Non-trivial real Milnor fibrations

Osamu Saeki  (Kyushu University)

Joint work with
R. Araújo dos Santos (ICMC-USP, Brazil),
M.A.B. Hohlenwerger (ICMC-USP, Brazil),
T.O. Souza (UFU, Brazil)

Singularities in Generic Geometry and its Applications

June 4, 2015, at Kobe
§ 1. Real Milnor Fibrations
Let \( f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0) \) be a polynomial map germ, \( n \geq p \geq 2 \).
Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be a polynomial map germ, $n \geq p \geq 2$. We assume that it has an isolated singularity at 0.
Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be a polynomial map germ, $n \geq p \geq 2$. We assume that it has an isolated singularity at $0$.

Then, for $0 < \forall \varepsilon << 1$, the link $K = f^{-1}(0) \cap S^{n-1}_\varepsilon$ is a codimension $p$ submanifold of $S^{n-1}_\varepsilon$. 

![Diagram of a link of a map germ](image)
Theorem 1.1 (Milnor, 1968)

*There exists a smooth locally trivial fibration* \( \varphi : S^{n-1}_\varepsilon \setminus K \rightarrow S^{p-1} \).
Theorem 1.1 (Milnor, 1968)

There exists a smooth locally trivial fibration \( \varphi : S^{n-1}_\varepsilon \setminus K \to S^{p-1} \).

More precisely, there is a trivialization of a tubular nbhd

\[
N(K) = K \times D^p \text{ s.t. } \varphi\big|_{N(K) \setminus K} \text{ coincides with }
\]

\[
\frac{f}{||f||} : N(K) \setminus K = K \times (D^p \setminus \{0\}) \to D^p \setminus \{0\} \to S^{p-1}.
\]
The closure $F_f$ of a fiber of Milnor fibration, $\varphi : S^{m-1}_\varepsilon \setminus K \to S^{p-1}$, is called the **Milnor fiber**, which is a compact manifold with boundary.
The closure $F_f$ of a fiber of Milnor fibration, $\varphi : S^{m-1}_\varepsilon \setminus K \to S^{p-1}$, is called the **Milnor fiber**, which is a compact manifold with boundary.

We have $\dim F_f = n - p$, $\dim K = n - p - 1$, $\partial F_f = K$. 
The closure $F_f$ of a fiber of Milnor fibration, $\varphi : S^{m-1}_\varepsilon \setminus K \to S^{p-1}$, is called the \textbf{Milnor fiber}, which is a compact manifold with boundary.

We have $\dim F_f = n - p$, $\dim K = n - p - 1$, $\partial F_f = K$.

Milnor fibration $\varphi$ is \textbf{trivial} if $F_f$ is homeomorphic to the disk $D^{n-p}$. 
The closure $F_f$ of a fiber of Milnor fibration, $\varphi : S_{\varepsilon}^{m-1} \setminus K \rightarrow S^{p-1}$, is called the **Milnor fiber**, which is a compact manifold with boundary.

We have $\dim F_f = n - p$, $\dim K = n - p - 1$, $\partial F_f = K$.

Milnor fibration $\varphi$ is **trivial** if $F_f$ is homeomorphic to the disk $D^{n-p}$. For example, the projection map germ has trivial Milnor fibration.
The closure $F_f$ of a fiber of Milnor fibration, $\varphi : S_{\varepsilon}^{m-1} \setminus K \to S^{p-1}$, is called the **Milnor fiber**, which is a compact manifold with boundary. We have $\dim F_f = n - p$, $\dim K = n - p - 1$, $\partial F_f = K$.

Milnor fibration $\varphi$ is **trivial** if $F_f$ is homeomorphic to the disk $D^{n-p}$. For example, the projection map germ has trivial Milnor fibration.

**Problem 1.2 (Milnor, 1968)** *For which dimensions $n \geq p \geq 2$ do non-trivial examples exist?*
Theorem 1.3 (Church–Lamotke, 1976 + Poincaré Conj.)

(a) For $0 \leq n - p \leq 2$, non-trivial examples occur precisely for $(n, p) = (2, 2), (4, 3), (4, 2)$.

(b) For $n - p \geq 4$, non-trivial examples occur for all $(n, p)$.

(c) For $n - p = 3$, non-trivial examples occur precisely for $(n, p) = (5, 2), (8, 5)$ and possibly for $(6, 3)$. 
Theorem 1.3 (Church–Lamotke, 1976 + Poincaré Conj.)
(a) For $0 \leq n - p \leq 2$, non-trivial examples occur precisely for $(n, p) = (2, 2), (4, 3), (4, 2)$.
(b) For $n - p \geq 4$, non-trivial examples occur for all $(n, p)$.
(c) For $n - p = 3$, non-trivial examples occur precisely for $(n, p) = (5, 2), (8, 5)$ and possibly for $(6, 3)$.

$(n, p) = (6, 3)$ was the unique unsolved dimension pair!
Theorem 1.3 (Church–Lamotke, 1976 + Poincaré Conj.)

(a) For $0 \leq n - p \leq 2$, non-trivial examples occur precisely for $(n, p) = (2, 2), (4, 3), (4, 2)$.

(b) For $n - p \geq 4$, non-trivial examples occur for all $(n, p)$.

