Forcing, Downward Löwenheim-Skolem and Omitting Types Theorems, Institutionally

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Abstract. In the context of proliferation of many logical systems in the area of mathematical logic and computer science, we present a generalization of forcing in institution-independent model theory which is used to prove two abstract results: Downward Löwenheim-Skolem Theorem (DLST) and Omitting Types Theorem (OTT). We instantiate these general results to many first-order logics, which are, roughly speaking, logics whose sentences can be constructed from atomic formulas by means of Boolean connectives and classical first-order quantifiers. These include first-order logic (**FOL**), logic of order-sorted algebras (**OSA**), preorder algebras (**POA**), as well as their infinitary variants $\mathbf{FOL}_{\omega_1,\omega}$, $\mathbf{OSA}_{\omega_1,\omega}$, $\mathbf{POA}_{\omega_1,\omega}$. In addition to the first technique for proving OTT, we develop another one, in the spirit of institution-independent model theory, which consists of borrowing the Omitting Types Property (OTP) from a simpler institution across an institution comorphism. As a result we export the OTP from **FOL** to partial algebras (**PA**) and higher-order logic with Henkin semantics (**HNK**), and from the institution of $\mathbf{FOL}_{\omega_1,\omega}$ to $\mathbf{PA}_{\omega_1,\omega}$ and $\mathbf{HNK}_{\omega_1,\omega}$. The second technique successfully extends the domain of application of OTT to non conventional logical systems for which the standard methods may fail.

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Introduction

A type is a set of formulas in a finite number of variables. A type Δ with free variables $x_1 \dots x_n$ is principal for a theory T if there exists a finite set of formulas p with free variables $x_1 \dots x_n$ such that $T \cup p \models \Delta$. A model M omits Δ if for all n-tuples $m_1, \dots, m_n$ of elements of M we have $M \not\models \Delta(m_1, \dots, m_n)$. The classical OTT states that if T is a complete theory and $(\Delta_n : n \in \mathbb{N})$ is a sequence of non-principal types of T, then there is a model of T omitting all the types Δ_n. The OTT was proved by Henkin [31] and Orey [38] using the method of diagrams and it has many applications in classical model theory. Forcing is a technique invented by Paul Cohen, for proving the independence of the continuum hypothesis from the other axioms of Zermelo-Fraenkel set theory [11, 12]. A. Robinson [41] developed an analogous theory of forcing in model theory, and Barwise [6] extended Robinson's theory to infinitary logic and used it to give a new proof of the OTT. An early contribution on forcing and the omitting types for infinitary logic $\mathcal{L}_{\omega_1,\omega}$ is [32]. For recent developments of the result see [2, 3, 4].

The framework adopted here is the theory of institutions [23] which is a category-based formalization of the intuitive notion of logical system. Institutions constitute a meta-theory on logical systems similar to the manner in which universal algebra constitute a meta-theory for groups and rings. The institution theory arose within computing science, by abstracting away from the realities of conventional logics, with the ambition of proving as much results as possible at the level of abstraction, independent of the details of any particular logical system. In addition to the large use in algebraic specification theory where institutions became the most fundamental mathematical structure underlying formal specification languages, there have been substantial developments towards an abstract institutional model theory [43, 45, 14, 15, 16, 28, 40, 29, 39, 10, 27]. See [19] for a monography dedicated to this topic.

The present paper studies the abstract notions of OTP and forcing in the framework of institutions, and points out many particular instances to concrete logics. In institutional model theory the forcing technique was introduced in [27] and it was used to prove an abstract first-order completeness theorem. The result was obtained as a consequence of the research on a syntactic forcing property. In the present paper we investigate a semantic forcing property which was studied in classical model theory by Robinson, Barwise and Keisler. As an outcome of this investigation we obtain institution-independent versions of some well-known results in classical model theory:

1. Downward Löwenheim-Skolem Theorem (”any consistent theory has a countable model”), and
2. Omitting Types Theorem (“any non-principal type has a model which omits it”).

The categorical assumptions used here are easy to check in concrete logics, and for this reason the abstract theorems can be instantiated to many institutions, some of them explicitly described here, and others just mentioned. Another advantage of the present research is the applicability to both finitary and infinitary cases which is due to the use of the forcing technique.

However, there are examples of more refined institutions which cannot be cast in this abstract framework and for which we believe that the standard methods of proving OTT cannot be replicated. Therefore, in addition to the first technique for establishing OTP, we develop another one, in the spirit of institutional model theory. Instead of developing directly the result within a given institution, one may "borrow" it from a simpler institution via an adequate encoding, expressed as an institution *comorphisms* [26]. More concretely, here we prove a generic theorem for OTP along an institution comorphism $\mathcal{I} \to \mathcal{I}'$ such that if $\mathcal{I}'$ has the OTP and the institution comorphism is *conservative*, then $\mathcal{I}$ can be established to have the OTP. We illustrate the applicability of our borrowing result with examples: we "export" the OTP from first-order logic to higher-order logic with Henkin semantics and from first-order logic to partial algebra.

The paper is organized as follows. The first technical section introduces the institution-theoretic preliminaries and recalls the necessary fundamental concepts of institution-independent model theory such as internal logic, basic sets of sentences, reachable models. The next section recalls the forcing technique in institutional model theory. In Section 3 we develop an institution-independent version of the OTT that is uniformly applicable to both finitary and inifinitary cases. Section 4 studies the translation of the OTP along institution comorphisms and illustrates its applicative power with examples which cannot be captured in the previous abstract setting. Section 5 concludes the paper and discusses the future work.

We assume that the reader is familiar with basic notions of category theory. See [33] for the standard definitions of category, functor, pushout, etc., which are omitted here. The category of sets is denoted by $\mathbb{S}et$, and $\mathbb{CAT}$ is the category of all categories.

1. Institutions

The concept of institution formalizes the intuitive notion of logical system, and has been defined by Goguen and Burstall in the seminal paper [23].

Definition 1.1. An *institution* $\mathcal{I} = (\mathbb{S}ig^{\mathcal{I}}, \mathbb{S}en^{\mathcal{I}}, \mathbb{M}od^{\mathcal{I}}, \models^{\mathcal{I}})$ consists of

1. a category $\mathbb{S}ig^{\mathcal{I}}$, whose objects are called *signatures*,
2. a functor $\mathbb{S}en^{\mathcal{I}} : \mathbb{S}ig^{\mathcal{I}} \to \mathbb{S}et$, providing for each signature Σ a set whose elements are called *(Σ-)sentences*,
3. a functor $\mathbb{M}od^{\mathcal{I}} : (\mathbb{S}ig^{\mathcal{I}})^{op} \to \mathbb{CAT}$, providing for each signature Σ a category whose objects are called *(Σ-)models* and whose arrows are called *(Σ-)morphisms*,
4. a relation $\models^{\mathcal{I}}_{\Sigma} \subseteq |\mathbb{M}od^{\mathcal{I}}(\Sigma)| \times \mathbb{S}en^{\mathcal{I}}(\Sigma)$ for each $\Sigma \in |\mathbb{S}ig^{\mathcal{I}}|$, called *($\Sigma$-)satisfaction*, such that for each morphism $\varphi : \Sigma \to \Sigma'$ in $\mathbb{S}ig^{\mathcal{I}}$, the following *satisfaction condition* holds:
$$M' \models^{\mathcal{I}}_{\Sigma'} \mathbb{S}en^{\mathcal{I}}(\varphi)(e) \text{ iff } \mathbb{M}od^{\mathcal{I}}(\varphi)(M') \models^{\mathcal{I}}_{\Sigma} e$$
for all $M' \in |\mathbb{M}od^{\mathcal{I}}(\Sigma')|$ and $e \in \mathbb{S}en^{\mathcal{I}}(\Sigma)$.

We denote the *reduct* functor $\mathbb{M}od^{\mathcal{I}}(\varphi)$ by $\_ \upharpoonright_{\varphi}$ and the sentence translation $\mathbb{S}en^{\mathcal{I}}(\varphi)$ by $\varphi(\_)$. When $M = M' \upharpoonright_{\varphi}$ we say that M is the φ-reduct of M' and M' is a φ-expansion of M. When there is no danger of confusion, we omit the superscript from the notations of the institution components; for example $\mathbb{S}ig^{\mathcal{I}}$ may be simply denoted by $\mathbb{S}ig$.

Example 1.1 (First-Order Logic (FOL) [23]). The signatures are triplets (S, F, P), where S is the set of sorts, $F = (F_{w\to s})_{(w,s)\in S^*\times S}$ is the ($S^*\times S$ -indexed) set of operation symbols, and $P = (P_w)_{w\in S^*}$ is the (S^*-indexed) set of relation symbols. If $w = \lambda$, an element of $F_{w\to s}$ is called a *constant symbol*, or a *constant*. By a slight notational abuse, we let F and P also denote $\bigcup_{(w,s)\in S^*\times S} F_{w\to s}$ and $\bigcup_{w\in S^*} P_w$ respectively. For all institutions defined in this paper we assume that the signatures consist of a countable number of symbols. This condition implies that the sets of sentences of any **FOL** signature are countable.

A signature morphism between (S, F, P) and (S', F', P') is a triplet $\varphi = (\varphi^{st}, \varphi^{op}, \varphi^{rl})$, where $\varphi^{st} : S \to S'$, $\varphi^{op} : F \to F'$, $\varphi^{rl} : P \to P'$ such that $\varphi^{op}(F_{w\to s}) \subseteq F'_{\varphi^{st}(w)\to\varphi^{st}(s)}$ and $\varphi^{rl}(P_w) \subseteq P'_{\varphi^{st}(w)}$ for all $(w, s) \in S^* \times S$. When there is no danger of confusion, we may let φ denote each of φ^{st}, φ^{rl} and φ^{op}.

