Cobordism Group of Morse Functions on Surfaces with Boundary

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§1. Introduction

§2. Reeb Graph and Reeb Space

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All manifolds and maps are differentiable of class $C^\infty$.

Let $N$ be a manifold \textbf{with boundary}.

A $C^\infty$ function $f : N \rightarrow \mathbb{R}$ is a \textbf{Morse function} if

(1) the critical points of $f$ and $f|_{\partial N}$ are all \textbf{non-degenerate} and have \textbf{distinct values}, and

(2) $f$ is a \textbf{submersion} on a neighborhood of $\partial N$.

($\iff$ critical points of $f|_{\partial N}$ are all \textbf{correct} critical points.)

\textbf{Fact.} $f$ is a Morse function iff it is $C^\infty$ \textbf{stable}.

It is also called a \textbf{correct function} in the literature.

A smooth map on a manifold with boundary is \textbf{admissible} if it is a submersion on a neighborhood of the boundary.

In this sense, every Morse function is admissible.
The critical points of $f$ and $f|_{\partial N}$ are non-degenerate. The critical values of $f$ and $f|_{\partial N}$ are all distinct. $f$ is a submersion near the boundary.
Let $N_0$ and $N_1$ be compact $n$-dim. manifolds with boundary. Morse functions $f_i: N_i \to \mathbb{R}$, $i = 0, 1$, are **admissibly cobordant** if

1. There exists a compact manifold $X^{n+1}$ with corners (cobordism between the manifolds $N_0$ and $N_1$) such that
   \[
   \partial X^{n+1} = N_0 \cup Q^n \cup N_1,
   \partial Q^n = \partial N_0 \cup \partial N_1 (Q^n \text{ is a cobordism between } \partial N_0 \text{ and } \partial N_1),
   \]
2. There exists $F: X^{n+1} \to \mathbb{R} \times [0, 1]$,
3. $F|_{N_0} = f_0: N_0 \to \mathbb{R} \times \{0\}$ and $F|_{N_1} = f_1: N_1 \to \mathbb{R} \times \{1\}$,
4. $F|_{X^{n+1}\setminus(N_0\cup N_1)}: X^{n+1}\setminus(N_0\cup N_1) \to \mathbb{R} \times (0, 1)$ is a proper admissible $C^\infty$ map which has only **folds** and **cusps** as its singularities.

In this case, we call $F$ an **admissible cobordism** between $f_0$ and $f_1$.

**fold**: $(x_1, x_2, \ldots, x_{n+1}) \mapsto (x_1, \pm x_2^2 \pm \cdots \pm x_{n+1}^2)$,

**cusp**: $(x_1, x_2, \ldots, x_{n+1}) \mapsto (x_1, x_2^3 + x_1 x_2 \pm x_3^2 \pm \cdots \pm x_{n+1}^2)$
Admissibility is important in the above definition. If we drop the condition, then any two Morse functions are cobordant!
If a Morse function is admissibly cobordant to the function on the empty set $\emptyset$, then it is **null-cobordant**.

“Admissible cobordism” defines an equivalence relation on the set of all Morse functions on compact $n$-dim. manifolds with boundary. The set of all equivalence classes forms an **additive group** under disjoint union:

1. the neutral element is the class of null-cobordant Morse functions,
2. $-[f : N \to \mathbb{R}] = [-f : N \to \mathbb{R}]$.

Denote by $\mathcal{B}_n$ the additive group of all admissible cobordism classes and call it the **$n$-dim. admissible cobordism group of Morse functions on manifolds with boundary**.
Theorem 1.1 The 2-dim. admissible cobordism group of Morse functions $b\mathcal{M}_2$ is cyclic of order two.

Remark 1.2 We had previously shown $\exists$ epimorphism $b\mathcal{M}_2 \rightarrow \mathbb{Z}_2$, using cohomology of the universal complex of singular fibers. This was presented in 13th International Workshop on $\mathbb{R}$ & $\mathbb{C}$ Singularities, São Carlos, in 2014. Prof. Terry Gaffney asked if it is an isomorphism. The above theorem affirmatively answers his question!
For Morse functions on manifolds \textbf{without boundary}, the \textbf{fold cobordism groups} have been studied.

Two Morse functions on closed $n$-dim. manifolds are \textbf{fold cobordant} if there exists a cobordism $F : X^{n+1} \to \mathbb{R} \times [0, 1]$ between them which has only \textbf{fold points} as its singularities. (No cusp is allowed.)