(c) For $n - p = 3$, non-trivial examples occur precisely for $(n, p) = (5, 2), (8, 5)$ and possibly for $(6, 3)$.

$(n, p) = (6, 3)$ was the unique unsolved dimension pair!


There exist polynomial map germs $(\mathbb{R}^6, 0) \to (\mathbb{R}^3, 0)$ with an isolated singularity at 0 with **non-trivial** Milnor fibration.
§2. Neuwirth–Stallings Pairs
Definition 2.1 (Looijenga, 1971)

\[ K = K^{n-p-1} : \text{oriented submanifold of } S^{n-1} \text{ with trivial normal bundle.} \]

We allow \( K = \emptyset \).

Suppose \( \exists \psi : S^{n-1} \setminus K \to S^{p-1} \) locally trivial fibration

s.t. for a trivialization \( N(K) = K \times D^p \) of a tubular nbhd of \( K \),

\( \psi|_{N(K) \setminus K} \) coincides with

\[ N(K) \setminus K = K \times (D^p \setminus \{0\}) \xrightarrow{\pi} S^{p-1}, \]

where \( \pi(x, y) = y/\|y\| \).
Definition 2.1 (Looijenga, 1971)

\[ K = K^{n-p-1} : \text{oriented submanifold of } S^{n-1} \text{ with trivial normal bundle}. \]

We allow \( K = \emptyset \).

Suppose \( \exists \psi : S^{n-1} \setminus K \to S^{p-1} \) locally trivial fibration
s.t. for a trivialization \( N(K) = K \times D^p \) of a tubular nbhd of \( K \),
\( \psi|_{N(K) \setminus K} \) coincides with

\[ N(K) \setminus K = K \times (D^p \setminus \{0\}) \xrightarrow{\pi} S^{p-1}, \]

where \( \pi(x, y) = y/\|y\| \).

Then, the pair \( (S^{n-1}, K^{n-p-1}) \) is called a Neuwirth–Stallings pair, or an NS-pair for short.
Definition 2.1 (Looijenga, 1971)
\[ K = K^{n-p-1} : \text{oriented submanifold of } S^{n-1} \text{ with trivial normal bundle.} \]
We allow \( K = \emptyset \).
Suppose \( \exists \psi : S^{n-1} \setminus K \to S^{p-1} \) locally trivial fibration
s.t. for a trivialization \( N(K) = K \times D^p \) of a tubular nbhd of \( K \),
\( \psi|_{N(K) \setminus K} \) coincides with
\[ N(K) \setminus K = K \times (D^p \setminus \{0\}) \xrightarrow{\pi} S^{p-1}, \]
where \( \pi(x, y) = y/\|y\| \).

Then, the pair \( (S^{n-1}, K^{n-p-1}) \) is called a Neuworth–Stallings pair, or an NS-pair for short.
It is also called a fibered knot or an open book structure.
Definition 2.1 (Looijenga, 1971)

$K = K^{n-p-1}$: oriented submanifold of $S^{n-1}$ with trivial normal bundle. We allow $K = \emptyset$.

Suppose $\exists \psi : S^{n-1} \setminus K \to S^{p-1}$ locally trivial fibration
s.t. for a trivialization $N(K) = K \times D^p$ of a tubular nbhd of $K$, $\psi|_{N(K) \setminus K}$ coincides with

$$N(K) \setminus K = K \times (D^p \setminus \{0\}) \xrightarrow{\pi} S^{p-1},$$

where $\pi(x, y) = y/\|y\|$.

Then, the pair $(S^{n-1}, K^{n-p-1})$ is called a Neuwirth–Stallings pair, or an NS-pair for short.
It is also called a fibered knot or an open book structure.
The closure $F$ of a fiber of $\psi$ is called a fiber.
Theorem 2.2 (Looijenga, 1971)

\((S^{n-1}, K^{n-p-1})\): an NS-pair with fiber \(F\) s.t. \(K^{n-p-1} \neq \emptyset\)

\[\Rightarrow \exists \text{ polynomial map germ } (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0) \text{ with an isolated singularity at } 0\]

s.t. the associated NS-pair (Milnor fibration) is isomorphic to the connected sum

\[(S^{n-1}, K^{n-p-1}) \# ((-1)^n S^{n-1}, (-1)^{n-p} K^{n-p-1}),\]

with fiber the boundary connected sum \(F \# (-1)^{n-p} F\).
Theorem 2.2 (Looijenga, 1971)

\((S^{n-1}, K^{n-p-1})\): an NS-pair with fiber \(F\) s.t. \(K^{n-p-1} \neq \emptyset\)

\[\Rightarrow \exists \ \text{polynomial map germ } (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0) \ \text{with an isolated singularity at } 0\]

s.t. the associated NS-pair (Milnor fibration) is isomorphic to the connected sum

\[\left( S^{n-1}, K^{n-p-1} \right) \# \left( (-1)^n S^{n-1}, (-1)^{n-p} K^{n-p-1} \right), \]

with fiber the boundary connected sum \(F^\# (-1)^{n-p} F\).

