \overline{\mathbb{C}[y]})$ by the isometry $T_C : L^2(\mathbb{R}) \to \overline{\mathbb{C}[y]}$. Let $K = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in \mathrm{Aut}(\mathbb{C}^2 \otimes L^2(\mathbb{R}))$. Then the eigenvalue problem $KQK\varphi = \lambda\varphi$ $(\varphi \in L^2(\mathbb{R}, \mathbb{C}^2))$ is equivalent to the equation $\pi'(\tilde{\mathcal{R}})u = 0$ $(u \in \overline{\mathbb{C}[y]})$.

Remark 2.1. We remark that the twist KQK and Q have the same spectrum.

Remark 2.2. Notice that $\pi'(\mathcal{R})$ is a 3rd order differential operator and the recurrence equation (or its corresponding continued fraction) in [32] is equivalent to this third order differential equation. It is also worth noting that the construction of the transcendental function $G^\pm(x)$ whose zeros give regular eigenvalues of the quantum Rabi model in [2, 3] resembles that of NcHO in [32].

Remark 2.3. The correspondence $\varphi \leftrightarrow u$ in the lemma above can be given explicitly. For the readers' convenience, we briefly summarize the correspondence given in [25].

Put

$$S_\pm := E + F \pm \frac{i}{\sqrt{\alpha\beta}} H \in \mathfrak{sl}_2$$

and

$$\tilde{\varphi} := \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} \begin{bmatrix} \sqrt{\alpha} & 0 \\ 0 & \sqrt{\beta} \end{bmatrix} \varphi.$$

Define (invertible) maps $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ and $T' : \mathbb{C}[y] \to \mathbb{C}[y]$ by

$$(Tf)(x) = e^{i(\sinh\kappa)x^2/2}(\cosh\kappa)^{1/4} f(\sqrt{\cosh\kappa}\, x) \quad \text{and} \quad (T'g)(y) = g\Big(\sqrt{\frac{\cosh\kappa}{i - \sinh\kappa}}\, y\Big).$$

The map T preserves the standard inner product on $L^2(\mathbb{R})$ (and intertwines the actions of $\mathfrak{sl}_2$, see p.361 in [25]). Also T' preserves the (Fisher) inner product on $\mathbb{C}[y]$ and can be extended to the isometry (using the same letter) $T' : \overline{\mathbb{C}[y]} \to \overline{\mathbb{C}[y]}$ (see, p.363 in [25]). Regarding T (resp. T') as an isometry on $L^2(\mathbb{R}, \mathbb{C}^2) = \mathbb{C}^2 \otimes L^2(\mathbb{R})$ (resp. $\mathbb{C}^2 \otimes \overline{\mathbb{C}[y]}$) in an obvious way, we set

$$\begin{bmatrix} u \\ \tilde{u} \end{bmatrix} := T' T_C T \tilde{\varphi}.$$

Then one knows that whenever $\alpha \neq \beta$ the eigenvalue problem $Q\varphi = \lambda\varphi$ can be written as

$$\begin{bmatrix} \frac{1}{\cosh\kappa}\pi'(H) - \frac{\alpha+\beta}{2\alpha\beta}\lambda & -\frac{\alpha+\beta}{2\alpha\beta}\varepsilon\lambda \\ -\frac{\alpha+\beta}{2\alpha\beta}\varepsilon\lambda & \frac{1}{\cosh\kappa}\pi'((\cosh 2\kappa)H - (\sinh 2\kappa)(E-F)) - \frac{\alpha+\beta}{2\alpha\beta}\lambda \end{bmatrix} \begin{bmatrix} u \\ \tilde{u} \end{bmatrix} = 0.$$

A standard argument allows us to see that the system of differential equations above for the vector function $\begin{bmatrix} u \\ \tilde{u} \end{bmatrix}$ is equivalent to the single differential equation for u

$$\left[\left\{\frac{1}{\cosh\kappa}\pi'((\cosh 2\kappa)H - (\sinh 2\kappa)(E-F)) - \frac{\alpha+\beta}{2\alpha\beta}\lambda\right\}\left\{\frac{1}{\cosh\kappa}\pi'(H) - \frac{\alpha+\beta}{2\alpha\beta}\lambda\right\} - \left\{\frac{\alpha+\beta}{2\alpha\beta}\varepsilon\lambda\right\}^2\right] u = 0,$$

by putting

$$\tilde{u} = \frac{2\alpha\beta}{(\alpha+\beta)\varepsilon}\left[\sqrt{1 - \frac{1}{\alpha\beta}}\,\pi'(H) - \frac{\alpha+\beta}{2\alpha\beta}\lambda\right] u.$$

Therefore, rewriting the above single equation, one concludes that $Q\varphi = \lambda\varphi\,(\varphi \in L^2(\mathbb{R}, \mathbb{C}^2))$ is equivalent to the equation $\pi'(\mathcal{R})u = 0\,(u \in \overline{\mathbb{C}[y]})$ (Lemma 8 and Corollary 9 in [25]).

2.2 Quasi-intertwiners arising from Laplace transforms

In order to obtain a complex analytic picture of the equation $\pi'(\mathcal{R})u = 0$ in Lemma 2.1 and to observe a connection between the NcHO and the quantum Rabi model through Heun's ODE, we introduce two representations of $\mathfrak{sl}_2$.

Let $a \in \mathbb{N}$. Define first the operator T_a acting on the space of Laurent polynomials $\mathbb{C}[y, y^{-1}]$ (or $y^2\mathbb{C}[y]$) by

$$T_a := -\frac{1}{2}\partial_y^2 + \frac{(a-1)(a-2)}{2} \cdot \frac{1}{y^2}.$$

Define a modified Laplace transform $\mathcal{L}_a$ by

$$(\mathcal{L}_a u)(z) = \int_0^\infty u(yz) e^{-\frac{y^2}{2}} y^{a-1} dy.$$

Then, one finds that

$$(\mathcal{L}_a T_a u)(z) = \big(-\frac{1}{2z}\partial_z + \frac{a-1}{2z^2}\big)(\mathcal{L}_a u)(z) + \frac{1}{2z}u'(0)\delta_{a,1} - \frac{a-1}{2z^2}u(0)\delta_{a,2},$$

where $\delta_{a,k} = 1$ when $k = a$ and 0 otherwise. This can be true whenever $u(0), u'(0)$ and $(\mathcal{L}_a u)(z)$ exist.

We now define a representation π'_a of $\mathfrak{sl}_2$ on $y^{a-1}\mathbb{C}[y]$ by

$$\pi'_a(H) = \pi'(H),\ \pi'_a(E) = \pi'(E),\ \pi'_a(F) = T_a = \pi'(F) + \frac{(a-1)(a-2)}{2} \cdot \frac{1}{y^2}. \tag{2.2}$$

Next, introduce another representation of $\mathfrak{sl}_2$ on $\mathbb{C}[z, z^{-1}]$ by

$$\varpi_a(H) = z\partial_z + \frac{1}{2},\ \varpi_a(E) = \frac{1}{2}z^2(z\partial_z + a),\ \varpi_a(F) = -\frac{1}{2z}\partial_z + \frac{a-1}{2z^2}. \tag{2.3}$$

Then one verifies the following

Lemma 2.2. (i) *Let $a \neq 1, 2$. Then one has*

$$\mathcal{L}_a \pi'_a(X) = \varpi_a(X)\mathcal{L}_a \ (X \in \mathfrak{sl}_2).$$

Furthermore, when $a = 1$ (resp. $a = 2$) the restriction of $\mathcal{L}_1$ (resp. $\mathcal{L}_2$) to the space of even (resp. odd) functions turns out to be an intertwiner between the two representations $\pi'(= \pi'_1)$ (resp. $= \pi'_2$)) and ϖ_1 (resp. ϖ_2). Precisely, $\mathcal{L}_j \, (j = 1, 2)$ possesses the following quasi-intertwining property:

$$\begin{aligned} \mathcal{L}_j \pi'(X) &= \varpi_j(X)\mathcal{L}_j, \ (\text{for } X = H, E), \\ (\mathcal{L}_1 \pi'(F)u)(z) &= \varpi_1(F)(\mathcal{L}_1 u)(z) + u'(0)/(2z), \\ (\mathcal{L}_2 \pi'(F)u)(z) &= \varpi_2(F)(\mathcal{L}_2 u)(z) - u(0)/(2z^2). \end{aligned}$$

(ii) *The map $\mathcal{L}_a$ gives $y^n \mapsto 2^{\frac{n+a}{2}-1}\Gamma\big(\frac{n+a}{2}\big)z^n$, whence if $u(y) = \sum_{n=0}^N u_n y^n \in \mathbb{C}[y]$ then $(\mathcal{L}_a u)(z) = 2^{\frac{a}{2}-1}\sum_{n=0}^N u_n \Gamma\big(\frac{n+a}{2}\big)(\sqrt{2}z)^n$.*

(iii) *If we define the inner product $(\ ,\ )_a$ in z-space such that $\{z^n \mid n \in \mathbb{Z}_+\}$ forms an orthogonal system and $(z^n, z^n)_a = \frac{\sqrt{\pi}n!}{2^{n+a-2}\Gamma\big(\frac{n+a}{2}\big)^2}$, then the modified Laplace transform $\mathcal{L}_a$ defines an isometry. For instance, when $a = 1, 2$, the inner products are given by $(z^n, z^n)_1 = \frac{2\Gamma(\frac{n}{2}+1)}{\Gamma(\frac{n+1}{2})} \sim \sqrt{2n}$ and $(z^n, z^n)_2 = \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2}+1)} \sim \sqrt{2/n}$ as $n \to \infty$, respectively.*

Proof. By the definition (2.2) (resp. (2.3)) of the representation π'_a (resp. ϖ_a), integral by parts at the definition of the Laplace integral $\mathcal{L}_a$ shows the assertion (i). Straight forward computation and the Stiring formula show (ii) and (iii). □

Since $\varpi_a(E)z^{-a} = 0$, one has the following second equivalence: the representation $(\pi'_a,\ y^{2-a}\mathbb{C}[y^2])$ can be considered as the Langlans quotient of the representations $(\varpi_a,\ \mathbb{C}[z^2, z^{-2}])$ or $(\varpi_a,\ z\mathbb{C}[z^2, z^{-2}])$ depending on the parity of a.