\textbf{Theorem 1.3 (Ikegami–Saeki 2003, Ikegami 2004)}

\textit{The fold cobordism group of Morse functions on closed surfaces is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}$.}

\textit{The fold cobordism group for oriented closed surfaces is isomorphic to $\mathbb{Z}$.}

The idea of our proof of the main theorem is based on [Ikegami–Saeki 2003].
§2. Reeb Graph and Reeb Space
Definition 2.1 \( f : N \rightarrow P \) smooth map

For \( x, x' \in N \), define \( x \sim_f x' \) if

(i) \( f(x) = f(x')(= y) \), and

(ii) \( x \) and \( x' \) belong to the same connected component of \( f^{-1}(y) \).

\[ W_f = N/\sim_f \] quotient space

\[ q_f : N \rightarrow W_f \] quotient map

\[ \exists! \bar{f} : W_f \rightarrow P \] that makes the diagram commutative:

\[
\begin{array}{ccc}
N & \xrightarrow{f} & P \\
\downarrow{q_f} & & \nearrow{\bar{f}} \\
W_f & & \\
\end{array}
\]

The above diagram is called the Stein factorization of \( f \).

\( W_f \) is called the Reeb space and \( \bar{f} \) the Reeb map.
Example

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\[ f : N \to \mathbb{R} \quad \text{Morse function on a compact surface with boundary} \]

**Lemma 2.2** Reeb space \( W_f \) is a finite graph whose vertices are the \( q_f \)-images of the critical points of \( f \) and \( f|_{\partial N} \).

\( W_f \) is also called a **Reeb graph**, and the continuous map \( \overline{f} : W_f \to \mathbb{R} \) a **Reeb function**.

Each edge corresponds to a **circle regular fiber** or an **arc regular fiber**. We label each edge by 0 or 1, where 0 (resp. 1) means that it corresponds to a **circle** regular fiber (resp. an **arc** regular fiber).

Such a Reeb graph is called a **labeled Reeb graph**.

Edge with label 0 ——— thick line : circle fiber
Edge with label 1 · · · · · dotted line : arc fiber
Around each vertex, \( \overline{f} : W_f \to \mathbb{R} \) is locally equivalent to one of the height functions below (or their negatives).
(The map \( \overline{f} \) is an embedding on each edge. )
Lemma 2.3  Let $f : X^3 \to P^2$ be an admissible $C^\infty$ stable map of a compact 3-dim. manifold with boundary into a surface without boundary. Then, the Reeb space $W_f$ is a compact 2-dim. polyhedron, which is labeled: each component of $W_f \setminus q_f(S(f))$ is labeled with 0 or 1, where $S(f)$ is the set of singular points of $f$.

Furthermore, around each point of $W_f$, the Reeb map $\overline{f} : W_f \to P^2$ is locally equivalent to one of the maps as depicted below, where the relevant map is the vertical projection to a plane.
Local forms of Reeb maps (I)

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Local forms of Reeb maps (II)

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§3. Proof
Definition 3.1
Let $G$ be a finite graph whose edges are labeled by 0 or 1. We assume that around each vertex of $G$, it is locally homeomorphic to one of the 9 local labeled Reeb graphs for Morse functions. Then, we call $G$ a labeled Reeb-like graph.

Let $r : G \to \mathbb{R}$ be a continuous function such that

1. around each vertex of $G$, $r$ is locally equivalent to one of the local Reeb functions of a Morse function, and
2. $r$ is an embedding on each edge.

Then, we call $r : G \to \mathbb{R}$ a Reeb-like function.

These are abstract generalizations of labeled Reeb graphs and Reeb functions for Morse functions on compact surfaces with boundary.
Definition 3.2  Two Reeb-like functions $r_i : G_i \to \mathbb{R}$, $i = 0, 1$, on labeled Reeb-like graphs are cobordant if

∃ compact 2-dim. polyhedron $W$,
∃ 1-dim. subpolyhedron $\Sigma(W)$, and
∃ continuous map $R : W \to \mathbb{R} \times [0, 1]$ s.t.

1. connected components of $W \setminus \Sigma(W)$ are labeled with 0 or 1,
2. $G_i$ are identified with “labeled” subcomplexes of $W$ with regular neighborhoods of the forms $G_i \times [0, \varepsilon]$,
3. $r_0 = R|_{G_0} : G_0 \to \mathbb{R} \times \{0\}$ and $r_1 = R|_{G_1} : G_1 \to \mathbb{R} \times \{1\}$,
4. around each point of $W \setminus (G_0 \cup G_1)$, $R$ is locally equivalent to the Reeb map $\overline{f} : W_f \to \mathbb{R} \times [0, 1]$ of a proper admissible $C^\infty$ stable map $f$ of a 3-dim. manifold with boundary into a surface.
This is an abstract generalization of the Reeb map of an admissible cobordism between two Morse functions on compact surfaces with boundary.