For our problem, it is enough to construct a non-trivial NS-pair \((S^5, K^2)\).
Theorem 2.2 (Looijenga, 1971)

\((S^{n-1}, K^{n-p-1})\): an NS-pair with fiber \(F\) s.t. \(K^{n-p-1} \neq \emptyset\)

\[\Rightarrow \exists \text{ polynomial map germ } (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0) \text{ with an isolated singularity at } 0\]

s.t. the associated NS-pair (Milnor fibration) is isomorphic to the connected sum

\[(S^{n-1}, K^{n-p-1}) \# ((-1)^n S^{n-1}, (-1)^{n-p} K^{n-p-1}),\]

with fiber the boundary connected sum \(F \# (-1)^{n-p} F\).

For our problem, it is enough to construct a non-trivial NS-pair \((S^5, K^2)\).

This is purely a differential topology problem!
1°. We classify smooth locally trivial fibrations

\[ F^3 \hookrightarrow E^5 \to S^2 \]

with fiber \( F^3 \) a compact (simply connected) 3-manifold with \( \partial F^3 = K^2 \) s.t. the boundary fibration \( \partial F^3 \hookrightarrow \partial E^5 \to S^2 \) is trivial.
1°. We **classify** smooth locally trivial fibrations

\[ F^3 \hookrightarrow E^5 \rightarrow S^2 \]

with fiber \( F^3 \) a compact (simply connected) 3-manifold with \( \partial F^3 = K^2 \) s.t. the boundary fibration \( \partial F^3 \hookrightarrow \partial E^5 \rightarrow S^2 \) is trivial.

Then, \( \partial E^5 = \partial F^3 \times S^2 = K^2 \times S^2 \).
1°. We **classify** smooth locally trivial fibrations

\[ F^3 \hookrightarrow E^5 \twoheadrightarrow S^2 \]

with fiber \( F^3 \) a compact (simply connected) 3-manifold with \( \partial F^3 = K^2 \) s.t. the boundary fibration \( \partial F^3 \hookrightarrow \partial E^5 \twoheadrightarrow S^2 \) is trivial.

Then, \( \partial E^5 = \partial F^3 \times S^2 = K^2 \times S^2 \).

Attach \( K^2 \times D^3 \) to \( E^5 \) along the boundaries to get a closed 5-dim. manifold \( \widetilde{E^5} \).
Strategy


1°. We **classify** smooth locally trivial fibrations

\[ F^3 \hookrightarrow E^5 \to S^2 \]

with fiber \( F^3 \) a compact (simply connected) 3-manifold with \( \partial F^3 = K^2 \) s.t. the boundary fibration \( \partial F^3 \hookrightarrow \partial E^5 \to S^2 \) is trivial.

Then, \( \partial E^5 = \partial F^3 \times S^2 = K^2 \times S^2 \).

Attach \( K^2 \times D^3 \) to \( E^5 \) along the boundaries to get a closed 5-dim. manifold \( \tilde{E}^5 \).

\( \tilde{E}^5 \) has a so-called “open book structure”. 
1°. We **classify** smooth locally trivial fibrations

\[ F^3 \hookrightarrow E^5 \to S^2 \]

with fiber \( F^3 \) a compact (simply connected) 3-manifold with \( \partial F^3 = K^2 \) s.t. the boundary fibration \( \partial F^3 \hookrightarrow \partial E^5 \to S^2 \) is trivial.

Then, \( \partial E^5 = \partial F^3 \times S^2 = K^2 \times S^2 \).

Attach \( K^2 \times D^3 \) to \( E^5 \) along the boundaries to get a closed 5-dim. manifold \( \tilde{E}^5 \).

\( \tilde{E}^5 \) has a so-called “open book structure”.

2°. We then **characterize** those fibrations with \( \tilde{E}^5 \cong S^5 \).

Such \( (\tilde{E}^5, K^2 \times \{0\}), \ 0 \in D^3 \), gives an NS-pair with fiber \( F^3 \).

If \( F^3 \neq D^3 \), we are done.
Let \((S^5, K^2)\) be an NS-pair. Since \(S^5\) never fibers over \(S^2\), we have \(K^2 \neq \emptyset\). We have a fibration \(\psi : S^5 \setminus \text{Int } N(K^2) \to S^2\) with fiber \(F\). By the homotopy exact sequence

\[
\pi_2(S^5 \setminus \text{Int } N(K^2)) \to \pi_2(S^2) \to \pi_1(F) \to \pi_1(S^5 \setminus \text{Int } N(K^2)),
\]
Let \((S^5, K^2)\) be an NS-pair. Since \(S^5\) never fibers over \(S^2\), we have \(K^2 \neq \emptyset\). We have a fibration \(\psi : S^5 \setminus \text{Int } N(K^2) \to S^2\) with fiber \(F\).

By the homotopy exact sequence

\[
\pi_2(S^5 \setminus \text{Int } N(K^2)) \to \pi_2(S^2) \to \pi_1(F) \to \pi_1(S^5 \setminus \text{Int } N(K^2)),
\]

we can deduce that \(F\) is simply connected.
Let \((S^5, K^2)\) be an NS-pair.
Since \(S^5\) never fibers over \(S^2\), we have \(K^2 \neq \emptyset\).
We have a fibration \(\psi : S^5 \setminus \text{Int } N(K^2) \to S^2\) with fiber \(F\).