Lemma 2.3. *The operator $\mathcal{L}_a$ gives the equivalence of irreducible modules of $\mathfrak{sl}_2$:*

$$\begin{aligned} &(\pi'_a,\ y^{a-1}\mathbb{C}[y^2]) \cong (\varpi_a,\ z^{a-1}\mathbb{C}[z^2]), \\ &(\pi'_a,\ y^{2-a}\mathbb{C}[y^2]) \cong (\varpi_a,\ z^a\mathbb{C}[z^2, z^{-2}]/z^{-a}\mathbb{C}[z^{-2}]). \end{aligned}$$

Moreover, the Casimir operator $Z_C := 4EF + H^2 - 2H \in \mathcal{ZU}(\mathfrak{sl}_2)$ takes the value $(a-1)(a-2) - \frac{3}{4}$ in both representations $(\pi'_a,\ y^{a-1}\mathbb{C}[y^2])$ and $(\pi'_a,\ y^{2-a}\mathbb{C}[y^2])$.

Remark 2.4. The lemma holds for $a = 1, 2$. Actually, for instance, by the quasi-intertwiner $\mathcal{L}_1$, we obtain the equivalence between the odd part of the (oscillator) representation $(\pi', y\mathbb{C}[y^2])$ and the Langlands quotient of the representation $(\varpi, z\mathbb{C}[z^2, z^{-2}])$ of $\mathfrak{sl}_2$.

Remark 2.5. There is a symmetry $a \leftrightarrow 3 - a$ for π'_a. Actually, when $a \notin \mathbb{Z}$, there is an equivalence between the two representations π'_a and π'_{3-a} in a suitable setting.

2.3 Heun differential operators

Recalling the operator $\mathcal{R} \in \mathcal{U}(\mathfrak{sl}_2)$, one observes (with $\theta_z = z\partial_z$) that

$$\varpi_a(\mathcal{R}) = \Big\{(z^2 + z^{-2} - 2\coth 2\kappa)(\theta_z + \frac{1}{2}) + (a - \frac{1}{2})(z^2 - z^{-2}) + \frac{2\nu}{\sinh 2\kappa}\Big\}(\theta_z + \frac{1}{2} - \nu) + \frac{2(\varepsilon\nu)^2}{\sinh 2\kappa}.$$

Also, one notes that

$$\varpi_a(\tilde{\mathcal{R}}) = (\theta_z + \frac{1}{2} - \nu)\Big\{(z^2 + z^{-2} - 2\coth 2\kappa)(\theta_z + \frac{1}{2}) + (a - \frac{1}{2})(z^2 - z^{-2}) + \frac{2\nu}{\sinh 2\kappa}\Big\} + \frac{2(\varepsilon\nu)^2}{\sinh 2\kappa}.$$

Therefore, conjugating by z^{a-1}, one obtains the following lemma for $\mathcal{R}$. The formula for $\tilde{\mathcal{R}}$ is similar.

Lemma 2.4. *For each integer a, one has*

$$z^{-a+1}\varpi_a(\mathcal{R})z^{a-1} = \Big\{(z^2+z^{-2}-2\coth 2\kappa)(\theta_z+a-\frac{1}{2}) + (a-\frac{1}{2})(z^2-z^{-2})+\frac{2\nu}{\sinh 2\kappa}\Big\}(\theta_z+a-\frac{1}{2}-\nu)+\frac{2(\varepsilon\nu)^2}{\sinh 2\kappa}. \qquad \square$$

Furthermore, notice that the operators $\varpi_a(H), \varpi_a(E)$ and $\varpi_a(F)$ are invariant under the symmetry $z \to -z$. This implies that the operator $\varpi_a(\mathcal{R})$ can be expressed in terms of the variable z^2. In fact, one has the following.

Proposition 2.5. *Let $w := z^2\coth\kappa$. Then the following relation holds.*

$$z^{-a+1}\varpi_a(\mathcal{R})z^{a-1} = 4(\tanh\kappa)\, w(w-1)(w-\coth^2\kappa)H^a(w,\partial_w),$$

where $H^a(w,\partial_w)$ is the Heun differential operator given by

$$H^a(w,\partial_w) = \frac{d^2}{dw^2}+\left(\frac{3-2\nu+2a}{4w}+\frac{-1-2\nu+2a}{4(w-1)}+\frac{-1+2\nu+2a}{4(w-\coth^2\kappa)}\right)\frac{d}{dw} + \frac{\frac{1}{2}(a-\frac{1}{2})(a-\frac{1}{2}-\nu)w-q_a}{w(w-1)(w-\coth^2\kappa)}.$$

Here the appearing accessory parameter q_a is given by

$$q_a = \Big\{-(a-\frac{1}{2}-\nu)^2+(\varepsilon\nu)^2\Big\}(\coth^2\kappa-1)-2(a-\frac{1}{2})(a-\frac{1}{2}-\nu).$$

Similarly, for $\tilde{\mathcal{R}}$, one has $z^{-a+1}\varpi_a(\tilde{\mathcal{R}})z^{a-1} = 4(\tanh\kappa)\, w(w-1)(w-\coth^2\kappa)\tilde{H}^a(w,\partial_w)$ with

$$\tilde{H}^a_\lambda(w,\partial_w) = \frac{d^2f}{dw^2}+\left(\frac{-1-2\nu+2a}{4w}+\frac{3-2\nu+2a}{4(w-1)}+\frac{3+2\nu+2a}{4(w-\coth^2\kappa)}\right)\frac{d}{dw} + \frac{\frac{1}{2}(a-\frac{1}{2})(a+\frac{3}{2}-\nu)w-q_a}{w(w-1)(w-\coth^2\kappa)}.$$

Proof. Since $w = z^2\coth\kappa$, one notices that $z\partial_z = 2w\partial_w$. Put $t = \coth^2\kappa$ for simplicity. Using the relations

$$\begin{aligned} z^2+z^{-2}-2\coth 2\kappa &= (\tanh\kappa)w^{-1}(w-1)(w-\coth^2\kappa),\\ z^2-z^{-2} &= (\tanh\kappa)w^{-1}(w^2-\coth^2\kappa),\\ 2/\sinh 2\kappa &= (\tanh\kappa)(\coth^2\kappa-1),\end{aligned}$$

one obtains

$$z^{-a+1}\varpi_a(\mathcal{R})z^{a-1} = (\tanh\kappa)\Big[\Big\{w^{-1}(w-1)(w-t)(2w\partial_w+a-\frac{1}{2}) + (a-\frac{1}{2})w^{-1}(w^2-t)+(t-1)\nu\Big\}(2w\partial_w+a-\frac{1}{2}-\nu)+(t-1)(\varepsilon\nu)^2\Big].$$

Taking into account the relation $[\partial_w, w] = 1$, one observes

$$\begin{aligned} & z^{-a+1}\varpi_a(\mathcal{R})z^{a-1}\\ =& (\tanh\kappa)\Big[4w(w-1)(w-t)\partial_w^2\\ &+\Big\{(2a-2\nu+3)(w-1)(w-t)+(2a-2\nu-1)w(w-t)+(2a+2\nu-1)w(w-1)\Big\}\partial_w\\ &+2w(a-\frac{1}{2})(a-\frac{1}{2}-\nu)+\Big\{-(a-\frac{1}{2}-\nu)^2+(\varepsilon\nu)^2\Big\}(t-1)-2(a-\frac{1}{2})(a-\frac{1}{2}-\nu)\Big].\end{aligned}$$

Factoring out the leading coefficient, one obtains the expression of $H^a(w,\partial_w)$. The expression of $\tilde{H}^a_\lambda(w,\partial_w)$ follows from the relation

$$\begin{aligned}&z^{-a+1}\varpi_a(\tilde{\mathcal{R}})z^{a-1}-z^{-a+1}\varpi_a(\mathcal{R})z^{a-1}\\ =&4(\tanh\kappa)\,w(w-1)(w-\coth^2\kappa)\left[\left\{-\frac{1}{w}+\frac{1}{w-1}+\frac{1}{w-t}\right\}+\frac{a-\frac{1}{2}}{(w-1)(w-t)}\right].\end{aligned}$$

This proves the proposition. □

3 Heun's operators description for NcHO

3.1 Equivalence of differential operators

The equivalence between the spectral problem of Q and the existence/non-existence of holomorphic solutions of a Heun's ODE in a certain complex domain is described in [25] for the odd parity. In the same way we have the equivalence for the even parity.