The above relation defines an equivalence relation for Reeb-like functions. Furthermore, the set of all cobordism classes forms an additive group under the disjoint union.

We denote by $b\mathcal{R}$ the additive group of all cobordism classes of Reeb-like functions on the labeled Reeb-like graphs and call it the cobordism group of Reeb-like functions.
We have the natural map $\rho: b\mathcal{N}_2 \rightarrow b\mathcal{R}$, which associates to each admissible cobordism class of a Morse function $f$ on a compact surface with boundary the cobordism class of the Reeb function $\overline{f}: W_f \rightarrow \mathbb{R}$.

It is straightforward to see that this defines a homomorphism of additive groups.

**Proposition 3.3** *The homomorphism $\rho: b\mathcal{N}_2 \rightarrow b\mathcal{R}$ is an isomorphism.*

**Surjectivity**: Given a labeled Reeb-like graph, one can construct an associated Morse function on a compact surface with boundary.

**Injectivity**: Given an abstract cobordism $W$ between two labeled Reeb graphs $R_{f_i}$ for Morse functions, we first modify the cobordism in such a way that it is a regular neighborhood of $\Sigma(W) \cup R_{f_0} \cup R_{f_1}$. Then, one can construct an associated admissible cobordism between the Morse functions.
It suffices to prove the following.

**Proposition 3.4** The cobordism group $b\mathcal{R}$ of Reeb-like functions is a **cyclic group of order two** generated by the cobordism class of the Reeb function of the Morse function as depicted below.

$$\begin{array}{c}
\bullet \\
\downarrow \\
D^2 \\
\downarrow \\
\bullet \\
\uparrow \\
f
\end{array}$$
Define \( \sigma : bR \rightarrow \mathbb{Z}_2 \) by setting \( \sigma([r : G \rightarrow \mathbb{R}]) \) to be the modulo two number of vertices of type (1), (2), (3), (4), (8) and (9).

\( \sigma \) is a well-defined homomorphism of abelian groups. In fact, the homomorphism corresponds to a certain cohomology class of the universal complex of singular fibers. The well-definedness of \( \sigma \) is a direct consequence of the fact that the representative of the cohomology class is a cocycle.
Lemma 3.5  Let $r_i : G_i \to \mathbb{R}$, $i = 0, 1$, be Reeb-like functions on labeled Reeb-like graphs. If $r_1$ is obtained from $r_0$ by the local moves as depicted below or their negatives, then $r_0$ and $r_1$ are cobordant.

In fact, there are many more; we use only the above ones in the proof.
Let \( r : G \to \mathbb{R} \) be an arbitrary Reeb-like function. We show \([r] = 0\) or \([r_6]\), where \(r_6\) is the Reeb function of the Morse function mentioned above (see below).

We first **cut** the edges of \(G\) by moves III and IV to get a disjoint union of **elementary Reeb-like functions** (or their negatives) as follows.

We see easily \([r_1] = [r_5] = 0\) and \([r_2] = [r_3] = [r_7] = 0\).
\[ r_4 = r_6 \]
\[ [r_6] + [r_8] = [r_6] + [r_9] \]

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Thus, we have shown that $[r_4] = [r_6] = [r_8] = [r_9]$ generates $b\mathbb{N}_2$. On the other hand, we have $\sigma([r_8]) = 1 \in \mathbb{Z}_2$ and $[r_8] + [r_8] = 0$ as below.

Hence, $\sigma$ is an isomorphism. □
§4. Low Dimensional Cases
Proposition 4.1  The $0$-dimensional admissible cobordism group of Morse functions $\mathcal{MN}_0$ is trivial.

Proposition 4.2  The $1$-dimensional admissible cobordism group of Morse functions $\mathcal{MN}_1$ is an infinite cyclic group generated by the admissible cobordism class of $f_0 : [-1, 2] \to \mathbb{R}$ given by $f_0(x) = x^2$, $x \in [-1, 2]$.

In fact, $(\#(\text{positive end points}) - \#(\text{negative end points}))/2$ gives an isomorphism.
Problem 4.3 Study the group structure of $b\mathcal{M}_n$, $n \geq 3$.

Problem 4.4 Study the group structure of the oriented version $b\Omega_n$.

Problem 4.5 Study the group structures of the admissible fold cobordism group $b\mathcal{F}_n$ and its oriented version.
Ending

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Thank you for your attention!