COMPUTING CONSTRAINED WILLMORE SURFACES

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ABSTRACT. An immersion $f\colon M \to \mathbb{R}^3$ of a surface M is called *constrained Willmore* if f minimizes the Willmore functional $W(f)$ compared to local variations of f that change the induced metric only conformally. We present a version of this variational problem in the context of Discrete Differential Geometry and use it to develop an efficient numerical algorithm. This algorithm allows us to experimentally explore the possible shapes of Willmore minimizers within a given conformal class.

1. Smooth Constrained Willmore Surfaces

Let M be a Riemann surface, i.e. an oriented surface with a conformal structure. Then a conformal immersion $f\colon M \to \mathbb{R}^3$ is called a *constrained Willmore immersion* if f is a critical point of the Willmore functional $W(f) = \int_M H^2$ among all conformal immersions of M into $\mathbb{R}^3$ [1].

Constrained Willmore surfaces are solutions of a certain variational problem under constraints. The Lagrange multiplier corresponding to the conformality constraint turns out to be a holomorphic quadratic differential on M.

2. Discrete Constrained Willmore Surfaces

Let M be a simplicial surface with vertex set V. A *metric* on M assigns a length to each edge of M. If we are given conformal factors at the vertices and multiply the length of each edge by the two factors at its end points, the resulting new metric is said to be conformally equivalent to the old one [3]. Suppose now we have a discrete version W of the Willmore functional, defined on maps $f\colon M \to \mathbb{R}^3$. We want to minimize W while allowing only variations of f that preserve the conformal class of the metric induced on M by f. For a critical point of this variational problem, the Lagrange multiplier corresponding to the conformality constraint can then be interpreted as a discrete analog of a holomorphic quadratic differential [2].

3. Numerical Strategy

We use a novel approach to constrained variational problems, called *Competive Gradient Descent* [4]. This method allows us to efficiently minimize Willmore energy while preserving conformality up to machine precision.

Key words and phrases. Constrained Willmore, Discrete Riemann Surfaces.
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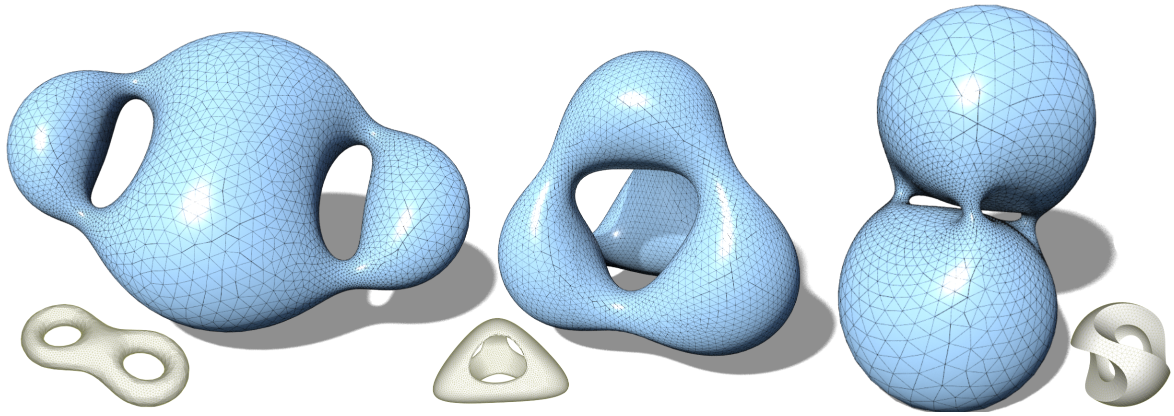

Figure 1. Constrained Willmore surfaces (blue) obtained by minimizing the Willmore functional while preserving the conformal type. The corresponding initial surfaces are shown in gray.

4. Applications

Minimizers of geometric energies under a conformality constraint can be practically useful for a variety of modelling tasks. They are a two-dimensional analog for elastic curves (minimizers of bending energy under a length constraint).

References

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