Proposition 3.1. *The element $\mathcal{R}\in\mathcal{U}(\mathfrak{sl}_2)$ satisfies the following equations:*

$$\begin{aligned}(\mathcal{L}_1\pi'(\mathcal{R})u)(z)\quad&=\varpi_1(\mathcal{R})(\mathcal{L}_1u)(z)+(\nu-\tfrac{3}{2})u'(0)z^{-1},\\ (\mathcal{L}_2\pi'(\mathcal{R})u)(z)\quad&=\varpi_2(\mathcal{R})(\mathcal{L}_2u)(z)-(\nu-\tfrac{1}{2})u(0)z^{-2}.\end{aligned}$$

In particular, the equation $(\pi'(\mathcal{R})u)(z)=0$ for the even and odd case is respectively equivalent to the equation

$$\varpi_1(\mathcal{R})(\mathcal{L}_1u)(z)=0 \text{ (the even case)}\quad\text{and}\quad\varpi_2(\mathcal{R})(\mathcal{L}_2u)(z)=0\text{ (the odd case)}.\tag{3.1}$$

Here

$$\begin{aligned}\varpi_1(\mathcal{R})\quad&=\Big\{(z^2+z^{-2}-2\coth2\kappa)(\theta_z+\tfrac{1}{2})+\tfrac{1}{2}(z^2-z^{-2})+\tfrac{2\nu}{\sinh2\kappa}\Big\}(\theta_z+\tfrac{1}{2}-\nu)+\tfrac{2(\varepsilon\nu)^2}{\sinh2\kappa},\\ \varpi_2(\mathcal{R})\quad&=\Big\{(z^2+z^{-2}-2\coth2\kappa)(\theta_z+\tfrac{1}{2})+\tfrac{3}{2}(z^2-z^{-2})+\tfrac{2\nu}{\sinh2\kappa}\Big\}(\theta_z+\tfrac{1}{2}-\nu)+\tfrac{2(\varepsilon\nu)^2}{\sinh2\kappa},\end{aligned}$$

where we have put $\theta_z=z\frac{\partial}{\partial z}$.

Proof. For each $j=1,2$, the relation between $\pi'(\mathcal{R})$ and $\varpi_j(\mathcal{R})$ follows from the *quiasi*-intertwining property of the operator $\mathcal{L}_j$ resulting from (i) in Lemma 2.2. Since $u'(0)=0$ (resp. $u(0)=0$) for the even (resp. odd) case, the equivalence between the equation $(\pi'(\mathcal{R})u)(z)=0$ and (3.1) follows immediately. The expression of $\varpi_j(\mathcal{R})$ in terms of the variable z is obtained by taking $j=a=1,2$ of $\varpi_a(\mathcal{R})$ in §2.3. □

For the even parity, by Proposition 2.5, one has

$$\varpi_1(\mathcal{R})=4\sqrt{\alpha\beta}^{-1}\,w(w-1)(w-\alpha\beta)H^+_\lambda(w,\partial_w),$$

where $H^+_\lambda(w,\partial_w):=H^1_\lambda(w,\partial_w)$ is the Heun differential operator given by (1.1) in the Introduction, that is,

$$H^+_\lambda(w,\partial_w)=\frac{d^2}{dw^2}+\left(\frac{\frac{1}{2}-p}{w}+\frac{-\frac{1}{2}-p}{w-1}+\frac{p+1}{w-\alpha\beta}\right)\frac{d}{dw}+\frac{-\frac{1}{2}(p+\frac{1}{2})w-q^+}{w(w-1)(w-\alpha\beta)},$$

where

$$p:=\frac{2\nu-3}{4}\quad(\nu=\frac{\alpha+\beta}{2\sqrt{\alpha\beta(\alpha\beta-1)}}\lambda)$$

and the accessory parameter $q^+:=q_1$ is given by

$$q^+=\left\{\Big(p+\frac{1}{2}\Big)^2-\Big(p+\frac{3}{4}\Big)^2\Big(\frac{\beta-\alpha}{\beta+\alpha}\Big)^2\right\}(\alpha\beta-1)-\frac{1}{2}\Big(p+\frac{1}{2}\Big).$$

By its expression, $H^+_\lambda(w,\partial_w)$ is a second-order linear differential operator with four regular singular points $0,1,\alpha\beta$ and ∞ on $\mathbb{P}^1(\mathbb{C})$. Notice that the parameter ν designates the exponents. From these observations, we may summarize the properties of the operator $\varpi_1(\mathcal{R})$ as follows.

Proposition 3.2. *The second-order linear differential operator $\varpi_1(\mathcal{R})$ with rational coefficients in z has six singular points $z = 0, \pm(\alpha\beta)^{-\frac{1}{4}}, \pm(\alpha\beta)^{\frac{1}{4}}, \infty$. Here, all these six points are of regular singular type. The exponents of those singularities can be read from the following Riemann scheme:*

$$\varpi_1(\mathcal{R}) : \begin{pmatrix} 0 & (\alpha\beta)^{-\frac{1}{4}} & -(\alpha\beta)^{-\frac{1}{4}} & (\alpha\beta)^{\frac{1}{4}} & -(\alpha\beta)^{\frac{1}{4}} & \infty & ;\, z \;\; q^+ \\ 0 & 0 & 0 & 0 & 0 & 1 & \\ 2p+1 & p+\frac{3}{2} & p+\frac{3}{2} & -p & -p & -2p-1 & \end{pmatrix}.$$

Proof. The former statement is immediate from the discussion above. We prove the latter. By the expression $H^a_\lambda(w, \partial_w)$ in Proposition 2.5 with $a = 1$, the second-order linear differential operator $H^+_\lambda(w, \partial_w)$ with four regular singularities on the Riemann sphere $\mathbb{C} \cup \{\infty\}$ has the following Riemann schemes (P-symbols) [34]:

$$\begin{pmatrix} 0 & 1 & \coth^2\kappa & \infty & ;\, w \;\; q^+ \\ 0 & 0 & 0 & \rho & \\ 1-\gamma & 1-\delta & 1-\epsilon & \mu & \end{pmatrix} = \begin{pmatrix} 0 & 1 & \coth^2\kappa & \infty & ;\, w \;\; q^+ \\ 0 & 0 & 0 & \frac{1}{2} & \\ p+\frac{1}{2} & p+\frac{3}{2} & -p & -p-\frac{1}{2} & \end{pmatrix}.$$

Since

$$\varpi_1(\mathcal{R}) = 4\coth^2\kappa\, z^2(z^2 - \tanh\kappa)(z^2 - \coth\kappa) H^+_\lambda(w, \partial_w)$$

and $\tanh\kappa = 1/\sqrt{\alpha\beta}$, one sees that the P-symbol of $\varpi_1(\mathcal{R})$ is given by

$$\begin{pmatrix} 0 & 1/\sqrt{\alpha\beta} & \sqrt{\alpha\beta} & \infty & ;\, z^2 \;\; q^+ \\ 0 & 0 & 0 & \frac{1}{2} & \\ p+\frac{1}{2} & p+\frac{3}{2} & -p & -p-\frac{1}{2} & \end{pmatrix}.$$

By unfolding $z^2 \mapsto z$, we have the assertion. □

Remark 3.1. For the readers' convenience, we recall the Riemann schema of the operator $\varpi_2(\mathcal{R})$ from [25]:

$$\varpi_2(\mathcal{R}) : \begin{pmatrix} 0 & (\alpha\beta)^{-\frac{1}{4}} & -(\alpha\beta)^{-\frac{1}{4}} & (\alpha\beta)^{\frac{1}{4}} & -(\alpha\beta)^{\frac{1}{4}} & \infty & ;\, z \;\; q^- \\ 1 & 0 & 0 & 0 & 0 & 2 & \\ 2p+1 & p+1 & p+1 & -(p+\frac{1}{2}) & -(p+\frac{1}{2}) & -2p-1 & \end{pmatrix}.$$

3.2 L^2-conditions and analytic continuation

The discussion of this part is analogues to that one made by Ochiai in [25]. However, in [25], it was necessary to deal with the third order equation (see Lemma 13 in [25]) because there was no chance to discuss the even eigenfunctions in the framework of a Heun ODE directly. Here, since we have taken another modified Laplace integral $\mathcal{L}_1$ (in place of $\mathcal{L}_2$), which intertwines eigenfunctions of the NCHO in the even parity and formal even power series solutions in z-space directly, it is sufficient to handle only a second order equation.

We denote by $\mathcal{O}_0$ the germ of the holomorphic function at the origin 0. In order to accomplish the proof of Theorem 1.1, we study the holomorphic even solution $\hat{u}(z) := (\mathcal{L}_1 u)(z) \in \mathcal{O}_0$ satisfying the equation $\varpi_1(\mathcal{R})(\hat{u}) = 0$ (see (3.1) in Proposition 3.1). We will take the discussion essentially due to [25], which establishes the relation between L^2-conditions on $\mathbb{R}$ (or convergence conditions in $\overline{\mathbb{C}[y]}$) and the holomorphic solutions of the equation with regular singularities in z-space.

One first notices that the solution $\hat{u} \in \mathcal{O}_0$ of $\varpi_1(\mathcal{R})(\hat{u}) = 0$ can be analytically continued along a path avoiding the singular points of the differential equation. Namely, in our case, the solution is holomorphic on the open disk of radius $(\alpha\beta)^{-\frac{1}{4}}$ (< 1). We now consider the behavior of the solution near the points $z = \pm(\alpha\beta)^{-\frac{1}{4}}$. Since the singularities of $\varpi_1(\mathcal{R})$ are located (only) at $0, \pm(\alpha\beta)^{-\frac{1}{4}}, \pm(\alpha\beta)^{\frac{1}{4}}$ and ∞ as in Proposition 3.2, we have the following two possibilities.