By the homotopy exact sequence

\[
\pi_2(S^5 \setminus \text{Int } N(K^2)) \to \pi_2(S^2) \to \pi_1(F) \to \pi_1(S^5 \setminus \text{Int } N(K^2)),
\]

we can deduce that \(F\) is **simply connected**.

The solution to the **Poincaré Conjecture** implies
\(F \cong S^3(k+1) = S^3 \setminus \bigcup^{k+1} \text{Int } D^3\).
Let \((S^5, K^2)\) be an NS-pair. Since \(S^5\) never fibers over \(S^2\), we have \(K^2 \neq \emptyset\). We have a fibration \(\psi : S^5 \setminus \text{Int} N(K^2) \to S^2\) with fiber \(F\).

By the homotopy exact sequence

\[
\pi_2(S^5 \setminus \text{Int} N(K^2)) \to \pi_2(S^2) \to \pi_1(F) \to \pi_1(S^5 \setminus \text{Int} N(K^2)),
\]

we can deduce that \(F\) is **simply connected**.

The solution to the **Poincaré Conjecture** implies \(F \cong S^3_{(k+1)} = S^3 \setminus \bigcup^{k+1} \text{Int} D^3\). So, our fibration is

\[
S^3_{(k+1)} \hookrightarrow E^5 \to S^2.
\]
Such fibrations are classified by $\pi_1(\text{Diff}(S^3_{(k+1)}, \partial S^3_{(k+1)}))$, where $\text{Diff}(S^3_{(k+1)}, \partial S^3_{(k+1)})$ is the group of diffeomorphisms which fix the boundary pointwise.
Such fibrations are classified by $\pi_1\left(\text{Diff}(S^3_{(k+1)}, \partial S^3_{(k+1)})\right)$, where $\text{Diff}(S^3_{(k+1)}, \partial S^3_{(k+1)})$ is the group of diffeomorphisms which fix the boundary pointwise.

Consider a disjoint union $\bigcup^{k+1} B^3$ “standardly” embedded in $S^3$. Let $\text{Diff}(S^3, \bigcup^{k+1} B^3)$ be the group of diffeomorphisms which fix $\bigcup^{k+1} B^3$ pointwise.
Such fibrations are classified by $\pi_1(\text{Diff}(S^3_{(k+1)}), \partial S^3_{(k+1)}))$, where $\text{Diff}(S^3_{(k+1)}, \partial S^3_{(k+1)})$ is the group of diffeomorphisms which fix the boundary pointwise.

Consider a disjoint union $\bigcup^{k+1} B^3$ “standardly” embedded in $S^3$. Let $\text{Diff}(S^3, \bigcup^{k+1} B^3)$ be the group of diffeomorphisms which fix $\bigcup^{k+1} B^3$ pointwise.

**Lemma 2.3 (Cerf, 1968)** The canonical map

$$\text{Diff}(S^3, \bigcup^{k+1} B^3) \rightarrow \text{Diff}(S^3_{(k+1)}, \partial S^3_{(k+1)})$$

is a weak homotopy equivalence.
Such fibrations are classified by $\pi_1(\text{Diff}(S^3_{(k+1)}, \partial S^3_{(k+1)}))$, where $\text{Diff}(S^3_{(k+1)}, \partial S^3_{(k+1)})$ is the group of diffeomorphisms which fix the boundary pointwise.

Consider a disjoint union $\bigcup^{k+1} B^3$ “standardly” embedded in $S^3$. Let $\text{Diff}(S^3, \bigcup^{k+1} B^3)$ be the group of diffeomorphisms which fix $\bigcup^{k+1} B^3$ pointwise.

**Lemma 2.3 (Cerf, 1968) The canonical map**

$$\text{Diff}(S^3, \bigcup^{k+1} B^3) \to \text{Diff}(S^3_{(k+1)}, \partial S^3_{(k+1)})$$

is a weak homotopy equivalence.

Thus, our fiber bundles are classified by $\pi_1(\text{Diff}(S^3, \bigcup^{k+1} B^3))$. 


Let \( \text{Emb}(\bigcup^{k+1} B^3, S^3) \) be the space of embeddings.
Let $\text{Emb}(\bigcup^{k+1} B^3, S^3)$ be the space of embeddings.

**Theorem 2.4 (Cerf–Palais)** We have the locally trivial fiber bundle

$$\text{Diff}(S^3, \bigcup^{k+1} B^3) \hookrightarrow \text{Diff}(S^3) \to \text{Emb}(\bigcup^{k+1} B^3, S^3).$$
Let $\text{Emb}(\bigcup^{k+1} B^3, S^3)$ be the space of embeddings.

