定値折り目特異点の消去と
特異レベルシェッツ束

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§1. Broken Lefschetz Fibrations
We will work in the **smooth category** ($= \text{real } C^\infty$ category).
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(1) A singularity of a $C^\infty$ map $M \to \Sigma$ that has the normal form

$$ (z, w) \mapsto zw $$

w.r.t. complex coordinates compatible with the orientations, is called a **Lefschetz singularity**.
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2. A singularity that has the normal form

\[
(x_1, x_2, x_3, x_4) \mapsto (x_1, x_2^2 + x_3^2 - x_4^2)
\]

is called an **indefinite fold singularity**.
Definition 1.2 (Auroux–Donaldson–Katzarkov, 2005, etc.)

Let $f : M^4 \to \Sigma^2$ be a $C^\infty$ map.

$f$ is a **broken Lefschetz fibration** (BLF, for short) if it has at most Lefschetz and indefinite fold singularities.
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A usual **Lefschetz fibration** (LF, for short) is a special case of a BLF. (LF $\iff$ BLF with $S_I(f) = \emptyset$)
Fibers of a BLF

Donaldson, Gompf, $\sim 2000$

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(↑ admitting 1-dim. zero locus)

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**broken Lefschetz fibrations** \(\iff\) **near-symplectic structures**

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Near-symplectic structure: \(\omega \in \Omega^2(M^4), \ d\omega = 0, \ \omega^2 \geq 0, \)
\[ \omega \ \text{vanishes along a 1-dim. submanifold \textit{“transversely”}.} \]
Theorem 1.3 (ADK, 2005) \( M^4 \): closed oriented 4-manifold, \( Z \subset M^4 \): 1-dim. closed submanifold

Then, the following two are equivalent.

(1) \( \exists \) near-symplectic form \( \omega \) on \( M^4 \) with zero locus \( Z \).

(2) \( \exists \) broken Lefschetz pencil (BLP) \( f \) over \( S^2 \) with \( S_1(f) = Z \) s.t. there is an \( h \in H^2(M^4; \mathbb{R}) \) satisfying \( h(C) > 0 \) for every component \( C \) of every fiber of \( f \).

Furthermore, if (2) holds, then a deformation class of near-symplectic forms that restrict to a volume form on each fiber away from \( Z \) is canonically associated to \( f \).
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\( \exists \text{BLP} \implies \exists \text{BLF} \) on a blown up 4-manifold

BLF is a special case of a BLP (BLF = BLP without base points).
Remark 1.4
Not every 4-manifold admits a symplectic structure.
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On the other hand, it is known that every closed oriented 4-manifold $M^4$ with $b_2^+(M^4) > 0$ admits a near-symplectic structure.
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In fact, there are a variety of such structures on a given 4-manifold $M^4$. 
§2. Singularities of Generic Maps
Let us discuss the relation to the singularity theory of $C^\infty$ maps.
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**Definition 2.5** (1) A singularity that has the normal form

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is called a **definite fold singularity**.
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(2) A singularity that has the normal form

$$(x_1, x_2, x_3, x_4) \mapsto (x_1, x_2^3 - 3x_1x_2 + x_3^2 \pm x_4^2)$$

is called a **cusp**.
Figure 1: **Indefinite fold**
Base Diagrams for Folds


1. Broken Lefschetz Fibrations

2. Singularities of Generic Maps

3. Elimination of Definite Fold

4. Moves for BLFs

5. Simplified BLFs

vanishing cycle

Figure 1: **Indefinite fold**

Figure 2: **Definite fold**
Figure 3: Indefinite cusp
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Figure 4: Definite cusp
Facts.

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**Levine (1965)** [Cusps can be eliminated in pairs.]
Every $C^\infty$ map $M^4 \to \Sigma^2$ is homotopic to an excellent map without a cusp if $\chi(M^4)$ is even, and with exactly one cusp if $\chi(M^4)$ is odd.
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BLFs may have Lefschetz critical points, but have no definite fold or cusp.
§3. Elimination of Definite Fold
Theorem 3.1 (S., 2006)
Every $C^\infty$ map $g : M^4 \to S^2$ is homotopic to an excellent map without definite fold singularities, and possibly with a cusp.
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Every $C^\infty$ map $g : M^4 \rightarrow S^2$ is homotopic to an excellent map without definite fold singularities, and possibly with a cusp.