1. The solution $\hat{u}(z)$ is holomorphic near the points $z = \pm(\alpha\beta)^{-\frac{1}{4}}$. In this case, general theory guarantees that it is continued to a single-valued holomorphic function on the disk $\{z \in \mathbb{C} \mid |z| < (\alpha\beta)^{\frac{1}{4}}\}$. In terms of the Taylor expansion $\hat{u}(z) = \sum_{n=0}^{\infty} a_n z^{2n}$, one has then

$$\forall\varepsilon > 0,\ \exists N \text{ such that } \forall n > N, \qquad |a_n| \le (1/\sqrt{\alpha\beta} + \varepsilon)^n.$$

2. The solution cannot be holomorphically continued to at least one of the points $z = \pm(\alpha\beta)^{-1/4}$, that is, the the radius of convergence of the Taylor series $\hat{u}(z) = \sum_{n=0}^{\infty} a_n z^{2n}$ is $(\alpha\beta)^{-1/4}$. In this case, one has

$$\forall \varepsilon > 0 \text{ and } \forall N,\ \exists n > N \text{ such that } \qquad |a_n| \geq (\sqrt{\alpha\beta} - \varepsilon)^n.$$

Proposition 3.3. *Consider the (even) formal series solution* $u(y) = \sum_{n=0}^{\infty} u_n y^{2n} \in \mathbb{C}[[y]]$ *of the equation* $\pi'(\mathcal{R})u = 0$. *Let* $\hat{u}(z) := (\mathcal{L}_1 u)(z) \in \mathbb{C}[[z]]$. *Then one has the following.*

(i) *A formal series solution* $\hat{u}(z) \in \mathbb{C}[[z]]$ *of the equation* $\varpi_1(\mathcal{R})f = 0$ *converges to a holomorphic function near the origin* 0, *that is,* $\hat{u}(z) \in \mathcal{O}_0$.

(ii) *The following conditions are equivalent:*

  (a) $u(y)$ *convergences in the Hilbert space* $\overline{\mathbb{C}[y]}$.

  (b) $\hat{u}(z)$ *converges to a holomorphic function on the unit disk.*

  (c) $\hat{u}(z)$ *can be holomorphically continued to a neighborhood of the closed interval* $[0,\, (\alpha\beta)^{-1/4}]$.

(iii) *The following conditions are equivalent:*

  (a) $u(y)$ *does not convergence in the Hilbert space* $\overline{\mathbb{C}[y]}$.

  (b) $\hat{u}(z)$ *cannot be convergent to any holomorphic function on the unit disk.*

  (c) $\hat{u}(z)$ *cannot be holomorphically continued to a neighborhood of the closed integral* $[0,\, (\alpha\beta)^{-1/4}]$.

(iv) *The unit disk in the statements in* (ii) and (iii) can be replaced by a connected and simply-connected domain Ω' of $\mathbb{C}$ which contains the three points $0, \pm(\alpha\beta)^{-1/4}$ and does not contain either of $\pm(\alpha\beta)^{1/4}$.

Proof. Since the operator $\varpi_1(\mathcal{R})$ is regular singular at the origin, one has that any formal series solution of $f \in \mathbb{C}[[z]]$ of the equation $\varpi_1(\mathcal{R})f = 0$ converges to a holomorphic function near the origin, whence the assertion (i) follows.

Recall the fact $(z^n, z^n)_1 = \frac{2\Gamma(\frac{n}{2}+1)}{\Gamma(\frac{n+1}{2})} \sim \sqrt{2n}\,(n \to \infty)$ on $\mathbb{C}[[z]]$ space, which makes $\mathcal{L}_1$ an isometry between $\mathbb{C}[[y]]$ and $\mathbb{C}[[z]]$. Setting $a_n = u_n \Gamma(n + \frac{1}{2}) 2^{n-\frac{1}{2}}$, one has $|u_n|^2 (y^{2n}, y^{2n}) = |a_n|^2 (z^{2n}, z^{2n})_1$ when $\hat{u}(z) = (\mathcal{L}_1 u)(z)$. It follows that the condition (ii-b) and (iii-b) imply

$$|u_n|^2 (y^{2n}, y^{2n}) \begin{cases} \leq & \frac{2\Gamma(n+1)}{\Gamma(n+\frac{1}{2})} (1/\sqrt{\alpha\beta} + \varepsilon)^{2n} \quad \text{for case (ii)}, \\ \geq & \frac{2\Gamma(n+1)}{\Gamma(n+\frac{1}{2})} (\sqrt{\alpha\beta} - \varepsilon)^{2n} \quad \text{for case (iii)}. \end{cases}$$

This immediately shows (ii-b) $\Rightarrow$ (ii-a) and (iii-b) $\Rightarrow$ (iii-a). Since there are only two possibilities for the behavior of $\hat{u}(z)$ as we described above, the converse (ii-a) $\Rightarrow$ (ii-b) and (iii-a) $\Rightarrow$ (iii-b) also follows. The claim (ii-b) $\Rightarrow$ (ii-c) is obvious. Conversely, since $\hat{u}(z)$ is even, the condition (ii-c) implies that this is also holomorphic at $z = -(\alpha\beta)^{-1/4}$. Since $\hat{u}(z)$ is a solution of the differential equation $\varpi_1(\mathcal{R})(\hat{u}) = 0$ it can be holomorphically continued to the regular singular points $z = \pm(\alpha\beta)^{1/4}$. In particular, it is holomorphic on the unit disk. This proves (ii-b). The same reasoning shows (iii-b) $\Leftrightarrow$ (iii-c).

The discussion above shows that the statement in (ii) (resp. (iii)) remains true if one replaces the unit disk by such domain Ω'. This proves (iv). □

3.3 Proof of Theorem 1.1

We are now in a position to give a proof of Theorem 1.1.

Let Ω' be a connected and simply-connected domain of $\mathbb{C}$ satisfying $0, \pm(\alpha\beta)^{-1/4} \in \Omega'$ while $\pm(\alpha\beta)^{1/4} \notin \Omega'$. Then, to summarize the equivalences of the several equations in question, one has firstly the equivalence $Q\varphi = \lambda\varphi$ $(\varphi \in L^2(\mathbb{R}, \mathbb{C}^2)) \Leftrightarrow \pi'(\mathcal{R})u = 0\,(u \in \overline{\mathbb{C}[y]})$, which is established in Lemma 2.1 (due to §2 in [25], see Remark 2.3). Successively, by Proposition 3.3, one has the equivalence $\pi'(\mathcal{R})u = 0\,(u \in \overline{\mathbb{C}[y]}) \Leftrightarrow \varpi_1(\mathcal{R})\hat{u}(z) = 0$ $(\hat{u} \in \mathcal{O}(\Omega'))$ for the even parity under the intertwining isometry $u \mapsto \hat{u} = \mathcal{L}_1 u$ obtained in Lemma 2.2. It follows that the eigenvalue problem $Q\varphi = \lambda\varphi$ of the NcHO for the even parity is equivalent to that of finding all the holomorphic even solutions $U(z) \in \mathcal{O}(\Omega')$ of the differential equation $\varpi_1(\mathcal{R})U(z) = 0$.

Notice that every solution $U(z) \in \mathcal{O}(\Omega')$ of $\varpi_1(\mathcal{R})U(z) = 0$ is the sum of an even solution and odd solution. By the map $z \to w = \sqrt{\alpha\beta}\, z^2$, the set of even functions in $\mathcal{O}(\Omega')$ is isomorphic to $\mathcal{O}(\Omega)$, where Ω is taken to be a connected and simply-connected domain in w-space containing 0 and 1, but does not contain the point $\alpha\beta(>1)$. Recall also that the differential operator $\varpi_1(\mathcal{R})$ is in terms of variable w expressed as

$$\varpi_1(\mathcal{R}) = 4\sqrt{\alpha\beta}^{-1}\, w(w-1)(w-\alpha\beta)H_\lambda^+(w,\partial_w) \;:\; \mathcal{O}(\Omega) \to \mathcal{O}(\Omega). \tag{3.2}$$

It is hence elementary to see the equivalence

$$\ker(\varpi_1(\mathcal{R}),\, \mathcal{O}(\Omega)) \cong \ker(H_\lambda^+,\, \mathcal{O}(\Omega)).$$

Therefore, the eigenvalue problem of Q for the even parity is equivalent to finding the holomorphic solutions $f(w) \in \mathcal{O}(\Omega)$ of the differential equation $H_\lambda^+(w,\partial_w)f(w) = 0$. This completes the proof of Theorem 1.1. □

Remark 3.2. By the same reasoning of the proof of Lemma 17 in [25], one sees that the map $\varpi_1(\mathcal{R}) : \mathcal{O}(\Omega) \to \mathcal{O}(\Omega)$ is continuous, having a closed range, and index zero. In particular, one knows that the injectivity, surjectivity and bijectivity of the operator $\varpi_1(\mathcal{R}) : \mathcal{O}(\Omega) \to \mathcal{O}(\Omega)$ are equivalent.

4 Degeneration of eigenstates

In this section, we discuss degeneration of eigenstates of the NcHO, that is, focus on eigenvalues of finite-type and their multiplicities. We give an example of finite-type eigenvalues and the proof of Theorem 1.2, which claims that the multiplicity of any eigenvalue of Q is at most 2 and actually may reach 2 in the same parity.