Let $\Sigma = (S, F, P)$. A Σ-model is a triplet

$$A = ((A_s)_{s\in S}, (A^{w,s}_\sigma)_{(w,s)\in S^*\times S,\sigma\in F_{w\to s}}, (A^w_\pi)_{w\in S^*,\pi\in P_w})$$

interpreting each sort s as a non-empty set A_s, each operation symbol $\sigma \in F_{w\to s}$ as a function $A^{w,s}_\sigma : A^w \to A_s$ (where A^w stands for $A_{s_1} \times \ldots \times A_{s_n}$ if $w = s_1 \ldots s_n$), and each relation symbol $\pi \in P_w$ as a relation $A^w_\pi \subseteq A^w$. When there is no danger of confusion we may let A_σ and A_π denote $A^{w,s}_\sigma$ and A^w_π respectively. Since the models consist of non-empty carriers, every injective signature morphism $\varphi : \Sigma \to \Sigma'$ is *conservative*, i.e. every Σ-model has a φ-expansion. Morphisms between models are the usual Σ-morphisms, i.e., S-sorted functions that preserve the structure.

The Σ-sentences are obtained from

- equality atoms $t_1 = t_2$, where $t_1, t_2 \in (T_{(S,F)})_s$, $T_{(S,F)}$ is the (S, F)-algebra of ground terms and $s \in S$, or
- relational atoms $\pi(t_1, \ldots, t_n)$, where $\pi \in P_{s_1\ldots s_n}$ and $t_i \in (T_{(S,F)})_{s_i}$ for all $i \in \{1, \ldots, n\}$,

by applying for a finite number of times:

- negation, disjunction, and
- existential quantification over finite sets of variables.

Satisfaction is the usual first-order satisfaction and is defined using the natural interpretations of ground terms t as elements A_t in models A. The definitions of functors $\mathbb{S}en$ and $\mathbb{M}od$ on morphisms are the natural ones: for any signature morphism $\varphi : \Sigma \to \Sigma'$, $\mathbb{S}en(\varphi) : \mathbb{S}en(\Sigma) \to \mathbb{S}en(\Sigma')$ translates sentences symbol-wise, and $\mathbb{M}od(\varphi) : \mathbb{M}od(\Sigma') \to \mathbb{M}od(\Sigma)$ is the forgetful functor.

Example 1.2 (First-Order Equational Logic (FOEQL)). This institution is obtained from **FOL** by restricting the syntax to signatures with no predicate symbols.

Example 1.3 (Preorder Algebra (POA) [21, 22]). The **POA** signatures are just the ordinary algebraic signatures. The **POA** models are *preordered algebras* which are interpretations of the signatures into the category of preorders $\mathbb{P}re$ rather than the category of sets $\mathbb{S}et$. This means that each sort gets interpreted as a preorder, and each operation as a preorder functor, which means a preorder-preserving (i.e. monotonic) function. A *preordered algebra morphism* is just a family of preorder functors (preorder-preserving functions) which is also an algebra morphism.

The sentences have two kinds of atoms: equations and *preorder atoms*. A preorder atom $t \leq t'$ is satisfied by a preorder algebra M when the interpretations of the terms are in the preorder relation of the carrier, i.e. $M_t \leq M_{t'}$. Full sentences are constructed from equational and preorder atoms by using Boolean connectives and first-order quantification.

Example 1.4 (Order-Sorted Algebra (OSA) [25]). An order-sorted signature $(S, \leq, F)$ consists of an algebraic signature (S, F) with a partial ordering $(S, \leq)$ such that the following *monotonicity* condition is satisfied: $\sigma \in F_{w_1 \to s_1} \cap F_{w_2 \to s_2}$ and $w_1 \leq w_2$ imply $s_1 \leq s_2$. A morphism of **OSA**-signatures $\varphi : (S, \leq, F) \to (S', \leq', F')$ is just a morphism of algebraic signatures $(S, F) \to (S', F')$ such that the ordering is preserved, i.e. $\varphi(s_1) \leq' \varphi(s_2)$ whenever $s_1 \leq s_2$. Given an order-sorted signature $(S, \leq, F)$, an order-sorted $(S, \leq, F)$-algebra is an (S, F)-algebra M such that $s_1 \leq s_2$ implies $M_{s_1} \subseteq M_{s_2}$, and $\sigma \in F_{w_1 \to s_1} \cup F_{w_2 \to s_2}$ and $w_1 \leq w_2$ imply $M_\sigma^{w_1, s_1} = M_\sigma^{w_2, s_2}|_{M_{w_1}}$. Given two order-sorted $(S, \leq, F)$-algebras M and N, an order-sorted $(S, \leq, F)$-morphism $h : M \to N$ is a (S, F)-morphism such that $s_1 \leq s_2$ implies $h_{s_1} = h_{s_2}|_{M_{s_1}}$.

An **OSA** signature $(S, \leq, F)$ is *regular* iff for each $\sigma \in F_{w_1 \to s_1}$ and $w_0 \leq w_1$ there is a unique least element in the set $\{(w, s) \mid \sigma \in F_{w \to s} \text{ and } w_0 \leq w\}$. For regular signatures $(S, \leq, F)$, any F-term t has a least sort $LS(t)$ and the initial $(S, \leq, F)$-algebra can be defined as a term algebra, cf. [25]. Let $(S, \leq, F)$ be an order-sorted signature. We say that the sorts s_1 and s_2 are in the same *connected component* of S iff $s_1 \equiv s_2$, where $\equiv$ is the least equivalence on S that contains $\leq$. A partial ordering $(S, \leq)$ is *filtered* iff for all $s_1, s_2 \in S$, there is some $s \in S$ such that $s_1 \leq s$ and $s_2 \leq s$. A partial ordering is *locally filtered* iff every connected component of it is filtered. An order-sorted signature $(S, \leq, F)$ is *locally filtered* iff $(S, \leq)$ is locally filtered, and it is *coherent* iff it is both locally filtered and regular. Hereafter we assume that all **OSA** signatures are coherent.

The atoms of the signature $(S, \leq, F)$ are equations of the form $t_1 = t_2$ such that the least sort of the terms t_1 and t_2 are in the same connected component. The sentences are closed formulas built by application of Boolean connectives and quantification to the equational atoms. Order-sorted algebras were extensively studied in [25, 24, 35].

Example 1.5 (Partial Algebra (PA)). Here we consider the institution **PA** as employed by the specification language CASL [5].

Its signatures consist of tuples (S, TF, PF), where TF is a family of sets of total function symbols and PF is a family of sets of partial function symbols such that $TF_{w \to s} \cap PF_{w \to s} = \emptyset$ for each arity w and sort s. Models consist of algebras interpreting each total symbol in TF as a total function and each partial symbol in PF as a partial function. A *partial algebra homomorphism* $h : A \to B$ is a family of (total) functions $(h_s : A_s \to B_s)_{s \in S}$ indexed by the set of sorts S of the signature such that $h_s(A_\sigma(a)) = B_\sigma(h_w(a))$ for each operation $\sigma : w \to s$ and each string of arguments $a \in A_w$ for which $A_\sigma(a)$ is defined.

We consider one kind of "base" sentences: *existence equality* $t \overset{e}{=} t'$. The existence equality $t \overset{e}{=} t'$ holds when both terms are defined and equal. The sentences are formed from these "base" sentences by Boolean connectives and quantification over variables interpreted as total functions. The definedness predicate and strong equality can be introduced as notations: $def(t)$ stands for $t \overset{e}{=} t$ and $t \overset{s}{=} t'$ stands for $(t \overset{e}{=} t') \vee (\neg def(t) \wedge \neg def(t'))$.

Example 1.6 (Higher-order logic with Henkin semantics (HNK)). Higher-order logic with Henkin semantics has been introduced and studied in [8] and [30]. In the present paper we consider a simplefied version close to the "higher-order algebra" of [42] which does not consider λ-abstraction.

For any set S of sorts, let $\overrightarrow{S}$ be the set of S-types defined as the least set such that $S \subseteq \overrightarrow{S}$ and $s_1 \to s_2 \in \overrightarrow{S}$ when $s_1, s_2 \in \overrightarrow{S}$. A **HNK**-signature is a tuple (S, F), where S is a set of sorts and F is a family of sets of constants $F = (F_s)_{s \in \overrightarrow{S}}$. A signature morphism $\varphi : (S, F) \to (S', F')$ consists of a function $\varphi^{st} : S \to S'$ and a family of functions between operation symbols $(\varphi^{op}_s : F_s \to F'_{\varphi^{type}(s)})_{s \in \overrightarrow{S}}$ where $\varphi^{type} : \overrightarrow{S} \to \overrightarrow{S'}$ is the natural extension of φ^{st} to $\overrightarrow{S}$. For every signature (S, F), a (S, F)-model interprets each

1. sort $s \in S$ as a set, and
2. function symbol $\sigma \in F_s$ as an element of M_s, where for each types $s_1, s_2 \in \overrightarrow{S}$, $M_{s_1 \to s_2} \subseteq [M_{s_1} \to M_{s_2}] = \{f \text{ function } \mid f : M_{s_1} \to M_{s_2}\}$.