In other words, we can eliminate **definite fold singularities** by homotopy.
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$S(g) (\subset M^4)$: set of **singular points**

$S_D(g) (\subset S(g))$: set of **definite fold** singular points
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**Step 1.** Modify $S_D(g)$ to a single “unknotted” component.
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For this, we use the proof of the following theorem.
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For this, we use the proof of the following theorem.

**Theorem 3.2 (S., 1995)**

\[ g : M^4 \rightarrow \Sigma^2 \text{ a } C^\infty \text{ map} \]

$L \subset M^4$: a non-empty closed 1-dim. submanifold

\[ \exists \text{ excellent map } f : M^4 \rightarrow \Sigma^2 \text{ homotopic to } g \text{ s.t. } S(f) = L \]

\[ \iff [L]_2 = 0 \text{ in } H_1(M^4; \mathbb{Z}_2) \]
Figure 5: Moves for modifying the **definite fold locus**
Step 2. Arrange $g$ so that $g|_{S_D(g)}$ is an embedding into $S^2$. 
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For Step 3, we need the following additional move.

Figure 6: Birth
Step 3. Definite fold circle $\leadsto$ Indefinite one (Williams, 2010)
Step 3. Definite fold circle $\sim\rightarrow$ Indefinite one (Williams, 2010)
Corollary 3.3 (Baykur, 2008)

Every closed oriented 4-manifold admits a BLF over $S^2$. 
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**Figure 7: Sinking and Unsinking (Lekili, 2009)**
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Figure 7: Sinking and Unsinking (Lekili, 2009)

Remark 3.4 For the existence of BLF, several proofs are known (Gay–Kirby, Baykur, Lekili, Akbulut–Karakurt).
We can also prove the following (cf. Lekili, 2009).
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**Theorem 3.5** \( g : M^4 \to S^2 \) a \( C^\infty \) map

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Using similar techniques in the context of near-symplectic structures (Perutz, 2006; Lekili, 2009), we can prove the following.
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Using similar techniques in the context of near-symplectic structures (Perutz, 2006; Lekili, 2009), we can prove the following.

**Theorem 3.6** \( M^4 \): closed oriented 4-manifold with \( b_2^+(M^4) > 0 \)

\( L \subset M^4 \): a non-empty closed 1-dim. submanifold

\( \exists \text{near-symplectic structure } \omega \) whose zero locus coincides with \( L \)

\( \iff [L]_2 = 0 \) in \( H_1(M^4; \mathbb{Z}_2) \)
Recent Result by Gay–Kirby


**Theorem 3.7 (Gay–Kirby, 2011)** \( g : M^4 \to \Sigma^2 \) a \( C^\infty \) map

\[ \exists f : M^4 \to \Sigma^2 \text{ BLF homotopic to } g \]

\[ \iff [\pi_1(\Sigma^2) : g_*\pi_1(M^4)] < +\infty \]

**Furthermore,** if \( g_* : \pi_1(M^4) \to \pi_1(\Sigma^2) \) is surjective, then we can arrange so that \( \forall \) fibers are connected.

**Remark 3.8** *Fiber connectedness* is very important!
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**Fiber connectedness** is very important!  
Recall the cohomological condition appearing in the ADK theorem on the existence and uniqueness of near-symplectic structures.
§ 4. Moves for BLFs
There is a set of “moves” for BLFs, called \textbf{Lekili’s moves}.
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\begin{tikzpicture}[scale=1,thick]
  \node at (0,0) [draw] {\textbf{Birth}};
  \node at (1,0) [draw] {\textbf{Merge}};
  \node at (2,0) [draw] {\textbf{Flip}};
  \node at (3,0) [draw] {\textbf{Wrinkle}};
  \node at (4,0) [draw] {\textbf{Sinking}};