**Theorem 2.4 (Cerf–Palais)** We have the locally trivial fiber bundle

$$\text{Diff}(S^3, \bigcup^{k+1} B^3) \hookrightarrow \text{Diff}(S^3) \to \text{Emb}(\bigcup^{k+1} B^3, S^3).$$

We can also prove

**Lemma 2.5** $\text{Emb}(\bigcup^{k+1} B^3, S^3) \simeq \mathbb{F}_{k+1}(S^3) \times O(3)^{k+1}$, where

$$\mathbb{F}_{k+1}(S^3) = \{(x_1, x_2, \ldots, x_{k+1}) \mid x_i \in S^3, x_j \neq x_\ell, j \neq \ell\},$$

which is called the configuration space.
By the homotopy exact sequence of the bundle

$$\text{Diff}(S^3, \bigcup^{k+1} B^3) \hookrightarrow \text{Diff}(S^3) \rightarrow \text{Emb}(\bigcup^{k+1} B^3, S^3)$$

together with the above lemma, we see that

$$\pi_2(F_{k+1}(S^3)) \cong \pi_1(\text{Diff}(S^3, \bigcup^{k+1} B^3)).$$
By the homotopy exact sequence of the bundle

$$\text{Diff}(S^3, \bigcup^{k+1} B^3) \hookrightarrow \text{Diff}(S^3) \rightarrow \text{Emb}(\bigcup^{k+1} B^3, S^3)$$

together with the above lemma, we see that

$$\pi_2(F_{k+1}(S^3)) \cong \pi_1(\text{Diff}(S^3, \bigcup^{k+1} B^3)).$$

Lemma 2.6 (Fadell–Husseini 2001) $$\pi_2(F_{k+1}(S^3)) \cong \mathbb{Z}^{k(k-1)/2}.$$
By the homotopy exact sequence of the bundle

\[ \text{Diff}(S^3, \bigcup^{k+1} B^3) \hookrightarrow \text{Diff}(S^3) \rightarrow \text{Emb}(\bigcup^{k+1} B^3, S^3) \]

together with the above lemma, we see that

\[ \pi_2(F_{k+1}(S^3)) \cong \pi_1(\text{Diff}(S^3, \bigcup^{k+1} B^3)). \]

Lemma 2.6 (Fadell–Husseini 2001) \[ \pi_2(F_{k+1}(S^3)) \cong \mathbb{Z}^k/(k-1)/2. \]

Thus, our bundles \( S_{(k+1)}^3 \hookrightarrow E^5 \rightarrow S^2 \) are in one-to-one correspondence with the elements in \( \mathbb{Z}^k/(k-1)/2 \).
This corresponds to a \( k \times k \) skew-symmetric integer matrix as follows.
Among the $k + 1$ boundary components of $E^5$, we choose $k$ of them and attach $k$ copies of the trivial fibration $D^3 \times S^2 \to S^2$ to get

$$S^3_{(1)} \hookrightarrow Y \xrightarrow{\tilde{\psi}} S^2,$$

where $S^3_{(1)} = S^3_{(k+1)} \cup (\bigcup^k D^3) \cong D^3$ and $Y = E^5 \cup (\bigcup^k D^3 \times S^2)$. 
Among the $k + 1$ boundary components of $E^5$, we choose $k$ of them and attach $k$ copies of the trivial fibration $D^3 \times S^2 \to S^2$ to get

$$S^3_{(1)} \hookrightarrow Y \xrightarrow{\tilde{\psi}} S^2,$$

where $S^3_{(1)} = S^3_{(k+1)} \cup (\bigcup^k D^3) \cong D^3$ and $Y = E^5 \cup (\bigcup^k D^3 \times S^2)$. This new fibration is trivial, since its boundary fibration is trivial.
Among the $k + 1$ boundary components of $E^5$, we choose $k$ of them and attach $k$ copies of the trivial fibration $D^3 \times S^2 \to S^2$ to get

\[ S^3_{(1)} \hookrightarrow Y \xrightarrow{\tilde{\psi}} S^2, \]

where $S^3_{(1)} = S^3_{(k+1)} \cup (\bigcup^k D^3) \cong D^3$ and $Y = E^5 \cup (\bigcup^k D^3 \times S^2)$. This new fibration is trivial, since its boundary fibration is trivial. Thus, the above fibration is identified with $Y \cong D^3 \times S^2 \to S^2$. 


Among the $k + 1$ boundary components of $E^5$, we choose $k$ of them and attach $k$ copies of the trivial fibration $D^3 \times S^2 \to S^2$ to get

$$S^3_{(1)} \hookrightarrow Y \xrightarrow{\tilde{\psi}} S^2,$$

where $S^3_{(1)} = S^3_{(k+1)} \cup (\cup^k D^3) \cong D^3$ and $Y = E^5 \cup (\cup^k D^3 \times S^2)$. This new fibration is trivial, since its boundary fibration is trivial. Thus, the above fibration is identified with $Y \cong D^3 \times S^2 \to S^2$. Furthermore, the $k$ copies of $D^3 \times S^2$ attached to $E^5$ give rise to $k$ disjoint embedded 2-spheres $\{0\} \times S^2$. 
Among the $k + 1$ boundary components of $E^5$, we choose $k$ of them and attach $k$ copies of the trivial fibration $D^3 \times S^2 \to S^2$ to get

$$S^3_{(1)} \hookrightarrow Y \xrightarrow{\tilde{\psi}} S^2,$$

where $S^3_{(1)} = S^3_{(k+1)} \cup (\bigcup^k D^3) \cong D^3$ and $Y = E^5 \cup (\bigcup^k D^3 \times S^2)$. This new fibration is trivial, since its boundary fibration is trivial. Thus, the above fibration is identified with $Y \cong D^3 \times S^2 \to S^2$. Furthermore, the $k$ copies of $D^3 \times S^2$ attached to $E^5$ give rise to $k$ disjoint embedded 2-spheres $\{0\} \times S^2$. We can attach $S^2 \times D^3$ to $Y = D^3 \times S^2$ to get $S^5$. 
Among the $k + 1$ boundary components of $E^5$, we choose $k$ of them and attach $k$ copies of the trivial fibration $D^3 \times S^2 \to S^2$ to get

$$S^3_{(1)} \hookrightarrow Y \xrightarrow{\tilde{\psi}} S^2,$$

where $S^3_{(1)} = S^3_{(k+1)} \cup (\bigcup^k D^3) \cong D^3$ and $Y = E^5 \cup (\bigcup^k D^3 \times S^2)$. This new fibration is trivial, since its boundary fibration is trivial. Thus, the above fibration is identified with $Y \cong D^3 \times S^2 \to S^2$.