An (S, F)-model morphism $h : M \to N$ interprets each type $s \in \overrightarrow{S}$ as a function $h_s : M_s \to N_s$ such that $h(M_\sigma) = N_\sigma$, for all function symbols $\sigma \in F$, and the following diagram commutes

$$\begin{array}{ccc} M_{s_1} & \xrightarrow{f} & M_{s_2} \\ {\scriptstyle h_{s_1}} \downarrow & & \downarrow {\scriptstyle h_{s_2}} \\ N_{s_1} & \xrightarrow[h_{s_1 \to s_2}(f)]{} & N_{s_2} \end{array}$$

for all types $s_1, s_2 \in \overrightarrow{S}$ and functions $f \in M_{s_1 \to s_2}$.

An (S, F)-equation is of the form $t_1 = t_2$, where t_1 and t_2 are terms of the same type. Sentences are constructed from equations by iteration of Boolean connectives and quantification over variables of any type.

Example 1.7 (Infinitary logic $\mathbf{FOL}_{\omega_1,\omega}$). This is the infinitary version of first-order logic allowing disjunctions of countable sets of sentences. Similarly we may define $\mathbf{POA}_{\omega_1,\omega}$, $\mathbf{OSA}_{\omega_1,\omega}$, $\mathbf{PA}_{\omega_1,\omega}$, and $\mathbf{HNK}_{\omega_1,\omega}$. Note that the set of sentences over a signature is uncountable even if the signatures consist of a countable number of symbols.

Example 1.8 (Institution of presentations). In any institution $\mathcal{I} = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$, a presentation is a pair (Σ, E) consisting of a signature $\Sigma \in |\mathbb{S}ig|$ and a set of Σ-sentences E. A presentation morphism $\varphi : (\Sigma, E) \to (\Sigma', E')$ is a signature morphism $\varphi : \Sigma \to \Sigma'$ such that $E' \models_{\Sigma'} \varphi(E)$. Note that presentation morphisms are closed under the composition from the category of signatures $\mathbb{S}ig$. The institution of presentations over $\mathcal{I}$, denoted by $\mathcal{I}^{pres} = (\mathbb{S}ig^{pres}, \mathbb{S}en^{pres}, \mathbb{M}od^{pres}, \models^{pres})$ is defined as follows:

- $\mathbb{S}ig^{pres}$ is the category of presentations of $\mathcal{I}$,
- $\mathbb{S}en^{pres}(\Sigma, E) = \mathbb{S}en(\Sigma)$,
- $\mathbb{M}od^{pres}(\Sigma, E)$ is the full subcategory of $\mathbb{M}od(\Sigma)$ of models satisfying E, and
- $M \models^{pres}_{(\Sigma,E)} e$ iff $M \models_\Sigma e$, for each (Σ, E)-model M and Σ-sentence e.

Let $\mathbb{S}ig^{cpres}$ be the full subcategory of $\mathbb{S}ig^{pres}$ consisting of countable presentations, i.e. presentations (Σ, E) for which the set of Σ-sentences E is countable. One can easily define the institution $\mathcal{I}^{cpres}$ of countable presentations over an arbitrary institution $\mathcal{I}$.

Definition 1.2 (Compactness). An institution $\mathcal{I} = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ is compact whenever $E \models_\Sigma e$ implies the existence of a finite subset $E_f \subseteq E$ such that $E_f \models_\Sigma e$.

According to [14], the institutions **FOL**, **POA** and **OSA** are compact. Their infinitary versions $\mathbf{FOL}_{\omega_1,\omega}$, $\mathbf{POA}_{\omega_1,\omega}$ and $\mathbf{OSA}_{\omega_1,\omega}$ are not.

Definition 1.3 (Finitary sentences). [18] In any institution a Σ-sentence ρ is *finitary* iff it can be written as $\varphi(\rho_f)$ where $\varphi : \Sigma_f \to \Sigma$ is a signature morphism such that Σ_f is a finitely presented signature [1] and ρ_f is a Σ_f sentence. An institution *has finitary sentences* when all its sentences are finitary.

This condition usually means that the sentences contain only a finite number of symbols and it holds for **FOL**, **POA**, **OSA**, **PA** and **HNK**. Only the infinitary logics, such as $\mathbf{FOL}_{\omega_1,\omega}$, do not fulfill this condition.

Definition 1.4 (Finitary signature morphisms). We say that a signature morphism $\varphi : \Sigma \to \Sigma'$ is finitary if it is finitely presented in the category $\Sigma/\mathbb{S}ig$.

In typical institutions the extensions of signatures with a finite number of symbols are finitary.

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[1] An object A in a category $\mathcal{C}$ is called *finitely presented* ([1]) if

- for each directed diagram $D : (J, \leq) \to \mathcal{C}$ with co-limit $(Di \overset{\mu_i}{\to} B)_{i\in J}$, and for each morphism $A \overset{g}{\to} B$, there exists $i \in J$ and $A \overset{g_i}{\to} Di$ such that $g_i; \mu_i = g$,

- for any two arrows g_i and g_j as above, there exists $i \leq k, j \leq k \in J$ such that $g_i; D(i \leq k) = g_j; D(j \leq k)$.

1.1. Internal logic

The following institutional notions dealing with the logical connectives and quantifiers were defined in [44]. Let Σ be a signature of an institution,

- a Σ-sentence $\neg e$ is a *(semantic) negation* of the Σ-sentence e when for every Σ-model M we have $M \models_\Sigma \neg e$ iff $M \not\models_\Sigma e$,
- a Σ-sentence $e_1 \vee e_2$ is a *(semantic) disjunction* of the Σ-sentences e_1 and e_2 when for every Σ-model M we have $M \models_\Sigma e_1 \vee e_2$ iff $M \models_\Sigma e_1$ or $M \models_\Sigma e_2$, and
- a Σ-sentence $(\exists\chi)e'$, where $\Sigma \xrightarrow{\chi} \Sigma' \in \mathbb{S}ig$ and $e' \in \mathbb{S}en(\Sigma')$, is a *(semantic) existential χ-quantification* of e' when for every Σ-model M we have $M \models_\Sigma (\exists\chi)e'$ iff $M' \models_{\Sigma'} e'$ for some χ-expansion M' of M.

Distinguished negation $\neg\_$, disjunction $\bigvee\_$ [2], and existential quantification $(\exists\chi)\_$ are called *first-order constructors* and they have the semantical meaning defined above. Throughout this paper we assume the following commutativity of the first-order constructors with the signature morphisms: for each $\varphi : \Sigma \to \Sigma_1$ and any Σ-sentence

- $\neg e$ we have $\varphi(\neg e) = \neg\varphi(e)$;
- $\bigvee E$ we have $\varphi(\bigvee E) = \bigvee \varphi(E)$;
- $(\exists\chi)e'$, where $\chi : \Sigma \to \Sigma'$, there exists $$\begin{array}{ccc} \Sigma' & \xrightarrow{\varphi'} & \Sigma'_1 \\ \chi \uparrow & pushout & \uparrow \chi' \\ \Sigma & \xrightarrow[\varphi]{} & \Sigma_1 \end{array}$$ s.t. $\varphi((\exists\chi)e') = (\exists\chi')\varphi'(e')$.

A variable for a **FOL** signature (S, F, P) is a triple $(x, s, (S, F, P))$, where x is the name of the variable and $s \in S$ is the sort of the variable. Let $\chi : (S, F, P) \hookrightarrow (S, F \cup X, P)$ be a signature inclusion, where X is a set of variables. For any $(S, F \cup X, P)$-sentence ρ, $(\forall X)\rho$ is an abbreviation of $(\forall\chi)\rho$. Consider a signature morphism $\varphi : (S, F, P) \to (S', F', P')$. Then $\varphi((\forall X)\rho) = (\forall X^\varphi)\varphi'(\rho)$ where $X^\varphi = \{(x, \varphi^{st}(s), (S', F', P')) \mid (x, s, (S, F, P)) \in X\}$ and $\varphi' : (S, F \cup X, P) \to (S', F' \cup X^\varphi, P')$ extends φ canonically by mapping each variable $(x, s, (S, F, P)) \in X$ to $(x, \varphi^{st}(s), (S', F', P'))$.

We assume another rather mild condition: for every signature morphism $\varphi : \Sigma \to \Sigma_1$, each Σ-sentence $(\exists\chi)e'$ and any Σ_1-sentence $(\exists\chi')e'_1$

- if $\varphi((\exists\chi)e') = (\exists\chi')e'_1$, where $\chi : \Sigma \to \Sigma'$ and $\chi' : \Sigma_1 \to \Sigma'_1$, then there exists $\varphi' : \Sigma' \to \Sigma'_1$ such that $$\begin{array}{ccc} \Sigma' & \xrightarrow{\varphi'} & \Sigma'_1 \\ \chi \uparrow & & \uparrow \chi' \\ \Sigma & \xrightarrow[\varphi]{} & \Sigma_1 \end{array}$$ is a pushout and $\varphi'(e') = e'_1$.

[2]We will use the symbol $\bigvee$ to represent the most general kind of disjunction even if it is finitary.

Very often quantification is considered only for a restricted class of signature morphisms. For example, quantification in **FOL** considers only the finitary signature extensions with variables. Based on these constructors for sentences we can also define $\bigwedge$, *true*, $(\forall\chi)\_$ using the classical definitions.