4.1 Polynomial solutions of $\varpi_1(\mathcal{R})f = 0$

Recall first Theorem 1.1 in [32] that the finite-type eigenvalues are of the form

$$\lambda = 2\frac{\sqrt{\alpha\beta(\alpha\beta-1)}}{\alpha+\beta}(N+\frac{1}{2}) \quad (N \in \mathbb{Z}_{\geq 0}).$$

This implies that $\nu = \lambda\delta\cosh\kappa = N + \frac{1}{2}$ if we have a polynomial solution of $\varpi_1(\mathcal{R})f = 0$. Suppose that $p(z) = \sum_{n=0}^{L} a_n z^{2n}$ $(a_L \neq 0)$ is a polynomial solution of the equation $\varpi_1(\mathcal{R})f = 0$ with $\nu = N + \frac{1}{2}$. Since

$$\begin{aligned}
&\varpi_1(\mathcal{R})z^{2n}\\
=&\Big\{(z^2+z^{-2}-2\coth 2\kappa)(2n+\frac{1}{2}) + \frac{1}{2}(z^2-z^{-2}) + \frac{2\nu}{\sinh 2\kappa}\Big\}(2n+\frac{1}{2}-\nu)z^{2n} + \frac{2(\varepsilon\nu)^2}{\sinh 2\kappa}z^{2n}\\
=&(2n+1)(2n-N)z^{2n+2}\\
+&\Big[\Big\{-2\coth 2\kappa(2n+\frac{1}{2}) + \frac{2N+1}{\sinh 2\kappa}\Big\}(2n-N) + \frac{\varepsilon^2(2N+1)^2}{2\sinh 2\kappa}\Big]z^{2n} + 2n(2n-N)z^{2n-2},
\end{aligned}$$

one observes

$$\begin{aligned}
\varpi_1(\mathcal{R})p(z) =& \sum_{n=1}^{L+1} a_{n-1}(2n-1)(2n-2-N)z^{2n}\\
&+\sum_{n=0}^{L} a_n\Big[\Big\{-2(2n+\frac{1}{2})\coth 2\kappa + \frac{2N+1}{\sinh 2\kappa}\Big\}(2n-N) + \frac{\varepsilon^2(2N+1)^2}{2\sinh 2\kappa}\Big]z^{2n}\\
&+\sum_{n=0}^{L-1} a_{n+1}2(n+1)(2n+2-N)z^{2n} = 0.
\end{aligned}$$

If we look at the coefficient of z^{2L+2} then $a_L(2L+1)(2L-N)=0$, whence necessarily $N=2L$ if $p\not\equiv 0$. Therefore the condition N to be even is necessary for having an even polynomial solution of $\varpi_1(\mathcal{R})f=0$, i.e., a finite-type eigenfunction of Q by Corollary 3.1.

Now we assume that $N=2L$. Then we have

$$\begin{cases} -2a_{L-1}(2L-1)+a_L\frac{\varepsilon^2(4L+1)^2}{2\sinh 2\kappa}=0,\\ -4a_{L-2}(2L-3)+a_{L-1}\Big[-2\big\{-2(2L-\frac{3}{2})\coth 2\kappa+\frac{4L+1}{\sinh 2\kappa}\big\}+\frac{\varepsilon^2(4L+1)^2}{2\sinh 2\kappa}\Big]=0,\\ a_{n-1}(2n-1)(2n-2-2L)+a_n\Big[2\big\{-2(2n+\frac{1}{2})\coth 2\kappa+\frac{4L+1}{\sinh 2\kappa}\big\}(n-L)+\frac{\varepsilon^2(4L+1)^2}{2\sinh 2\kappa}\Big]\\ \qquad\qquad +4a_{n+1}(n+1)(n+1-L)=0 \quad (1\le n\le L-2),\\ a_0\Big[-2L\big\{-\coth 2\kappa+\frac{4L+1}{\sinh 2\kappa}\big\}+\frac{\varepsilon^2(4L+1)^2}{2\sinh 2\kappa}\Big]+4a_1(1-L)=0. \end{cases}$$

Notice that there are no monomial solutions, and in particular no constant solutions to $\varpi_1(\mathcal{R})f=0$, whenever $\varepsilon\neq 0$, i.e. $\alpha\neq\beta$. Let us hence consider the simplest case, that is, $L=1$. Then $N=2$ and the equations above reduce to the following

$$\begin{cases} -2a_0+a_1\frac{5^2\varepsilon^2}{2\sinh 2\kappa}=0,\\ a_0\Big[-2\big\{-\coth 2\kappa+\frac{5}{\sinh 2\kappa}\big\}+\frac{25\varepsilon^2}{2\sinh 2\kappa}\Big]=0. \end{cases}$$

Hence, if

$$-4(-\cosh 2\kappa+5)+25\varepsilon^2=0 \tag{4.1}$$

holds, then the polynomial $p(z)=a_0+a_1z^2=a_0\Big(1+\frac{4\sinh 2\kappa}{25\varepsilon^2}z^2\Big)$ is a non-trivial solution of $\varpi_1(\mathcal{R})p=0$. We now observe the existence of solutions for (4.1). For simplicity we put $\alpha=1$ and $\beta>1$. Since $\varepsilon^2=\Big(\frac{\alpha-\beta}{\alpha+\beta}\Big)^2$ and $\cosh 2\kappa=\frac{\alpha\beta+1}{\alpha\beta-1}$, the equation (4.1) turns to be

$$25\Big(\frac{\beta-1}{\beta+1}\Big)^2+4\frac{\beta+1}{\beta-1}-20=0.$$

Define next the cubic polynomial

$$f(\beta)=25(\beta-1)^3+4(\beta+1)^3-20(\beta+1)^2(\beta-1).$$

Then, since $f(1)>0$, $f(2)<0$, $f(8)<0$, $f(9)>0$, it follows immediately that we have 2 solutions of (4.1), one in the interval $(1,2)$ and another one in $(8,9)$. This shows that there exists a pair (α,β) such that $Q\varphi=5\frac{\sqrt{\alpha\beta(\alpha\beta-1)}}{\alpha+\beta}\varphi$ and $\varphi(-x)=\varphi(x)$.

The general theorem in [32] indicates (actually, Theorem 1.2, whose proof will be given in the subsequent section, implies) that the multiplicity of the eigenvalue $5\frac{\sqrt{\alpha\beta(\alpha\beta-1)}}{\alpha+\beta}$ is 2 for this $Q=Q_{(\alpha,\beta)}$. Hence, the eigenvalue curves can be indeed crossing as the numerical graph in [23] (see Figure 1 on p.648) has indicated.

In general, we define the tri-diagonal $(L+1)\times(L+1)$-matrix $B_{2L}(\alpha,\beta)=(B_{ij})_{0\le i,j\le L}$ by

$$\begin{cases} B_{0,0}=\Big[-2L\big\{-\coth 2\kappa+\frac{4L+1}{\sinh 2\kappa}\big\}+\frac{\varepsilon^2(4L+1)^2}{2\sinh 2\kappa}\Big],\ B_{0,1}=4(1-L),\\ B_{n-1,n}=(2n-1)(2n-2-2L),\\ B_{n,n}=\Big[2\big\{-2(2n+\frac{1}{2})\coth 2\kappa+\frac{4L+1}{\sinh 2\kappa}\big\}(n-L)+\frac{\varepsilon^2(4L+1)^2}{2\sinh 2\kappa}\Big],\\ B_{n+1,n}=4(n+1)(n+1-L)=0\,(n=1,2,\dots L-2),\\ B_{L-2,L-1}=-4(2L-3),\ B_{L-1,L-1}=\Big[-2\big\{-2(2L-\frac{3}{2})\coth 2\kappa+\frac{4L+1}{\sinh 2\kappa}\big\}+\frac{\varepsilon^2(4L+1)^2}{2\sinh 2\kappa}\Big],\\ B_{L-1,L}=-2(2L-1),\ B_{L,L}=\frac{\varepsilon^2(4L+1)^2}{2\sinh 2\kappa}. \end{cases}$$

Note that $B_{ij}=0$ if $|i-j|>1$.

Since there can not be two independent polynomial solutions of $\varpi_1(\mathcal{R})f=0$, we notice that the rank of the matrix satisfies $L\le \operatorname{rank}(B_{2L}(\alpha,\beta))\le L+1$. Clearly one has the following.

Proposition 4.1. *Let $L \in \mathbb{N}$. If $\alpha, \beta\,(\alpha \neq \beta)$ satisfy the algebraic equation* $\det(B_{2L}(\alpha,\beta)) = 0$, *then* $\lambda = 2\frac{\sqrt{\alpha\beta(\alpha\beta-1)}}{\alpha+\beta}(2L+\frac{1}{2}) \in \Sigma_0^+$. □

Remark 4.1. Since $\alpha \neq \beta$, one has that the coefficient $B_{L,L} \neq 0$. Thus, if we set $\tilde{B}_{2L}(\alpha,\beta) = (B_{ij})_{0\leq i,j\leq L-1}$ we may consider the equation $\det(\tilde{B}_{2L}(\alpha,\beta)) = 0$ in place of $\det(B_{2L}(\alpha,\beta)) = 0$.

Remark 4.2. The odd cases corresponding to Proposition 4.1 can be established in the same way.

Remark 4.3. It would be an interesting problem to see whether these eigenvalues and corresponding eigenfunctions can be obtained within a framework of finite dimensional representation theory of an appropriate Lie algebra as in [40], in which we have proved that the Judd (isolated exact) solutions/eigenstates ([20]) of quantum Rabi model are obtained as vectors in finite dimensional irreducible representations of Lie algebra $\mathfrak{sl}_2$.