Furthermore, the $k$ copies of $D^3 \times S^2$ attached to $E^5$ give rise to $k$ disjoint embedded 2-spheres $\{0\} \times S^2$.

We can attach $S^2 \times D^3$ to $Y = D^3 \times S^2$ to get $S^5$.

Then, the $k$ disjoint sections give $k$ disjoint (oriented) 2-spheres $S^2_i$, $i = 1, 2, \ldots, k$, embedded in $S^5$. 
Among the $k + 1$ boundary components of $E^5$, we choose $k$ of them and attach $k$ copies of the trivial fibration $D^3 \times S^2 \to S^2$ to get

$$S^3_{(1)} \hookrightarrow Y \xrightarrow{\tilde{\psi}} S^2,$$

where $S^3_{(1)} = S^3_{(k+1)} \cup (\bigcup^k D^3) \cong D^3$ and $Y = E^5 \cup (\bigcup^k D^3 \times S^2)$. This new fibration is trivial, since its boundary fibration is trivial. Thus, the above fibration is identified with $Y \cong D^3 \times S^2 \to S^2$.

Furthermore, the $k$ copies of $D^3 \times S^2$ attached to $E^5$ give rise to $k$ disjoint embedded 2-spheres $\{0\} \times S^2$.

We can attach $S^2 \times D^3$ to $Y = D^3 \times S^2$ to get $S^5$.

Then, the $k$ disjoint sections give $k$ disjoint (oriented) 2-spheres $S^2_i$, $i = 1, 2, \ldots, k$, embedded in $S^5$.

Thus, we have the **linking matrix** $L = (lk(S^2_i, S^2_j))_{1 \leq i, j \leq k}$. 

---

**Linking matrix**

We adopt the convention that the diagonal entries \( \text{lk}(S_i^2, S_i^2) \) are all zero.
We adopt the convention that the diagonal entries $\text{lk}(S^2_i, S^2_i)$ are all zero. Then, $L$ is a $k \times k$ skew-symmetric integer matrix.
We adopt the convention that the diagonal entries $\text{lk}(S^2_i, S^2_i)$ are all zero. Then, $L$ is a $k \times k$ skew-symmetric integer matrix.

All these arguments imply that we have the following one-to-one correspondence by the **linking matrix**:

\[
\begin{align*}
\{\text{isomorphism classes of our fiber bundles} \ S^3_{(k+1)} \hookrightarrow E^5 \rightarrow S^2\} & \\
\updownarrow & \\
\{k \times k \text{ skew-symmetric integer matrices} \ L = (\text{lk}(S^2_i, S^2_j))_{1 \leq i, j \leq k}\}
\end{align*}
\]
We see easily that $\widetilde{E}^5$ is simply connected.
It is known that a smooth homotopy 5-sphere is always
diffeomorphic to $S^5$. 
We see easily that $\tilde{E}^5$ is simply connected. It is known that a smooth homotopy 5-sphere is always diffeomorphic to $S^5$. Therefore, by a standard (but tedious) computation based on Mayer-Vietoris exact sequence for homology, we get the following.
We see easily that $\widetilde{E}^5$ is simply connected. It is known that a smooth homotopy 5-sphere is always diffeomorphic to $S^5$. Therefore, by a standard (but tedious) computation based on Mayer-Vietoris exact sequence for homology, we get the following.

**Theorem 2.7**

We have $\widetilde{E}^5 \cong S^5$, i.e., our fiber bundle comes from an NS pair iff $\det L = \pm 1$. 
It is easy to construct $k \times k$ skew-symmetric matrix of determinant $\pm 1$ as long as $k$ is even. Thus, by the Looijenga construction, we get the following.
It is easy to construct $k \times k$ skew-symmetric matrix of determinant $\pm 1$ as long as $k$ is even. Thus, by the Looijenga construction, we get the following.

**Corollary 2.8**

For $\forall k = 0, 2, 4, \ldots$, $\exists$ NS-pair $(S^5, L_{k+1})$ with $L_{k+1} \cong \bigcup_{i=1}^{k+1} S^2$.

Consequently, $\exists$ polynomial map germ $(\mathbb{R}^6, 0) \to (\mathbb{R}^3, 0)$ with an isolated singularity at $0$ s.t. associated NS-pair is isomorphic to $(S^5, L_{k+1}\#(-L_{k+1}))$.

In particular, $L_{k+1}\#(-L_{k+1})$ consists of $2k + 1$ components.
It is easy to construct $k \times k$ skew-symmetric matrix of determinant $\pm 1$ as long as $k$ is even.
Thus, by the **Looijenga construction**, we get the following.