1.2. Basic sets of sentences

A set of sentences $E \subseteq \mathbb{S}en(\Sigma)$ is called *basic* [14] if there exists a Σ-model M_E such that

$$M \models E \text{ iff there exists a morphism } M_E \to M$$

for all Σ-models M. If in addition the morphism $M_E \to M$ is unique then the set E is called *epi basic.*

Lemma 1.1. *Let (S,F,P) be a* **FOL***-signature such that all sorts are inhabited, i.e. for all sorts $s \in S$ we have $(T_{(S,F)})_s \neq \emptyset$. Then any set of atoms $E \subseteq \mathbb{S}en^{\mathbf{FOL}}(S,F,P)$ is basic.*

Proof. Let E be a set of atomic (S,F,P)-sentences. Since all sorts of (S,F,P) are inhabited, $T_{(S,F)}$ is a (S,F,P)-model interpreting each $\pi \in P$ as the empty set. The basic model M_E is constructed as follows: on the quotient $(T_{(S,F)})/_{\equiv_E}$ of the term model $T_{(S,F)}$ by the congruence generated by the equational atoms of E, we interpret each relation symbol $\pi \in P$ by $(M_E)_\pi = \{(t_1/_{\equiv_E}, \ldots, t_n/_{\equiv_E}) \mid \pi(t_1, \ldots, t_n) \in E\}$. □

The proof of the above lemma is from [14]. The only difference is that the semantics is restricted in this paper to models with non-empty carrier sets. A similar construction as the preceding holds for **OSA** provided that the order-sorted signatures are coherent. By defining an appropriate notion of congruence for **POA**-models compatible with the preorder (see [20] or [10]) one may obtain the same result for **POA**. In **PA** any set of ground existence equations is basic provided that all sorts of the underlying signature are inhabited by terms formed of total function symbols (see [10] for a proof of this fact). In **HNK** this property does not hold [9].

1.3. Substitutions

Consider $\Sigma \overset{\chi_1}{\hookrightarrow} \Sigma(C_1)$ and $\Sigma \overset{\chi_2}{\hookrightarrow} \Sigma(C_2)$ two inclusion signature morphisms, where $\Sigma = (S,F,P)$ is a **FOL** signature, C_i is a family of constant symbols disjoint from the constants of F, $\Sigma(C_i) = (S, F \cup C_i, P)$. A substitution between χ_1 and χ_2 can be represented by a function $\theta : C_1 \to T_{(S,F)}(C_2)$. One can easily notice that θ can be extended to a function

$$\mathbb{S}en(\theta) : \mathbb{S}en(\Sigma(C_1)) \to \mathbb{S}en(\Sigma(C_2))$$

that preserves Σ and replaces all symbols in C_1 by the corresponding $(S, F \cup C_2)$-terms according to θ. On the semantics side, θ determines a functor

$$\mathbb{M}od(\theta) : \mathbb{M}od(\Sigma(C_2)) \to \mathbb{M}od(\Sigma(C_1))$$

such that for all $\Sigma(C_2)$-models M we have

- $\mathbb{M}od(\theta)(M)_z = M_z$, for each sort $z \in S$, or operation symbol $z \in F$, or relation symbol $z \in P$, and
- $\mathbb{M}od(\theta)(M)_z = M_{\theta(z)}$ for each $z \in C_1$.

Lemma 1.2. [17] *For every* **FOL** *signature* Σ *and each substitution* $\theta : C_1 \to \Sigma(C_2)$

$$\mathbb{M}od(\theta)(M) \models \rho \text{ iff } M \models \mathbb{S}en(\theta)(\rho)$$

for all $\Sigma(C_2)$*-models* M *and all* $\Sigma(C_1)$*-sentences* ρ.

Let $\mathbf{FOL}'$ be the institution obtained from **FOL** by generalizing the notion of signature morphism, allowing mappings of constants to terms. Similarly we define $\mathbf{OSA}'$ and $\mathbf{POA}'$. Our abstract results are applicable to these generalized institutions rather than their standard versions.

Definition 1.5. Consider an institution $\mathcal{I} = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ and two signature morphisms $\chi_1 : \Sigma \to \Sigma_1$ and $\chi_2 : \Sigma \to \Sigma_2$. A signature morphism $\theta : \Sigma_1 \to \Sigma_2$ such that $\chi_1; \theta = \chi_2$ is called a substitution between χ_1 and χ_2. Let $\mathcal{D} \subseteq \mathbb{S}ig$ be a subcategory of signature morphisms. A $\mathcal{D}$-substitution is just a substitution between two signature morphisms in $\mathcal{D}$.

A more general treatment of substitutions may be found in [17]. Here we treat substitutions in the comma category of signature morphisms.

1.4. Reachable models

We give an institution-independent characterization of the models which consist of interpretations of terms.

Definition 1.6. [27] Let $\mathcal{I} = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ be an institution and $\mathcal{D}$ a broad subcategory of signature morphisms. We say that a Σ-model M is $\mathcal{D}$-reachable if for all signature morphisms $\chi : \Sigma \to \Sigma'$ in $\mathcal{D}$, each χ-expansion M' of M determines a substitution $\theta : \chi \to 1_\Sigma$ such that $M \upharpoonright_\theta = M'$.

This definition makes sense in institutions with generalized signature morphisms which capture the notion of substitution. In concrete examples $\mathcal{D}$ consists of signature morphisms used for quantifications, i.e. extensions of signatures with a finite number of variables. A model M is reachable if the elements of M are exactly the interpretations of the terms.

Proposition 1.3. [27] *In* $\mathbf{FOL}'$, $\mathbf{OSA}'$ *and* $\mathbf{POA}'$ *a model is* $\mathcal{D}$*-reachable iff its elements consist only of interpretations of terms, where* $\mathcal{D}$ *is the class of signature morphisms used for quantification, i.e. signature extensions with a finite number of variables.*

Remark 1.1. Let (S, F, P) be a first-order signature such that all sorts are inhabited. For any set $E \subseteq \mathbb{S}en(S, F, P)$ of atomic sentences there exists a model M_E, defining E as a basic set of sentences, which is $\mathcal{D}$-reachable, where $\mathcal{D}$ consists of signature extensions with a finite number of variables.

A similar remark as above holds $\mathbf{POA}'$ and $\mathbf{OSA}'$. Proposition 1.3 is not applicable to partial algebras (see [10]).

2. Forcing and Generic Models

All the results in this section can be found in [27].

Definition 2.1 (First-order institutions). Let $\mathcal{I} = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ be an institution, $\mathbb{S}en_0 \subseteq \mathbb{S}en$ a sub-functor, and $\mathcal{D} \subseteq \mathbb{S}ig$ a subcategory of signature morphisms. We say that $\mathcal{I}$ is a $\mathcal{D}$-first-order institution over $\mathcal{I}_0 = (\mathbb{S}ig, \mathbb{S}en_0, \mathbb{M}od, \models)$ if each sentence of $\mathcal{I}$ is constructed from the sentences of $\mathcal{I}_0$ by applying negation, disjunction of countable sets of sentences and existential quantification over the signatures morphisms in $\mathcal{D}$.

For example **FOL** is a $\mathcal{D}$-first-order institution over $\mathbf{FOL}_0$, the restriction of **FOL** to atomic sentences, where $\mathcal{D}$ is the subcategory of signature morphisms which consists of signature extensions with a finite number of variables. $\mathbf{FOL}_{\omega_1,\omega}$ is also $\mathcal{D}$-first-order institution over $\mathbf{FOL}_0$.

Definition 2.2 (First-order fragments). Let $\mathcal{I} = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ be an institution, $\mathcal{D} \subseteq \mathbb{S}ig$ a broad subcategory of signature morphisms, $\Sigma \in |\mathbb{S}ig|$ a signature, and Γ is a set of Σ-sentences. The *$\mathcal{D}$-first-order fragment ($\mathcal{D}$-fragment, for short) over* Γ is the least set of sentences containing $\Gamma \cup \mathbb{S}en_0(\Sigma)$ which is closed under

1. negation, i.e. if $e \in \mathcal{L}$ then $\neg e \in \mathcal{L}$.
2. "sub-sentence" relation, i.e.
    - if $\neg e \in \mathcal{L}$ then $e \in \mathcal{L}$,
    - if $\bigvee E \in \mathcal{L}$ then $e \in \mathcal{L}$ for all $e \in E$, and
    - if $(\exists \chi)e' \in \mathcal{L}$, where $\chi \in \mathcal{D}$ and $\theta : \chi \to 1_\Sigma$, then $\theta(e') \in \mathcal{L}$.

Framework 1. Throughout this section we consider a $\mathcal{D}$-first-order institution $\mathcal{I} = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ over $\mathcal{I}_0 = (\mathbb{S}ig, \mathbb{S}en_0, \mathbb{M}od, \models)$, and we work within a $\mathcal{D}$-fragment $\mathcal{L}$ over some set Γ of Σ-sentences.

Definition 2.3. A forcing property is a tuple $\mathbb{P} = \langle P, \leq, f \rangle$ such that:

1. $\langle P, \leq \rangle$ is a partially ordered set with a least element 0,
2. f is a function which associates with each $p \in P$ a set $f(p)$ of sentences in $\mathbb{S}en_0(\Sigma)$,
3. $f(p) \subseteq f(q)$ whenever $p \leq q$, and
4. for all $E \subseteq \mathbb{S}en_0(\Sigma)$, $e \in \mathbb{S}en_0(\Sigma)$ and $p \in P$ if $E \subseteq f(p)$ and $E \models e$ then there is $q \geq p$ such that $e \in f(q)$.

The elements of P are called *conditions*. We will define the forcing relation $\Vdash \subseteq P \times \mathcal{L}$ associated to a forcing property $\mathbb{P} = \langle P, \leq, f \rangle$.