4.2 Proof of Theorem 1.2

We now prove that the multiplicity m_λ of the eigenvalue λ of Q is at most 2. When $\alpha = \beta$, since Q is unitarily equivalent to a couple of quantum harmonic oscillators, i.e. $m_\lambda \equiv 2$ ([31, 32]), one may assume that $\alpha \neq \beta$. Since one knows from [32] that the multiplicity of each eigenvalue is at most 3, it is enough to show that $m_\lambda \neq 3$ for every λ.

Suppose $m_\lambda = 3$. Then we have either the case $\lambda \in \Sigma_0^+ \cap \Sigma_\infty^-$ or $\lambda \in \Sigma_0^- \cap \Sigma_\infty^+$. Let us assume $\lambda \in \Sigma_0^+ \cap \Sigma_\infty^-$. This implies that we have $\dim_{\mathbb{C}}\{f \in \mathcal{O}(\Omega) \,|\, H_\lambda^+ f = 0\} = 2$, Ω being the domain in Theorem 1.1, and λ is of the form $\lambda = 2\frac{\sqrt{\alpha\beta(\alpha\beta-1)}}{\alpha+\beta}(2L+\frac{1}{2})$ for some $L \in \mathbb{N}$ ([32] I). Recall the relation $p = L - \frac{1}{2}$. Then, it follows that the parameter p in the Riemann scheme of the Heun operator H_λ^+ satisfies $p + \frac{1}{2} \in \mathbb{N}$.

Let $f_1(w)$ and $f_2(w)$ respectively be a polynomial and a holomorphic solution of $H_\lambda^+(w,\partial_w)f = 0$ corresponding to the eigenvalue $\lambda \in \Sigma_0^+ (\subset \Sigma_\infty^+)$ of even eigenfunctions of Q in Theorem 1.1. Let $\tilde{f}_j(z)\,(j = 1,2)$ be the respective solutions of the equation $\varpi_1(\mathcal{R})\tilde{f}(z) = 0$, that is, $f_j(w) = \tilde{f}_j(z)\,(w = z^2\coth\kappa = z^2\sqrt{\alpha\beta})$. Put $u_j = \mathcal{L}_1^{-1}\tilde{f}_j$. Note that u_1 is an even polynomial in $\mathbb{C}[y]$. We may then construct a constant-term-free even solution $u^+ \in \overline{\mathbb{C}[y]}$ of $\pi'(\mathcal{R})u^+ = 0$ by a suitably chosen linear combination of u_1 and u_2. Then, by Proposition 3.1, one verifies that $\varpi_2(\mathcal{R})(\mathcal{L}_2 u^+)(z) = 0$. If we put $\tilde{g}^+(z) = (\mathcal{L}_2 u^+)(z)$, then $\tilde{g}^+(z) \in \mathcal{O}(\Omega')$, Ω' being a connected and simply connected domain satisfying $0, \pm(\alpha\beta)^{-\frac{1}{4}} \in \Omega'$ while $\pm(\alpha\beta)^{\frac{1}{4}} \notin \Omega'$, and $\tilde{g}^+(0) = 0$. Define $g^+(w)$ by the equation $g^+(w) = z^{-1}\tilde{g}^+(z)$. Then $g^+(w) \in \sqrt{w}\mathcal{O}(\Omega)$ is a solution of $H_\lambda^- g(w) = 0$. Note that $g^+(w)$ is holomorphic at $w = 1$.

We recall now the Riemann scheme of H_λ^-

$$\begin{pmatrix} 0 & 1 & \alpha\beta & \infty & ;\, w \;\; q^- \\ 0 & 0 & 0 & \frac{3}{2} & \\ p & p+1 & -p-\frac{1}{2} & -p & \end{pmatrix}.$$

Regarding this Riemann scheme, one knows that $g^+(w)$ is a (global) solution whose exponent is $p\,(\in \{\frac{1}{2}, \frac{3}{2}, \ldots\})$ at $w = 0$, whereas should be 0 at $w = 1$. On the other hand, since $\lambda \in \Sigma_\infty^-$, there exists a non-zero solution $g^-(w)$ of $H_\lambda^- g^-(w) = 0$, which is holomorphic on the domain Ω. This in particular implies that the exponents of $g^-(w)$ are 0 both at $w = 0$ and 1. It follows that two local (independent) solutions at $w = 1$ are holomorphic. This contradicts the fact that there is a local solution of $H_\lambda^- g(w) = 0$ whose exponent at the point $w = 1$ is $p + 1\,(\in \{\frac{3}{2}, \frac{5}{2}, \ldots\})$. Therefore we have $\Sigma_0^+ \cap \Sigma_\infty^- = \emptyset$. Similarily one can show that $\Sigma_0^- \cap \Sigma_\infty^+ = \emptyset$. This completes the proof of the fact $m_\lambda \leq 2$. The rest of the assertions of the theorem is clear. □

Remark 4.4. One knows [23] that the eigenvalue curves of the NCHO are continuous with respect to the variable α/β. Hence the fact $\Sigma_\infty^+ \cap \Sigma_\infty^- \neq \phi$ immediately follows from the numerical examples *Figure 1.* Approximate N-th eigenvalues λ_N of Q in [23].

4.3 Heun polynomials for $H_\lambda^+ f = 0$ for $\lambda \in \Sigma_0^+$

In this subsection assume again that $\alpha \neq \beta$. As in the study of the connection problem for the Heun differential equation in [26], which corresponds to the odd parity case, from the equation

$H_\lambda^+ f = 0$, one can determine a shape of the solution corresponding to the eigenvalues in Σ_0^+. In the terminology of [34] (see p.41) these solutions are given by Heun polynomials. Here, the Heun polynomial, which we denote by Hp (see [34]) is, by definition, a solution of the Heun equation given by the form

$$Hp(w) = w^{\sigma_1}(w-1)^{\sigma_2}(w-\alpha\beta)^{\sigma_3}p(w),$$

where $p(w)$ is a polynomial in w, and σ_1, σ_2 and σ_3 are, each of them, one of the two possible exponents at the corresponding singularity.

In order to discuss the Heun polynomials, as in the preceding subsection, we consider the monodromy representation of the equation $H_\lambda^+ f = 0$. Take a base point near the origin and denote B_0, B_1, B_2 and B_3 the monodromy matrix around the singularities $0, 1, \alpha\beta$ and ∞, respectively. Note that $B_0 B_1 B_2 B_3 = I$. To be more precise, if we denote $(f_1(w), f_2(w))$ the basis of local solutions at $w = 0$, then the analytic continuation of (f_1, f_2) along the path around $w = 0$ is $(f_1, f_2)B_0$ (see, e.g. [14] for monodromy representations). The proof of the following theorems owes the idea to [26]. Actually, the technical discussion developed in [26] works nicely also to the even parity case, that is, the equation $H_\lambda^+ f = 0$. However, for the readers' convenience we shall present the proof.

Let us first recall the Riemann scheme of the Heun equation $H_\lambda^+ f = 0$, where one notes that $p = \frac{\alpha+\beta}{\sqrt{\alpha\beta(\alpha\beta-1)}}\lambda - \frac{3}{4}$.

$$\begin{pmatrix} 0 & 1 & \alpha\beta & \infty & ;w \;\; q^+ \\ 0 & 0 & 0 & \frac{1}{2} & \\ p+\frac{1}{2} & p+\frac{3}{2} & -p & -\left(p+\frac{1}{2}\right) & \end{pmatrix}. \tag{4.2}$$

From this, one sees that the eigenvalues of B_0 and B_1 are 1, whence they are unipotent. When $B_0 \neq I$, there exists a logarithmic solution at the singular point $w = 0$. When $B_0 = I$, the point $w = 0$ is an apparent singular point, that is, all of the solutions at $w = 0$ are meromorphic near the point $w = 0$. (This is also the case for B_1 at $w = 1$.) The monodromy matrices B_2 and B_3 have two distinct eigenvalues, 1 and -1, and thus are semisimple.

Theorem 4.2. *Let Ω be a connected and simply-connected domain of $\mathbb{C}$ satisfying $0, 1 \in \Omega$ while $\alpha\beta \notin \Omega$. Consider the differential equation $H_\lambda^+ f = 0$. Suppose that $\lambda \in \Sigma_0^+$. Then, there exist Heun polynomials $Hp_1(w)$ and $Hp_2(w)$ such that $\{f \in \mathcal{O}(\Omega) \mid H_\lambda^+ f = 0\} = \mathbb{C}Hp_1 \oplus \mathbb{C}Hp_2$. More precisely, $Hp_1(w)$ is equal to a polynomial $p_1(w)$ of degree at most $p + \frac{1}{2}$ and $Hp_2(w) = (w-\alpha\beta)^{-p}p_2(w)$, $p_2(w)$ being a polynomial of degree at most $p - \frac{1}{2}$, and these polynomials $p_j(w)\,(j = 1, 2)$ are unique up to scalar multiples.*