**Corollary 2.8**

For $\forall k = 0, 2, 4, \ldots$, $\exists$ NS-pair $(S^5, L_{k+1})$ with $L_{k+1} \cong \bigcup^{k+1} S^2$.
Consequently, $\exists$ polynomial map germ $(\mathbb{R}^6, 0) \to (\mathbb{R}^3, 0)$ with an isolated singularity at 0 s.t. associated NS-pair is isomorphic to $(S^5, L_{k+1}#(-L_{k+1}))$.
In particular, $L_{k+1}#(-L_{k+1})$ consists of $2k + 1$ components.

This answers **Milnor’s non-triviality question** for $(n, p) = (6, 3)$. 
§3. Topology of Milnor Fibers
If $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is a complex polynomial map germ with an isolated singularity at 0, then the Milnor fiber is homotopy equivalent to the **bouquet of spheres**

$$\vee^\mu S^{n-1}.$$
If \( f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0) \) is a complex polynomial map germ with an isolated singularity at 0, then the Milnor fiber is homotopy equivalent to the bouquet of spheres
\[ \vee^\mu S^{n-1}. \]

In the real case, it is easy to construct an example of a polynomial map germ with an isolated singularity whose Milnor fiber is NOT homotopy equivalent to the bouquet of spheres.
If \( f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0) \) is a\textbf{ complex} polynomial map germ with an isolated singularity at 0, then the Milnor fiber is homotopy equivalent to the \textbf{bouquet of spheres} \( \bigvee^\mu S^{n-1} \).

In the \textbf{real} case, it is easy to construct an example of a polynomial map germ with an isolated singularity whose Milnor fiber is NOT homotopy equivalent to the bouquet of spheres.

However, we have the following.
Proposition 3.1 Let $f : (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^n, 0)$ be a polynomial map germ with an isolated singularity at the origin, $n \geq 2$.

Then, the Milnor fiber $F_f$ has the homotopy type of $\bigvee^\mu S^{n-1}$, where it means a point when $\mu = 0$. 
Proposition 3.1 Let \( f : (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^n, 0) \) be a polynomial map germ with an isolated singularity at the origin, \( n \geq 2 \).

Then, the Milnor fiber \( F_f \) has the homotopy type of \( \bigvee_S S^{n-1} \), where it means a point when \( \mu = 0 \).

Proposition 3.2 Let \( f : (\mathbb{R}^{2n+1}, 0) \to (\mathbb{R}^n, 0) \) be a polynomial map germ with an isolated singularity at the origin, \( n \geq 3 \).

Then, \( H_{n-1}(F_f) \) is torsion free if and only if \( F_f \) has the homotopy type of \( \bigvee(S^{n-1}) \vee \bigvee(S^n) \), where it means a point when \( \mu = 0 \).
Non-trivial examples due to Church–Lamotke (1976) have **contractible Milnor fibers** (but with non-simply connected links).
Non-trivial examples due to Church–Lamotke (1976) have contractible Milnor fibers (but with non-simply connected links).
Using our techniques for \((n, p) = (6, 3)\), we get the following.
Non-trivial examples due to Church–Lamotke (1976) have **contractible Milnor fibers** (but with non-simply connected links).

Using our techniques for \((n, p) = (6, 3)\), we get the following.

**Proposition 3.3** For each pair of dimensions \((2n, p)\), \(2 \leq p \leq n\),

\[ \exists \text{polynomial map germ } \left( \mathbb{R}^{2n}, 0 \right) \to \left( \mathbb{R}^p, 0 \right), \text{ with an isolated singularity at } 0 \text{ s.t. the Milnor fiber is homotopy equivalent to} \]

\[ \bigvee_{\mu} S^{n-1}, \quad \mu > 0. \]
Proposition 3.4

For each pair of dimensions \((2n + 1, p), 2 \leq p \leq n\), \(\exists\) polynomial map germ \((\mathbb{R}^{2n+1}, 0) \rightarrow (\mathbb{R}^p, 0)\) with an isolated singularity at 0 s.t. the Milnor fiber is homotopy equivalent to

\[
\left( \bigvee_{\mu} S^{n-1} \right) \vee \left( \bigvee_{\mu} S^n \right), \quad \mu > 0.
\]
Proposition 3.4

For each pair of dimensions \((2n + 1, p), 2 \leq p \leq n\), \(\exists\) polynomial map germ \((\mathbb{R}^{2n+1}, 0) \to (\mathbb{R}^p, 0)\) with an isolated singularity at 0 s.t. the Milnor fiber is homotopy equivalent to

\[
\left( \bigvee S^{m-1} \right) \vee \left( \bigvee S^m \right), \quad \mu > 0.
\]