Definition 2.4. Let $\mathbb{P} = \langle P, \leq, f \rangle$ be a forcing property. The relation $p \Vdash e$, read p forces e, is defined by induction on e, for $p \in P$ and $e \in \mathcal{L}$, as follows:

- *For* $e \in \mathbb{S}en_0(\Sigma)$*:* $p \Vdash e$ if $e \in f(p)$.
- *For* $\neg e \in \mathcal{L}$*:* $p \Vdash \neg e$ if there is no $q \geq p$ such that $q \Vdash e$.
- *For* $\bigvee E \in \mathcal{L}$*:* $p \Vdash \bigvee E$ if $p \Vdash e$ for some $e \in E$.
- *For* $(\exists \chi)e \in \mathcal{L}$*:* $p \Vdash (\exists \chi)e$ if $p \Vdash \theta(e)$ for some substitution $\theta : \chi \to 1_\Sigma$.

We say that p *weakly forces* e, in symbols $p \Vdash^w e$, iff $p \Vdash \neg\neg e$. The above definition is a generalization of the forcing studied in [41], [6] and [32].

Lemma 2.1. *Let $\mathbb{P} = \langle P, \leq, f\rangle$ be a forcing property and $e \in \mathcal{L}$.*

1. *$p \Vdash^w e$ iff for each $q \geq p$ there is a condition $r \geq q$ such that $r \Vdash e$.*
2. *If $p \leq q$ and $p \Vdash e$ then $q \Vdash e$.*
3. *If $p \Vdash e$ then $p \Vdash^w e$.*
4. *We can not have both $p \Vdash e$ and $p \Vdash \neg e$.*

Definition 2.5. Let $\mathbb{P} = \langle P, \leq, f\rangle$ be a forcing property. A subset $G \subseteq P$ is said to be generic iff

1. $p \in G$ and $q \leq p$ implies $q \in G$.
2. $p, q \in G$ implies that there exists $r \in G$ such that $p \leq r$ and $q \leq r$.
3. for each sentence $e \in \mathcal{L}$ there exists a condition $p \in G$ such that either $p \Vdash e$ or $p \Vdash \neg e$.

Lemma 2.2. *Let $\mathbb{P} = \langle P, \leq, f\rangle$ be a forcing property. If $\mathcal{L}$ is countable then every p belongs to a generic set.*

Definition 2.6. Let $\mathbb{P} = \langle P, \leq, f\rangle$ be a forcing property.

1. M is a model for $G \subseteq P$ if for every sentence $e \in \mathcal{L}$
$$M \models e \text{ iff } G \Vdash e$$
where $G \Vdash e$ iff $p \Vdash e$ for some $p \in G$.
2. M is a generic model for $p \in P$ if there is a generic set $G \subseteq P$ such that $p \in G$ and M is a model for G.

Proposition 2.3. *Assume that*

1. *$\mathcal{I}_0$ is compact,*
2. *all sets $E \subseteq \mathbb{S}en_0(\Sigma)$ are basic, and for each $E \subseteq \mathbb{S}en_0(\Sigma)$ there exists a basic model M_E that is $\mathcal{D}$-reachable.*

Let $\mathbb{P} = \langle P, \leq, f\rangle$ be a forcing property. Then there is a $\mathcal{D}$-reachable model for every generic set G.

Theorem 2.4. *(Generic model theorem) Under the conditions of Proposition* 2.3*, if $\mathcal{L}$ is countable then there is a generic $\mathcal{D}$-reachable model for each condition $p \in P$.*

The following is a corollary of the generic model theorem.

Corollary 2.5. *Under the conditions of Theorem* 2.4*, for every condition $p \in P$ and any sentence $e \in \mathcal{L}$ we have that*

$$p \Vdash^w e \text{ iff } M \models e \text{ for each generic model } M \text{ for } p \text{ which is } \mathcal{D}\text{-reachable}$$

3. Omitting Types

In this section we investigate a forcing property studied by Robinson [41] and Barwise [6] in classical model theory and we use it to prove a very general form of OTT.

3.1. Preliminaries

We define OTP at the abstract level of institutions.

Definition 3.1. Given an institution $\mathcal{I} = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$, for each signature morphism $\chi : \Sigma \to \Sigma'$, a Σ-model M *χ-realizes* a set Δ of Σ'-sentences, if there exists a χ-expansion M' of M which satisfies Δ. We say that M *χ-omits* Δ if does not χ-realize Δ.

The key theorem of this section gives sufficient institution-independent conditions for a theory $\Gamma \subseteq \mathbb{S}en(\Sigma)$ to have a model which omits Δ. As in classical model theory, the central idea is the notion of a theory locally omitting a set of sentences.

Definition 3.2. Let $\mathcal{I} = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ be an institution. For each signature morphism $\chi : \Sigma \to \Sigma'$, a set Γ of Σ-sentences *locally χ-realizes* a set Δ of Σ'-sentences iff there is a finite set p of Σ'-sentences such that:

1. $\chi(\Gamma) \cup p$ is consistent [3], and
2. $\chi(\Gamma) \cup p \models_{\Sigma'} \Delta$

Γ *locally χ-omits* Δ if does not locally χ-realize Δ.

Definition 3.3. An institution $\mathcal{I} = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ has the $\mathcal{D}$-Omitting Types Property (*$\mathcal{D}$-OTP*), where $\mathcal{D} \subseteq \mathbb{S}ig$ is a broad subcategory of signature morphisms, when for any

- countable and consistent set of sentences $\Gamma \subseteq \mathbb{S}en(\Sigma)$,
- sequence of signature morphisms $(\Sigma \xrightarrow{\chi_n} \Sigma_n \in \mathcal{D})_{n\in\mathbb{N}}$, and
- countable sets of sentences $(\Delta_n \subseteq \mathbb{S}en(\Sigma_n))_{n\in\mathbb{N}}$

if Γ locally χ_n-omits Δ_n, for all $n \in \mathbb{N}$, then there is a Σ-model M of Γ which χ_n-omits Δ_n, for all $n \in \mathbb{N}$.

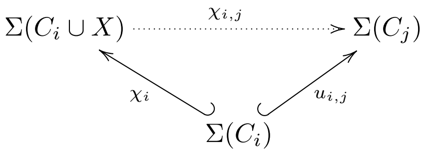
In classical model theory, the models of interest are constructed in an extension $\mathcal{L}_C$ of the initial language $\mathcal{L}$ with an infinite but countable set of constants C (see [32, 38, 31, 7]). The following definition is from [27] and it gives the categorical properties of the extension $\mathcal{L} \hookrightarrow \mathcal{L}_C$ that we need to obtain our results.

Definition 3.4. Let $\mathcal{I} = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ be an institution, and $\mathcal{D} \subseteq \mathbb{S}ig$ a subcategory of signature morphisms. We say that a conservative signature morphism $\Sigma \xrightarrow{v} \Sigma'$ is a $\mathcal{D}$-extension of Σ if it is the vertex of a directed co-limit $(u_i \xrightarrow{v_i} v)_{i\in J}$ of a directed diagram $(u_i \xrightarrow{u_{i,j}} u_j)_{(i\leq j)\in(J,\leq)}$ in $\Sigma/\mathbb{S}ig$ such that

1. for all $i \in J$, $u_i \in \mathcal{D}$ and v_i is conservative,
2. for all $(i \leq j) \in (J, \leq)$, $u_{i,j}$ is conservative, and
3. for all $\Sigma_i \xrightarrow{\chi_i} \Sigma'_i \in \mathcal{D}$ there exists a conservative substitution $\chi_{i,j} : \chi_i \to u_{i,j}$.

[3] A set E of Σ-sentences is consistent if there exists a Σ-model M satisfying E.

Take for example **FOL** and assume that $\mathcal{D}$ is the class of signature extensions with a finite number of variables. Let $\Sigma = (S, F, P)$ be a **FOL** signature, and C a set of variables for Σ such that C_s is infinite and countable for all sorts $s \in S$. The inclusion $\Sigma \overset{u}{\hookrightarrow} \Sigma(C)$, where the signature $\Sigma(C) = (S, F \cup C, P)$, is the vertex of the directed co-limit $((\Sigma \overset{u_i}{\hookrightarrow} \Sigma(C_i)) \overset{v_i}{\hookrightarrow} (\Sigma \overset{v}{\hookrightarrow} \Sigma(C)))_{C_i \subseteq C finite}$ of the directed diagram $((\Sigma \overset{u_i}{\hookrightarrow} \Sigma(C_i)) \overset{u_{i,j}}{\hookrightarrow} (\Sigma \overset{u_j}{\hookrightarrow} \Sigma(C_j))_{C_i \subseteq C_j \subseteq C finite}$. Since C is infinite, for every signature extension $\chi_i : \Sigma(C_i) \hookrightarrow \Sigma(C_i \cup X)$, where X is a finite set of new variables for $\Sigma(C_i)$, there exists an injective mapping $\chi_{i,j} : C_i \cup X \to C_j$ for some $j \in J$ such that the restriction $\chi_{i,j} \restriction_{C_i} : C_i \to C_j$ is the inclusion. Hence, the following diagram commutes.



Since $\chi_{i,j} : \Sigma(C_i \cup X) \to \Sigma(C_j)$ is injective and the models have non empty carriers, $\chi_{i,j} : \Sigma(C_i \cup X) \to \Sigma(C_j)$ is conservative.

3.2. A semantic forcing property

We generalize the semantic forcing property defined in [32] to the abstract level of institutions.