Proof. By the assumption and Theorem 1.1, one sees that $\dim_{\mathbb{C}}\{f \in \mathcal{O}(\Omega) \mid H_\lambda^+ f = 0\} = 2$. Also, by Theorem 1.2, since $p = L - \frac{1}{2}$ when $\lambda = 2\frac{\sqrt{\alpha\beta(\alpha\beta-1)}}{\alpha+\beta}\left(2L + \frac{1}{2}\right)$ $(L \in \mathbb{N})$, one notices that $p + \frac{1}{2} \in \mathbb{N}$ in the Reimann scheme of $H_\lambda^+ f = 0$ (4.2). Since there exists two dimensional holomorphic solutions on Ω, $B_0 = B_1 = I$. Thus, one has $B_2 = B_3^{-1}$. Hence it follows that the monodromy representation factors through the cyclic group of order two. Thus, we may choose a basis such that $B_2 = B_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Then, the solution corresponding to the vector $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is invariant under the monodromy representation. It follows that this solution is meromorphic on the Riemannian sphere $\mathbb{C} \cup \{\infty\}$, whence it is a rational function of w. Write this solution by $q_1(w)$. Denote by $f_2(w)$ the solution corresponding to the vector $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Then, $f_2(w)$ changes sign along the path around $w = \alpha\beta, \infty$. The sign is invariant along the path around $w = 0, 1$. Hence it follows that $f_2(w)$ can be written as $\sqrt{w-\alpha\beta}\,q_2(w)$ for some rational function $q_2(w)$. These rational functions $q_j(w)$ are obviously holomorphic, except at the singular points arising from the differential equation $H_\lambda^+ f = 0$. From the Riemann scheme (4.2), the exponents at $w = 0$ and 1 are known to be nonnegative, whence these two solutions are holomorphic at these points. This implies that $q_j(w)\,(j = 1, 2)$ are holomorphic at $w = 0, 1$. The exponents of the solution $q_1(w)$ is 0 at $w = \alpha\beta$ and is $-(p + \frac{1}{2}) \in \mathbb{Z}_{<0}$ at ∞. It follows that $q_1(w)$ is a polynomial of degree at most $p + \frac{1}{2}$. The exponents of the solution $f_2(w) = \sqrt{w-\alpha\beta}\,q_2(w)$ is $-p$ at $w = \alpha\beta$ and $\frac{1}{2}$ at ∞. Namely, the exponent of $q_2(w)$ is $-p - \frac{1}{2}$ at $w = \alpha\beta$ and 0 at ∞. Therefore, we conclude that there exists a polynomial $p_2(w)$ such that $q_2(w) = p_2(w)(w-\alpha\beta)^{-p-\frac{1}{2}}$. This completes the proof of the theorem. □

Remark 4.5. Since the degree of polynomial $p_2(w)$ is at most $p-\frac{1}{2}$, we have $p \geq \frac{1}{2}$, whence $L \geq 1$.

Furthermore, as in [26], we have two converse statements of Theorem 4.2. In other words, the existence of a solution of the form either rational solution or non-rational (algebraic) solution stated in Theorem 4.2 implies essentially that the dimension of the space of holomorphic solutions of $H_\lambda^+ f = 0$ on Ω is 2. First, the case of non-rational solution can be described as follows.

Theorem 4.3. *Suppose that the Heun equation $H_\lambda^+ f = 0$ has a solution of the form $q(w)(w-\alpha\beta)^{\frac{1}{2}}$ at the origin, where $q(w)$ is a non-zero rational function. Then, one has $\dim_{\mathbb{C}}\{f \in \mathcal{O}(\Omega) \mid H_\lambda^+ f = 0\} = 2$. In particular, all of the assertions stated in Theorem 4.2 are true and $\lambda \in \Sigma_0^+$.*

Proof. Choose the basis such that $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ corresponding to the solution $q(w)(w-\alpha\beta)^{\frac{1}{2}}$. The behavior along the analytic continuation of the multi-valued function $q(w)(w-\alpha\beta)^{\frac{1}{2}}$ around the singular points $0, 1, \alpha\beta$ and ∞ determine the second columns of the monodromy matrices B_0, B_1, B_2 and B_3 as $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$, respectively. Therefore, since one knows all the eigenvalues of $B_i (i = 0, 1, 2, 3)$ from the Riemann scheme (4.2), they are of the forms

$$B_0 = \begin{bmatrix} 1 & 0 \\ c_0 & -1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 & 0 \\ c_1 & -1 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 1 & 0 \\ c_2 & -1 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 1 & 0 \\ c_3 & -1 \end{bmatrix}, \tag{4.3}$$

where $c_i (i = 0, 1, 2, 3)$ are certain unknown constants. The exponent of the solution corresponding to e_2 is $-p$ at $w = \alpha\beta$ and $\frac{1}{2}$ at $w = \infty$. Since the solution $q(w)(w-\alpha\beta)^{\frac{1}{2}}$ corresponding to e_2 is non-logarithmic and holomorphic with nonnegative exponent at $w = 0$ and 1, the rational function $q(w)$ is of the form $q(w) = p(w)(w-\alpha\beta)^{-p-\frac{1}{2}}$, $p(w)$ being a polynomial of degree at most $p-\frac{1}{2}$.

Suppose now $c_0 \neq 0$. Then, at the singular point $w = 0$, there exists a non-zero holomorphic solution and logarithmic solution of $H_\lambda^+ f = 0$. Also, one sees that the holomorphic solution corresponding to e_2 is fixed by B_0 and is determined uniquely up to a scalar multiple. Hence the general theory implies that the difference of the exponents at this singular point should be an integer, and that the holomorphic solution corresponds to the larger exponent. In the present case, as indicated in (4.2), under the assumption, the exponent corresponding to the holomorphic solution must be $p+\frac{1}{2}$. This shows that $q(w)$ has a zero with multiplicity $p+\frac{1}{2}$ at $\alpha\beta$. However, this is impossible, because the degree of $p(w)$ is at most $p-\frac{1}{2}$. Hence we conclude that $c_0 = 0$, whence the matrix $B_0 = I$. Similarly, $B_1 = I$, because in this case also if $c_1 \neq 0$ the exponent turns out to be $p+\frac{3}{2}$ which is greater than $p-\frac{1}{2}$. This shows that the dimension of the space of holomorphic solutions on Ω equals 2. Hence the theorem follows. □

The second converse is the case where one has a rational solution of the Heun equation $H_\lambda^+ f = 0$. Recall that the relation $p = \frac{\alpha+\beta}{\sqrt{\alpha\beta(\alpha\beta-1)}}\lambda - \frac{3}{4}$.

Theorem 4.4. *Assume $p+\frac{1}{2} \in \mathbb{N}$. Suppose that the Heun equation $H_\lambda^+ f = 0$ has a non-zero rational solution of w at the origin. Then, one has $\dim_{\mathbb{C}}\{f \in \mathcal{O}(\Omega) \mid H_\lambda^+ f = 0\} = 2$. In particular, all of the assertions stated in Theorem 4.2 are true.*

Proof. From the Riemann scheme (4.2) of H_λ^+, there could be a logarithmic solution at $w = 0$ (resp. 1). Suppose that $w = 0$ is logarithmic. Then, the rational solution has exponent $p+\frac{1}{2}$ at $w = 0$. Since the sum of the exponents of a non-zero rational solution is at least $p+\frac{1}{2}+0+0+\{-(p+\frac{1}{2})\} = 0$, such a function is unique, up to a scalar multiple, and is a multiple of $w^{p+\frac{1}{2}}$. However, since the Heun operator H_λ^+ comes from the non-commutative harmonic oscillator, by the formula (1.4) of the accessory parameter q^+, one can easily verifies that the monomial $w^{p+\frac{1}{2}}$ can not be a solution of the equation $H_\lambda^+ f = 0$. Thus, one concludes that $w = 0$ is an apparent singular points, that is, $B_0 = I$. We can discuss similarly for the case $w = 1$. Suppose the point $w = 1$ is logarithmic. Then the meromorphic solution at $w = 1$ is unique, up to a scalar multiple, and the corresponding exponent is $p+\frac{3}{2}$, whence the rational solution has exponent $p+\frac{3}{2}$ at $w = 1$. This implies that the sum of the exponents of a non-zero rational solution is at least $0 + p + \frac{3}{2} + 0 + \{-(p + \frac{1}{2})\} > 0$, but there can not exist such rational function. This contradicts the assumption. It hence follows that $B_1 = 1$. Hence there exists a two diminutional space of holomorphic solutions on Ω. This shows the theorem. □

Remark 4.6. All the statements about $\lambda \in \Sigma_0^-$ (or the differential operator $H_\lambda^-(w, \partial_w)$) above follow from the corresponding theorems in [26].

5 Connection with the quantum Rabi model via confluence process

In this section we will observe the relation between the NcHO and the quantum Rabi model. Precisely, we find that the quantum Rabi model (see [20, 2, 8, 41]) can be obtained from $\mathcal{R} \in \mathcal{U}(\mathfrak{sl}_2)$ by a suitable choice of a triple $(\kappa, \varepsilon, \nu) \in \mathbb{R}^3$.

The quantum Rabi model is defined by the Hamiltonian

$$H_{\text{Rabi}}/\hbar = \omega\psi^\dagger\psi + \Delta\sigma_z + g\sigma_x(\psi^\dagger + \psi).$$

Here $\psi = (x + \partial_x)/\sqrt{2}$ (resp. $\psi^\dagger = (x - \partial_x)/\sqrt{2}$) is the annihilation (resp. creation) operator for a bosonic mode of frequency ω, $\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ are the Pauli matrices for the two-level system, 2Δ is the energy difference between the two levels, and g denotes the coupling strength between the two-level system and the bosonic mode. For simplicity and without loss of generality we may set $\hbar = 1$ and $\omega = 1$.