These provide NEW NON-TRIVIAL examples!
We first construct a fiber bundle

$$S^n \to E^{2n-1} \to S^{n-1}$$

such that it is trivial on the boundary, using a $k \times k$ integer matrix $L$ which is $(-1)^n$-symmetric whose diagonal entries all vanish.
We first construct a fiber bundle

\[ S_{(k+1)}^n \hookrightarrow E^{2n-1} \rightarrow S^{n-1} \]

such that it is trivial on the boundary, using a \( k \times k \) integer matrix \( L \) which is \((-1)^n\)-symmetric whose diagonal entries all vanish. This is possible by a homotopy theoretic argument involving certain configuration spaces, similar to the previous case.
We first construct a fiber bundle

\[ S_{(k+1)}^{m} \hookrightarrow E^{2n-1} \rightarrow S^{n-1} \]

such that it is trivial on the boundary, using a \( k \times k \) integer matrix \( L \) which is \( (-1)^n \)-symmetric whose diagonal entries all vanish. This is possible by a homotopy theoretic argument involving certain \textbf{configuration spaces}, similar to the previous case. If the matrix \( L \) has determinant \( \pm 1 \), then we see that the associated closed manifold \( \tilde{E}^{2n-1} \) is a homotopy \( (2n - 1) \)-sphere.
We first construct a fiber bundle

\[ S^m_{(k+1)} \rightarrow E^{2n-1} \rightarrow S^{n-1} \]

such that it is trivial on the boundary, using a \( k \times k \) integer matrix \( L \) which is \((-1)^n\)-symmetric whose diagonal entries all vanish. This is possible by a homotopy theoretic argument involving certain **configuration spaces**, similar to the previous case.

If the matrix \( L \) has determinant \( \pm 1 \), then we see that the associated closed manifold \( \tilde{E}^{2n-1} \) is a homotopy \((2n - 1)\)-sphere. This may not be diffeomorphic to \( S^{2n-1} \).
We first construct a fiber bundle
\[ S^m_{(k+1)} \rightarrow E^{2n-1} \rightarrow S^{n-1} \]
such that it is trivial on the boundary, using a \( k \times k \) integer matrix \( L \) which is \((-1)^n\)-symmetric whose diagonal entries all vanish. This is possible by a homotopy theoretic argument involving certain configuration spaces, similar to the previous case.
If the matrix \( L \) has determinant ±1, then we see that the associated closed manifold \( \tilde{E}^{2n-1} \) is a homotopy \((2n - 1)\)-sphere. This may not be diffeomorphic to \( S^{2n-1} \).
However, \( \tilde{E}^{2n-1} \#(-\tilde{E}^{2n-1}) \) is diffeomorphic to \( S^{2n-1} \).
Then, the **Looijenga construction** leads to a polynomial map germ $f : (\mathbb{R}^{2n}, 0) \rightarrow (\mathbb{R}^n, 0)$ with an isolated singularity at 0 whose Milnor fibration is non-trivial.
Then, the Looijenga construction leads to a polynomial map germ $f : (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^n, 0)$ with an isolated singularity at $0$ whose Milnor fibration is non-trivial. Considering the composition with a canonical projection

$$(\mathbb{R}^{2n}, 0) \to (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$$

for $n > p \geq 2$, we get again a non-trivial example.
Then, the **Looijenga construction** leads to a polynomial map germ 
\( f : (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^n, 0) \) with an isolated singularity at 0 whose Milnor fibration is non-trivial.

Considering the **composition with a canonical projection**

\[
(\mathbb{R}^{2n}, 0) \to (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)
\]

for \( n > p \geq 2 \), we get again a non-trivial example.

For \((\mathbb{R}^{2n+1}, \mathbb{R}^p)\), we first apply the **spinning construction** to the non-trivial NS-pair \((S^{2n-1}, K^{n-1})\) constructed above, to get a non-trivial NS-pair \((S^{2n}, \tilde{K}^n)\).
Then, the **Looijenga construction** leads to a polynomial map germ $f : (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^n, 0)$ with an isolated singularity at 0 whose Milnor fibration is non-trivial. Considering the **composition with a canonical projection**

$$(\mathbb{R}^{2n}, 0) \to (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$$

for $n > p \geq 2$, we get again a non-trivial example.

For $(\mathbb{R}^{2n+1}, \mathbb{R}^p)$, we first apply the **spinning construction** to the non-trivial NS-pair $(S^{2n-1}, K^{n-1})$ constructed above, to get a non-trivial NS-pair $(S^{2n}, \tilde{K}^n)$. Applying the **Looijenga construction** to this, we get a polynomial map germ $(\mathbb{R}^{2n+1}, 0) \to (\mathbb{R}^n, 0)$ with non-trivial Milnor fibration.
Then, the **Looijenga construction** leads to a polynomial map germ $f : (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^n, 0)$ with an isolated singularity at 0 whose Milnor fibration is non-trivial. Considering the composition with a canonical projection

$$(\mathbb{R}^{2n}, 0) \to (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$$

for $n > p \geq 2$, we get again a non-trivial example.

For $(\mathbb{R}^{2n+1}, \mathbb{R}^p)$, we first apply the **spinning construction** to the non-trivial NS-pair $(S^{2n-1}, K^{n-1})$ constructed above, to get a non-trivial NS-pair $(S^{2n}, \tilde{K}^n)$. Applying the **Looijenga construction** to this, we get a polynomial map germ $(\mathbb{R}^{2n+1}, 0) \to (\mathbb{R}^n, 0)$ with non-trivial Milnor fibration. Then, consider the composition with a canonical projection $\mathbb{R}^n \to \mathbb{R}^p$. 
Thank you for your attention !