Framework 2. Throughout this section, we work within a $\mathcal{D}$-first-order institution $\mathcal{I} = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ over $\mathcal{I}_0$, with $\mathbb{S}en_0$ the sub-functor of $\mathbb{S}en$, such that

1. the sentences in $\mathcal{I}_0$ are finitary,
2. for all $\Sigma \in |\mathbb{S}ig|$ and $E_0 \subseteq \mathbb{S}en_0(\Sigma)$ we have $\mathbb{S}en_0(\Sigma) \cap fo(E_0) = E_0$, where $fo(E_0)$ is the least set of Σ-sentences obtained from E_0 by applying negation, disjunction and existential quantification over the signature morphisms in $\mathcal{D}$, and
3. every signature morphism in $\mathcal{D}$ is conservative and finitary.

The second condition above means that the sentences in $\mathbb{S}en_0(\Sigma)$ are not obtained by applying Boolean connectives and quantification. We have the following consequence of the finiteness of the "atomic" sentences.

Lemma 3.1. *For any $\mathcal{D}$-extension $v : \Sigma \to \Sigma'$ as in Definition* 3.4 *we have* $Sen_0(\Sigma') = \bigcup_{i \in J} v_i(\mathbb{S}en_0(\Sigma_i))$.

Similar results as the above Lemma can be found also in [39] or [27].

If $v : \Sigma \to \Sigma'$ is a $\mathcal{D}$-extension as in Definition 3.4 then we denote by $\mathcal{L}_v$ the set of sentences $\bigcup_{i \in J} v_i(\mathbb{S}en(\Sigma_i))$. We have the following consequence of Lemma 3.1 and the finiteness of signature morphisms in $\mathcal{D}$.

Proposition 3.2. [27] *$\mathcal{L}_v$ is a $\mathcal{D}$-fragment over $\mathbb{S}en_0(\Sigma)$, where $v : \Sigma \to \Sigma'$ is a $\mathcal{D}$-extension as in Definition* 3.4.

Let $v : \Sigma \to \Sigma'$ be a conservative signature morphism, $\mathcal{M} \subseteq \mathbb{M}od(\Sigma)$ a non-empty class of models, and $\mathcal{L} \subseteq \mathbb{S}en(\Sigma')$ a $\mathcal{D}$-fragment. We define the followings:

1. $P = \{p \subseteq \mathcal{L} \text{ finite } \mid \text{there is } M' \in \mathbb{M}od(\Sigma') \text{ s.t. } M' \upharpoonright_v \in \mathcal{M} \text{ and } M' \models p\}$,
2. $\leq$ is the inclusion between sets of sentences, and
3. $f : P \to \mathcal{P}(\mathbb{S}en_0(\Sigma))$ by $f(p) = p \cap \mathbb{S}en_0(\Sigma')$ for all $p \in P$.

Proposition 3.3. *$\mathbb{P}(v, \mathcal{M}, \mathcal{L}) = \langle P, \leq, f\rangle$ is a forcing property.*

Proof. Since $\mathcal{M}$ is not empty and v is conservative, there exists a Σ'-model M' such that $M' \upharpoonright_v \in \mathcal{M}$. Since $M' \models \emptyset$, we have that $\emptyset \in P$ which implies that P has a least element.

Assume a condition $p \in P$, a set of sentences $E \subseteq f(p)$ and a sentence $e \in \mathbb{S}en_0(\Sigma)$ such that $E \models e$. There is $M' \in \mathbb{M}od(\Sigma')$ such that $M' \upharpoonright_v \in \mathcal{M}$ and $M' \models p$. We have $M' \models E$ which implies $M' \models e$. Hence, $p \cup \{e\} \in P$ and $e \in f(p \cup \{e\})$. □

In [32] the conditions of P are called *finite pieces of* $\mathcal{M}$.

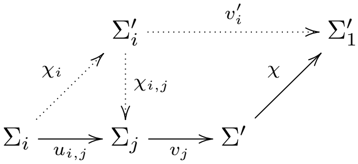
Lemma 3.4. *Let $v : \Sigma \to \Sigma'$ be a $\mathcal{D}$-extension as in Definition* 3.4*, $\mathcal{M}$ a non-empty class of Σ-models, and $\mathcal{L} \subseteq \mathcal{L}_v$ a $\mathcal{D}$-fragment. $\mathbb{P}(v, \mathcal{M}, \mathcal{L})$ has the following properties:*

1. *if $p \in P$ and $\bigvee E \in p$ then $p \cup \{e\} \in P$ for some $e \in E$.*
2. *if $p \in P$ and $(\exists \chi)e \in p$, where $\chi : \Sigma' \to \Sigma'_1$, there exists a substitution $\theta : \chi \to 1_{\Sigma'}$ such that $p \cup \{\theta(e)\} \in P$.*

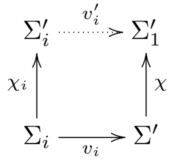
Proof. We proceed as follows:

1. Assume $\bigvee E \in p$, where p is a condition in P. There is a Σ'-model M' such that $M' \upharpoonright_v \in \mathcal{M}$ and $M' \models p$. Since $M \models \bigvee E$, $M' \models e$ for some $e \in E$. We have $M' \models p \cup \{e\}$, and we get $p \cup \{e\} \in P$.
2. Assume $(\exists \chi)e \in p$, where $p \in P$. There is a Σ'-model M' such that $M' \upharpoonright_v \in \mathcal{M}$ and $M' \models p$. Since $p \subseteq \bigcup_{i \in J} v_i(\mathbb{S}en(\Sigma_i))$ is finite there exists $p_i \subseteq \mathbb{S}en(\Sigma_i)$, where $i \in J$, such that $v_i(p_i) = p$. We have $v_i((\exists \chi_i)e_i) = (\exists \chi)e$ for some $(\exists \chi_i)e_i \in p_i$, where $\chi_i : \Sigma_i \to \Sigma'_i$. By our assumptions, there exists a signature morphism $v'_i : \Sigma'_i \to \Sigma'_1$ such that
$$\begin{array}{ccc} \Sigma'_i & \xrightarrow{v'_i} & \Sigma'_1 \\ {\scriptstyle \chi_i}\uparrow & & \uparrow{\scriptstyle \chi} \\ \Sigma_i & \xrightarrow[v_i]{} & \Sigma' \end{array}$$
is a pushout and $e = v'_i(e_i)$. By Definition 3.4, there exists $(i \leq j) \in (J, \leq)$ and a substitution $\chi_{i,j} : \chi_i \to u_{i,j}$ with

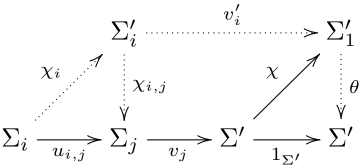
$\chi_{i,j} : \Sigma'_i \to \Sigma_j$ conservative.



Since



is a pushout and $\chi_i; (\chi_{i,j}; v_j) = v_i; 1_{\Sigma'}$, there exists $\theta : \Sigma'_1 \to \Sigma'$ such that $v'_i; \theta = (\chi_{i,j}; v_j)$ and $\chi; \theta = 1_{\Sigma'}$.



From $M' \models_{\Sigma'} p$ and $p = v_i(p_i)$, by the satisfaction condition, $M' \upharpoonright_{v_i} \models_{\Sigma_i} p_i$. Since $(\exists \chi_i) e_i \in p_i$, we have $M' \upharpoonright_{v_i} \models_{\Sigma_i} (\exists \chi_i) e_i$. There exists a χ_i-expansion M'_i of $M' \upharpoonright_{v_i}$ such that $M'_i \models_{\Sigma'_i} e_i$. Since $\chi_{i,j}$ is conservative, there exists a $\chi_{i,j}$-expansion M_j of M'_i, and by the satisfaction condition, $M_j \models_{\Sigma_j} \chi_{i,j}(\chi_i(p_i) \cup \{e_i\})$. Since v_j is conservative, there exists a v_j-expansion M'' of M_j, and by satisfaction condition, $M'' \models_{\Sigma'} (\chi_{i,j}; v_j)(\chi_i(p_i) \cup \{e_i\}) = p \cup \{\theta(e)\}$. Note that $M'' \upharpoonright_{v_i} = M' \upharpoonright_{v_i}$ which implies $M'' \upharpoonright_v = (M'' \upharpoonright_{v_i}) \upharpoonright_{u_i} = (M' \upharpoonright_{v_i}) \upharpoonright_{u_i} = M' \upharpoonright_v \in \mathcal{M}$. Hence, $M'' \models_{\Sigma'} p \cup \{\theta(e)\}$ implies $p \cup \{\theta(e)\} \in P$.

□

Proposition 3.5. *Let $v : \Sigma \to \Sigma'$ be a $\mathcal{D}$-extension as in Definition* 3.4, *$\mathcal{M}$ a non-empty class of Σ-models, and $\mathcal{L} \subseteq \mathcal{L}_v$ a $\mathcal{D}$-fragment. For every sentence $e \in \mathcal{L}$ and each condition $p \in P$ we have*

$$\textit{there exists } q \geq p \textit{ such that } q \Vdash e \textit{ iff } p \cup \{e\} \in P$$

Proof. We proceed by induction on the structure of the sentence e.

For $e \in \mathbb{S}en_0(\Sigma')$: If there is $q \geq p$ such that $q \Vdash e$ then $e \in q$. We have $p \cup \{e\} \subseteq q$ and by the definition of $\mathbb{P}(v, \mathcal{M}, \mathcal{L})$ we obtain $p \cup \{e\} \in P$. For the converse implication take $q = p \cup \{e\}$.