  % Birth
  \draw [->] (-0.5,0) -- (0.5,0);
  % Merge
  \draw [<-] (1.5,0) -- (2.5,0);
  % Flip
  \draw [->] (2.5,0) -- (3.5,0);
  % Wrinkle
  \draw [<-] (3.5,0) -- (4.5,0);
  % Sinking
  \draw [<-] (4.5,0) -- (5.5,0);
\end{tikzpicture}

\textbf{Figure 8: Lekili’s moves}
Theorem 4.1 (Williams, 2010; Gay–Kirby, 2011)
If two BLFs $M^4 \rightarrow \Sigma^2$ are homotopic, then one is obtained from the other by a finite sequence of Lekili’s moves (Birth, Merge, Flip, Wrinkle, and Sink operations, and their inverses), together with “Isotopies”.

Uniqueness

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If one can describe the change in the corresponding near-symplectic structures, one would be able to define a gauge theoretic invariant for 4-manifolds $\Rightarrow$ **Lagrangian matching invariant** (Perutz, 2007)
Uniqueness

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If one can describe the change in the corresponding near-symplectic structures, one would be able to define a gauge theoretic invariant for 4-manifolds $\Rightarrow$ Lagrangian matching invariant (Perutz, 2007)

It is conjectured that Lagrangian matching invariants equal the Seiberg–Witten invariants.
Problem 4.2 (Baykur)

*Find a sufficient sequence of moves that guarantees to stay within the class of fibrations* without null-homologous fiber components.
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*How about the class of fibrations with connected fibers?*
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*How about the class of fibrations with connected fibers?*

**Note.**

These guarantee that if we start with a **near-symplectic BLF**, then we can perform the moves within the subclass of **near-symplectic BLFs**.
Theorem 4.3 (Gay–Kirby, 2011)

\( f_0, f_1 : M^4 \to \Sigma^2 \) excellent maps without definite folds
s.t. all the fibers are connected.

\[ \Rightarrow \exists \text{generic homotopy } f_t \text{ between } f_0 \text{ and } f_1 \]

s.t. \( \forall \text{fibers of } f_t \text{ are connected.} \)
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Idea: A careful application of the classical Cerf theory.
An Answer

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s.t. \( \forall \) fibers of \( f_t \) are connected.

Idea: A careful application of the classical Cerf theory.

cf. The proof that the Kirby moves are enough for converting one
framed link diagram to another for a given 3-manifold.
§5. Simplified BLFs
Let \( f : M^4 \to S^2 \) be a BLF.
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1. $S_I(f) \cong S^1$,
2. $f|_{S_I(f)}$ is an embedding onto the equator of $S^2$,
3. all fibers are connected.
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Then, $f$ is a simplified broken Lefschetz fibration (SBLF, for short).
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(3) $\forall$ fibers are connected.

Then, $f$ is a **simplified broken Lefschetz fibration** (SBLF, for short).

It is known that **every closed oriented 4-manifold admits a SBLF** (Gay–Kirby, etc.).
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Then, one can represent the 4-manifold by a finite sequence of simple closed curves on a fiber surface. \(\rightsquigarrow\) **surface diagram** of a 4-manifold.
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Then, one can represent the 4-manifold by a finite sequence of simple closed curves on a fiber surface. → **surface diagram** of a 4-manifold

**Theorem 5.1 (Williams, 2011)**
Surface diagram of a given closed oriented 4-manifold is unique up to certain moves, called stabilization, handleslide, multislide, and shift.
(1) Every closed oriented 4-manifold admits a lot of **BLFs**; when $b_2^+(M^4) > 0$, a lot of BLFs with associated **near-symplectic structures**.

(2) Two BLFs in a fixed homotopy class are related by **Lekili’s moves**. They are also related in the class of BLFs with **connected fibers**. This would lead to prove the conjecture that the **Lagrangian matching invariant** defined for near-symplectic structures equals the **Seiberg-Witten invariant**.

(3) The **indefinite locus** of a BLF can be prescribed, and the **zero locus** of a near-symplectic structure as well.

(4) **Surface diagrams** arising from SBLFs may be useful to describe a given 4-manifold, like Heegaard diagrams or framed link diagrams for 3-manifolds.
Thank you for your attention!