In order to observe the relation between the NcHO and the quantum Rabi model, we will consider the confluent Heun differential equation which is derived by the standard confluence procedure from the Heun differential equation defined by $\mathcal{R}$ in Lemma 2.1 via the representation $\pi_a'(\cong \varpi_a)$ of $\mathfrak{sl}_2$. Roughly speaking, our observation shows that the quantum Rabi model can be obtained by a confluence process by $\mathcal{R}$ through their respective Heun's pictures:

$$\begin{array}{ccccc} \text{NcHO} & \xleftarrow{\ \pi'\ } & \underset{\mathcal{U}(\mathfrak{sl}_2)}{\overset{\mathcal{R}}{\Cap}} & \xrightarrow[\pi_a'(\cong\varpi_a)]{\ \mathcal{L}_a\ } & \text{Heun ODE} \\ & & & & \Big\Downarrow {\scriptstyle \text{confluence process}} \\ & & & & \text{Confluent Heun ODE} \sim \text{quantum Rabi Model} \end{array}$$

In this picture, under the action defined by the representation (a flat picture of principal series) π_a' on $\mathbb{C}[y, y^{-1}]$ (and ϖ_a) of $\mathfrak{sl}_2$ (see §5.1 below), which is not equivalent in general to the oscillator representation π', $\mathcal{R}$ provides a target Heun operator for obtaining the confluent Heun operator corresponding to the quantum Rabi model through the Laplace transform $\mathcal{L}_a$.

5.1 Confluent Heun equations derived from the quantum Rabi model

From now on we assume $a \in \mathbb{R}$, not necessarily an integer. The analysis of the quantum Rabi model has extensively used the Bargmann representation of bosonic operators which is realized by the following Bargmann transform $\mathcal{B}$ (from real coordinate x to complex variable z) [1, 37].

$$(\mathcal{B}f)(z) = \sqrt{2}\int_{-\infty}^{\infty} f(x)e^{2\pi xz - \pi x^2 - \frac{\pi}{2}z^2}dx.$$

Here the Bargmann space is by definition a Hilbert space of entire functions equipped with the inner product

$$(f|g) = \frac{1}{\pi}\int_{\mathbb{C}} \overline{f(z)}g(z)e^{-|z|^2}d(\text{Re}(z))d(\text{Im}(z)).$$

The main advantage is simply due to the fact that

$$\psi^\dagger = (x - \partial_x)/\sqrt{2} \to z \quad \text{and} \quad \psi = (x + \partial_x)/\sqrt{2} \to \partial_z.$$

Remark 5.1. This makes the quantum Rabi model to be a first order differential operator. The same situation, however, does not appear for NcHOs. This explains one of the reasons why the analysis of NcHOs is rather difficult.

Then the Schrödinger equation $H_{\text{Rabi}}\varphi = E\varphi$ of the quantum Rabi model is reduced to the following 2nd order differential equation:

$$\frac{d^2 f}{dz^2} + p(z)\frac{df}{dz} + q(z)f = 0,$$

where

$$p(z) = \frac{(1-2E-2g^2)z - g}{z^2 - g^2}, \quad q(z) = \frac{-g^2z^2 + gz + E^2 - g^2 - \Delta^2}{z^2 - g^2}.$$

Write $f(w) = e^{-gz}\phi(x)$, where $x = (g+z)/(2g)$. Substituting f into the equation above, one finds that the function ϕ satisfies the following confluent Heun equation (by a calculation similar to that in [41]). Then one has $H_1^{\text{Rabi}}\phi = 0$, where

$$H_1^{\text{Rabi}} := \frac{d^2}{dx^2} + \Big(-4g^2 + \frac{1-(E+g^2)}{x} + \frac{1-(E+g^2+1)}{x-1}\Big)\frac{d}{dx} + \frac{4g^2(E+g^2)x + \mu}{x(x-1)},$$

with the accessory parameter $\mu = (E+g^2)^2 - 4g^2(E+g^2) - \Delta^2$.

Setting $f(z) = e^{gz}\phi(x)$, where $x = (g-z)/2g$, one obtains another equation as

$$H_2^{\text{Rabi}} := \frac{d^2}{dx^2} + \Big(-4g^2 + \frac{1-(E+g^2+1)}{x} + \frac{1-(E+g^2)}{x-1}\Big)\frac{d}{dx} + \frac{4g^2(E+g^2-1)x + \mu}{x(x-1)}.$$

Remark 5.2. Each equation $H_j^{\text{Rabi}}\phi = 0$ $(j = 1, 2)$ has a one dimensional family of analytic solutions. The suitable linear combination of these analytic solutions gives symmetric (resp. anti-symmetric) solutions (cf. [41]).

5.2 Confluence process of the Heun equation

Put $t = \coth^2\kappa (> 1)$. The Heun operator $H^a(w, \partial_w)$ derived from $\varpi_a(\mathcal{R})$ is give by

$$H^a(w,\partial_w) = \frac{d^2}{dw^2} + \left(\frac{3-2\nu+2a}{4w} + \frac{-1-2\nu+2a}{4(w-1)} + \frac{-1+2\nu+2a}{4(w-t)}\right)\frac{d}{dw} + \frac{\frac{1}{2}(a-\frac{1}{2})(a-\frac{1}{2}-\nu)w - q_a}{w(w-1)(w-t)}.$$

The corresponding generalized Riemann scheme (see §1.5 in [35]) is expressed as

$$\left(\begin{matrix} 1 & 1 & 1 & 1 & \\ 0 & 1 & t & \infty & ;\ w\ \ q_a \\ 0 & 0 & 0 & a-\frac{1}{2} & \\ \frac{1+2\nu-2a}{4} & \frac{5+2\nu-2a}{4} & \frac{5-2\nu-2a}{4} & \frac{-1-2\nu+2a}{4} & \end{matrix}\right).$$

Here the first line indicates the s-rank of each singularity (see §1.1 in [35]). Replace a (resp. ν) by $a+p$ (resp. $\nu+p$) in the expression of $H^a(w,\partial_w)$ above. It then follows that with

$$A := \frac{1}{4}(-1-2\nu+2a),\ B := a+p+\frac{1}{2},\ C := \frac{1}{4}(3-2\nu+2a) = 1+A,\ D := A,$$

we have

$$w(w-1)(w-t)H^a(w,\partial_w) = w(w-1)(w-t)\partial_w^2 + \Big[C(w-1)(w-t) + Dw(w-t) + (A+B+1-C-D)w(w-1)\Big]\partial_w + ABw - q_a.$$

Let us consider a confluence process of the singular points at $w = t$ and $w = \infty$ ([35] p.100, Table 3.1.2). The corresponding process is given by $t := \rho^{-1}, B := r\rho^{-1}$ and $\rho \to 0$ (equivalently $p \to \infty$):

$$\begin{aligned}
&- \lim_{\rho \to 0} w(w-1)(w-t)\rho H^a(w, \partial_w) \\
=&w(w-1)\partial_w^2 + \big[C(w-1) + Dw - rw(w-1)\big] - rAw + \lim_{\rho \to 0} \rho q_a.
\end{aligned}$$

Now we take $\varepsilon = k\rho$ for some constraint k. Then

$$\begin{aligned}
\lim_{\rho \to 0} \rho q_a &= \lim_{\rho \to 0} \Big[\Big\{ -(a - \frac{1}{2} - \nu)^2 + (\varepsilon(\nu + p))^2 \Big\}(1 - \rho) - \Big\{2\rho(a + p - \frac{1}{2})\Big\} \cdot (a - \frac{1}{2} - \nu)\Big] \\
&= -(2A)^2 - 4A + k^2.
\end{aligned}$$

Hence one obtains the following confluent Heun equation.

$$\frac{d^2\phi}{dw^2} + \Big[-r + \frac{1+A}{w} + \frac{A}{w-1}\Big]\frac{d\phi}{dw} + \frac{-rAw - (2A)^2 - 4A + k^2}{w(w-1)}\phi = 0,$$

whose generalized Riemann's scheme is given as

$$\begin{pmatrix} 1 & 1 & 2 & \\ 0 & 1 & \infty & ;w \quad -q \\ 0 & 0 & A & \\ -A & 1-A & 1+A & \\ & & 0 & \\ & & t & \end{pmatrix} \quad \text{with} \quad A = \frac{1}{4}(-1 - 2\nu + 2a).$$

Notice that $w = \infty$ is an irregular singularity with s-rank 2 (see e.g. [35], p.33).

Let us compare this equation with the confluent Heun operator H_1^{Rabi} for the quantum Rabi model above. Then, taking $r = 4g^2$, $A = -(E + g^2)$ with a suitable choice of k (i.e. $k^2 = 5A^2 + 4(1 - g^2)A - \Delta^2$) in this equation gives the latter.

Remark 5.3. Recall the operator $\tilde{\mathcal{R}} \in \mathcal{U}(\mathfrak{sl}_2)$. Then, one has the confluent Heun operator from the Heun operator $\tilde{H}^a_\lambda(w, \partial_w)$ corresponding to $\varpi_a(\tilde{\mathcal{R}})$ as

$$\tilde{H}^a_\lambda(w, \partial_w) \;\to\; \frac{d^2}{dw^2} + \Big[-r + \frac{A}{w} + \frac{1+A}{w-1}\Big]\frac{d}{dw} + \frac{-r(1+A)w - (2A)^2 - 4A + k^2}{w(w-1)}.$$

A confluence procedure for $\varpi_a(\tilde{\mathcal{R}})$, similar to the one we have taken in the case of $\varpi_a(\mathcal{R})$, yields H_2^{Rabi} of the preceding subsection.

Remark 5.4. One can find an element $\mathcal{K}$ (resp. $\tilde{\mathcal{K}}$) $\in \mathcal{U}(\mathfrak{sl}_2)$ of order two such that $\varpi_a(\mathcal{K})$ (resp. $\varpi_a(\tilde{\mathcal{K}})$) essentially (i.e. up to the accessory parameter) provides the confluent Heun operator H_1^{Rabi} (resp. H_2^{Rabi}) [40].

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