For $\neg e$: By the induction hypothesis applied to e, for all $q \in P$ we have

$$\text{for every } r \geq q, r \nVdash e \iff q \cup \{e\} \notin P$$

which implies that for all $q \in P$ we have

$$q \Vdash \neg e \iff q \cup \{e\} \notin P$$

We need to prove

$$\text{there exists } q \geq p \text{ such that } q \cup \{e\} \notin P \iff p \cup \{\neg e\} \in P$$

Assume that there is $q \geq p$ such that $q \cup \{e\} \notin P$. There exists a Σ'-model M' such that $M' \upharpoonright_v \in \mathcal{M}$ and $M' \models_{\Sigma'} q$ and since $q \cup \{e\} \notin P$ we have $M' \models_{\Sigma'} \neg e$. We obtain $M' \models_{\Sigma'} q \cup \{\neg e\}$ and in particular $M' \models_{\Sigma'} p \cup \{\neg e\}$ meaning that $p \cup \{\neg e\} \in P$. For the converse implication, take $q = p \cup \{\neg e\}$.

For $\bigvee E$*:* If there is $q \geq p$ such that $q \Vdash \bigvee E$, then there is $e \in E$ such that $q \Vdash e$. By the induction hypothesis, $p \cup \{e\} \in P$. There exists a Σ'-model M' such that $M' \upharpoonright_v \in \mathcal{M}$ and$M' \models_{\Sigma'} p \cup \{e\}$. We obtain $M' \models_{\Sigma'} p \cup \{\bigvee E\}$ meaning that $p \cup \{\bigvee E\} \in P$.

For the converse implication assume that $p \cup \{\bigvee E\} \in P$. By Lemma 3.4 (1) there is $e \in E$ such that $p \cup \{\bigvee E, e\} \in P$. By induction hypothesis applied to e we have $q \Vdash e$ for some $q \geq p \cup \{\bigvee E\}$. Hence there exists $q \geq p$ such that $q \Vdash \bigvee E$.

For $(\exists \chi)e$*:* Assume that there is $q \geq p$ such that $q \Vdash (\exists \chi)e$. By the definition of forcing relation there exists a substitution $\chi' : \chi \to 1_{\Sigma'}$ such that $q \Vdash \chi'(e)$. By induction $p \cup \{\chi'(e)\} \in P$. There exists a Σ'-model M' such that $M' \upharpoonright_v \in \mathcal{M}$ and $M' \models_{\Sigma'} p \cup \{\chi'(e)\}$. We obtain $M' \models_{\Sigma'} p \cup \{(\exists \chi)e\}$ meaning that $p \cup \{(\exists \chi)e\} \in P$.

For the converse implication assume that $p \cup \{(\exists \chi)e\} \in P$ where $\chi : \Sigma \to \Sigma'$. By Lemma 3.4 (2), there exists a substitution $\chi' : \chi \to 1_\Sigma$ such that $p \cup \{(\exists \chi)e, \chi'(e)\} \in P$. By the induction hypothesis applied to $\chi'(e)$, there exists $q \geq p \cup \{(\exists \chi)e\}$ such that $q \Vdash \chi'(e)$. Therefore, by the definition of forcing relation $q \Vdash (\exists \chi)e$. □

Corollary 3.6. *For each forcing property* $\mathbb{P}(v, \mathcal{M}, \mathcal{L})$*, where* v*,* $\mathcal{M}$ *and* $\mathcal{L}$ *are as in Proposition* 3.5*, we have:*

1. *For each condition* $p \in P$*, any generic model* M *for* p *satisfies* p*, and*
2. *If* $\mathcal{M} = \mathbb{M}od(\Sigma, \Gamma)$ *and* $v(\Gamma) \subseteq \mathcal{L}$ *then every generic model satisfies* $v(\Gamma)$*.*

Proof. We proceed as follows:

1. Let $G \subseteq P$ be a generic set such that $p \in G$ and M is a model for G. We prove that $M \models e$ for all $e \in p$.

   Let e be an arbitrary sentence in p. Since $G \subseteq P$ is a generic set there exists $q \in G$ such that either $q \Vdash e$ or $q \Vdash \neg e$. Suppose that $q \Vdash \neg e$ then there is $r \in G$ such that $r \geq p$ and $r \geq q$. By Lemma 2.1 (2), $r \Vdash \neg e$. By Proposition 3.5, since $e \in r$ there exists $s \geq r$ such that $s \Vdash e$. Using Lemma 2.1 (2) again, we get $s \Vdash \neg e$, which is a contradiction. Therefore $q \Vdash e$ and since M is a model for G we have that $M \models e$.
2. Let M be a generic model for G and $e \in \Gamma$. Since $v(e) \in v(\Gamma) \subseteq \mathcal{L}$ and G is generic, there exists $p \in G$ such that either $p \Vdash v(e)$ or $p \Vdash \neg v(e)$. Assuming that $p \Vdash \neg v(e)$ since $p \cup \{v(e)\} \in P$, by Proposition 3.5, there exists $q \geq p$ such that $q \Vdash v(e)$. By Lemma 2.1 (2), $q \Vdash \neg v(e)$, which is

a contradiction. Therefore, $p \Vdash v(e)$, which implies $M \models_{\Sigma'} v(e)$. Since e was arbitrary, we get $M \models_{\Sigma'} v(\Gamma)$.

□

3.3. Main results

Based on the forcing property that we have defined we obtain two main results: Downward Löwenheim-Skolem Theorem and Omitting Types Theorem.

Theorem 3.7 (Downward Löwenheim-Skolem Theorem). *Consider a $\mathcal{D}$-first-order institution $\mathcal{I} = (\mathbb{S}ig, Sen, \mathbb{M}od, \models)$ with $\mathbb{S}en_0$ the sub-functor of $\mathbb{S}en$ such that all conditions in Framework 2 are fulfilled:*

1. *the sentences in $\mathcal{I}_0$ are finitary,*
2. *for all $\Sigma \in |\mathbb{S}ig|$ and $E_0 \subseteq \mathbb{S}en_0(\Sigma)$ we have $\mathbb{S}en_0(\Sigma) \cap fo(E_0) = E_0$, and*
3. *every signature morphism in $\mathcal{D}$ is conservative and finitary.*

In addition we assume that

4. *for all signature morphisms $\Sigma \xrightarrow{\chi} \Sigma' \in \mathcal{D}$, the substitutions $\theta : \chi \to 1_\Sigma$ form a countable set, and*
5. *every signature Σ has a $\mathcal{D}$-extension $v : \Sigma \to \Sigma'$ such that any set of sentences $E \subseteq \mathbb{S}en_0(\Sigma')$ is basic.*

For every countable and consistent set Γ of Σ-sentences there exists a $\mathcal{D}$-extension $v : \Sigma \to \Sigma'$ and a $\mathcal{D}$-reachable Σ'-model M' such that $M' \upharpoonright_v \models_\Sigma \Gamma$.

Proof. Let Γ be a countable and consistent set of Σ-sentences, and $v : \Sigma \to \Sigma'$ a $\mathcal{D}$-extension as in Definition 3.4 such that any set $E \subseteq \mathbb{S}en_0(\Sigma')$ is basic. We denote by $\mathcal{L}$ the $\mathcal{D}$-fragment over $v(\Gamma)$.

By condition 4, one can easily prove by induction that $\mathcal{L}$ is countable. Since Γ is consistent $\mathcal{M} = \mathbb{M}od(\Sigma, \Gamma)$ is not empty. Consider the forcing property $\mathbb{P}(v, \mathcal{M}, \mathcal{L})$ defined in the previous subsection. By Theorem 2.4, there exists a generic $\mathcal{D}$-reachable Σ'-model M' for $\emptyset$. By Corollary 3.6, $M' \models v(\Gamma)$, and by the satisfaction condition, $M' \upharpoonright_v \models \Gamma$. □

By applying Theorem 3.7 to first-order logic we obtain the traditional Löwenheim-Skolem Theorem in the many-sorted form.

Corollary 3.8. *In* **FOL** *any consistent set of sentences has a countable model.*

Proof. Let Σ be a first-order signature, and $\Gamma \subseteq \mathbb{S}en^{\mathbf{FOL}}(\Sigma)$ a consistent set of sentences. It follows that Γ is consistent also in $\mathbf{FOL}'$.

We set the parameters of Theorem 3.7 for $\mathbf{FOL}'$. Since Σ consists of a countable set of symbols, Γ is countable too. Let $\mathcal{D}$ be the subcategory of $\mathbf{FOL}'$ signature morphisms which consists of signature extensions with a finite number of variables. Note that for all signature morphisms $(S, F, P) \hookrightarrow (S, F \cup X, P) \in \mathcal{D}$, where X is a finite set of variables for (S, F, P), the substitutions $\theta : X \to T_{(S,F)}$ form a countable set. Let $\mathcal{C}$ be a set of variables for Σ, such that C_s is infinite and countable for all sorts s. The inclusion $\Sigma \hookrightarrow \Sigma(C)$ is a $\mathcal{D}$-extension. Since all sorts of $\Sigma(C)$ are inhabited, by Lemma 1.1, each set of atoms $E \subseteq \mathbb{S}en^{\mathbf{FOL}'}(\Sigma(C))$ is basic.

By Theorem 3.7, there is a $\mathcal{D}$-reachable model $M \in |\mathbb{M}od^{\mathbf{FOL}'}(\Sigma(C))|$ which satisfies Γ. By Proposition 1.3, the elements of M consist only of interpretations of terms, and since $\Sigma(C)$ is countable, $M \in |\mathbb{M}od^{\mathbf{FOL}'}(\Sigma(C))|$ is countable too. It follows that $M \in |\mathbb{M}od^{\mathbf{FOL}}(\Sigma(C))|$, and since $M \models^{\mathbf{FOL}}_{\Sigma(C)} \Gamma$, we obtain $M \upharpoonright_{\Sigma(C)} \models^{\mathbf{FOL}}_{\Sigma} \Gamma$. □

Similar corollaries as above hold also for **POA** and **OSA**. As for their infinitary variants, we obtain that any consistent and countable set of sentences has a countable model.

Theorem 3.9 (Omitting Types Theorem). *Let $\mathcal{I} = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ be a $\mathcal{D}$-first-order institution with $\mathbb{S}en_0$ the sub-functor of $\mathbb{S}en$ such that all conditions in Framework 2 are fulfilled:*

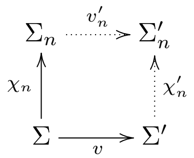
1. *the sentences in $\mathcal{I}_0$ are finitary,*
2. *for all $\Sigma \in |\mathbb{S}ig|$ and $E_0 \subseteq \mathbb{S}en_0(\Sigma)$ we have $\mathbb{S}en_0(\Sigma) \cap fo(E_0) = E_0$, and*
3. *every signature morphism in $\mathcal{D}$ is conservative and finitary.*

In addition we assume that

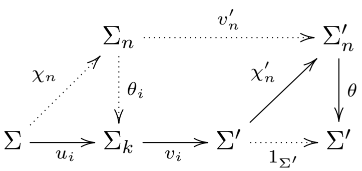
4. *for all signature morphisms $\Sigma \xrightarrow{\chi} \Sigma' \in \mathcal{D}$, the substitutions $\theta : \chi \to 1_\Sigma$ form a countable set,*
5. *$\mathcal{I}$ has existential quantification along $\mathcal{D}$-substitutions, i.e. for all $\mathcal{D}$-substitutions $\theta : (\Sigma \xrightarrow{\chi} \Sigma_1) \to (\Sigma \xrightarrow{\varphi} \Sigma_2)$ and Σ_2-sentences e_2 there exists a Σ_1-sentence e_1 semantically equivalent to $(\exists\theta)e_2$, and*
6. *every signature Σ has a $\mathcal{D}$-extension $v : \Sigma \to \Sigma'$ such that any set of sentences $E \subseteq \mathbb{S}en_0(\Sigma')$ is basic.*

The institution $\mathcal{I}$ has the $\mathcal{D}$-OTP.

Proof. Let $\Gamma \subseteq \mathbb{S}en(\Sigma)$, $\Sigma \xrightarrow{\chi_n} \Sigma_n \in \mathcal{D}$ and $\Delta_n \subseteq \mathbb{S}en(\Sigma_n)$ be as in Definition 3.3. Let $v : \Sigma \to \Sigma'$ be a $\mathcal{D}$-extension of Σ as in Definition 3.4 such that all sets $E \subseteq \mathbb{S}en_0(\Sigma')$ are basic. For all $n \in \mathbb{N}$ let



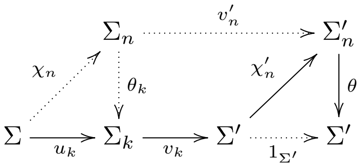
be a pushout such that $v((\exists\chi_n)\rho) = (\exists\chi'_n)v'_n(\rho)$ for all sentences $(\exists\chi_n)\rho \in \mathbb{S}en(\Sigma)$. For the sake of simplicity we will make the following notations $\Gamma' = v(\Gamma)$, $\Delta'_n = v'_n(\Delta_n)$ and $\mathcal{L}_v = \bigcup_{i\in J} v_i(\mathbb{S}en(\Sigma_i))$. Let $\mathcal{L}$ be the $\mathcal{D}$-fragment over $\Gamma' \cup (\bigcup_{n\in\mathbb{N},\theta:\chi'_n\to 1_{\Sigma'}} \theta(\Delta'_n))$. We prove that $\mathcal{L} \subseteq \mathcal{L}_v$.



Let $n \in \mathbb{N}$ and $\theta : \chi'_n \to 1_{\Sigma'}$. Since χ_n is finitary, there exists $i \in J$ and $\theta_i : \chi_n \to u_i$ such that $\theta_i ; v_i = v'_n ; \theta$. We have $\theta(\Delta'_n) = \theta(v'_n(\Delta_n)) = v_i(\theta_i(\Delta_n)) \subseteq \mathcal{L}_v$. It follows that $\Gamma' \cup (\bigcup_{n \in \mathbb{N}, \theta : \chi'_n \to 1_{\Sigma'}} \theta(\Delta'_n)) \subseteq \mathcal{L}_v$ which implies $\mathcal{L} \subseteq \mathcal{L}_v$.

Since for all $n \in \mathbb{N}$ the substitutions $\theta : \chi'_n \to 1_{\Sigma'}$ form a countable set, $\mathcal{L}$ is countable. Γ is consistent which implies that $\mathbb{M}od(\Sigma, \Gamma)$ is not empty. Consider the forcing property $\mathbb{P}(v, \mathcal{M}, \mathcal{L})$ with $\mathcal{M} = \mathbb{M}od(\Sigma, \Gamma)$.

We prove that for all natural numbers $n \in \mathbb{N}$, conditions $p \in P$, and substitutions $\theta : \chi'_n \to 1_{\Sigma'}$ there exists $\delta' \in \Delta'_n$ such that $p \cup \{\neg\theta(\delta')\} \in P$. Let $n \in \mathbb{N}$, $p \in P$ and $\theta : \chi'_n \to 1_{\Sigma'}$. Since χ_n is finitary, there exists $i \in J$ and $\theta_i : \chi_n \to u_i$ such that $\theta_i ; v_i = v'_n ; \theta$. Since $p \subseteq \mathcal{L}$ is finite and $\mathcal{L} \subseteq \mathcal{L}_v$, there exists $j \in J$ and $p_j \subseteq \mathbb{S}en(\Sigma_j)$ such that $p = v_j(p_j)$. Let $k \in J$ such that $k \geq i$ and $k \geq j$. We have $\theta_k ; v_k = v'_n ; \theta$ and $p = v_k(p_k)$, where $\theta_k = \theta_i ; u_{i,k}$ and $p_k = u_{j,k}(p_j)$.



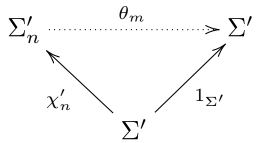
By the definition of $\mathbb{P}(v, \mathcal{M}, \mathcal{L})$, the set $\Gamma' \cup p$ is consistent. By the satisfaction condition, $u_k(\Gamma) \cup p_k$ is consistent. By our hypothesis, there is a Σ_n-sentence ρ_n semantically equivalent with $(\exists \theta_k) \bigwedge p_k$. It follows that $\chi_n(\Gamma) \cup \{\rho_n\}$ is consistent. Since Γ locally χ_n-omits Δ_n and $\chi_n(\Gamma) \cup \{\rho_n\}$ is consistent, there exists $\delta \in \Delta_n$ such that $\chi_n(\Gamma) \cup \{\rho_n\} \not\models_{\Sigma_n} \delta$ which implies that $\chi_n(\Gamma) \cup \{\rho_n, \neg\delta\}$ is consistent. It follows that $u_k(\Gamma) \cup p_k \cup \{\neg\theta_k(\delta)\}$ is consistent. Since v_k is conservative, $\Gamma' = v_k(u_k(\Gamma))$, $v_k(p_k) = p$ and $v_k(\neg\theta_k(\delta)) = \neg\theta(v'_n(\delta))$, the set $\Gamma' \cup p \cup \{\neg\theta(v'_n(\delta))\}$ is consistent. Note that $\delta' = v'_n(\delta) \in \Delta'_n$ and the set $\Gamma' \cup p \cup \{\neg\theta(\delta')\}$ is consistent.

We construct a generic set G such that all generic $\mathcal{D}$-reachable models for G will χ'_n-omit Δ'_n, for all $n \in \mathbb{N}$. Note that the substitutions $\theta : \chi'_n \to 1_{\Sigma'}$, where $n \in \mathbb{N}$, form a countable set. Let $\{\theta_m \mid m \in \mathbb{N}\}$ be an enumeration of all such substitutions. Since the fragment $\mathcal{L}$ is also countable let $\{e_m \mid m \in \mathbb{N}\}$ be an enumeration of $\mathcal{L}$. We form an increasing chain of conditions $p_0 \leq p_1 \ldots \leq p_m \ldots$ such that for all $m \in \mathbb{N}$

1. $p_{m+1} \Vdash e_m$ or $p_{m+1} \Vdash \neg e_m$, and
2. there exists $\delta' \in \Delta'_n$ such that $\neg\theta_m(\delta') \in p_{m+1}$, where $\theta_m : \chi'_n \to 1_{\Sigma'}$.

Let $p_0 = \emptyset$ and assuming that we already have the condition p_m we construct p_{m+1} as follows: if $p_m \Vdash \neg e_m$ then take $q = p_m$ else take $q \geq p_m$ such that $q \Vdash e_m$; assuming that $\theta_m : \chi'_n \to 1_{\Sigma'}$, by the first part of the proof, there exists $\delta' \in \Delta'_n$ such that $q \cup \{\neg\theta_m(\delta')\} \in P$; take $p_{m+1} = q \cup \{\neg\theta_m(\delta')\}$. The set $G = \{q \mid q \leq p_m$ for some $m \in \mathbb{N}\}$ is generic. Let M be a $\mathcal{D}$-reachable model for G. We show that M χ'_n-omits Δ'_n for all $n \in \mathbb{N}$. Let $n \in \mathbb{N}$ and M_n a χ'_n-expansion of M. Since M is $\mathcal{D}$-reachable there exists a substitution

$\theta_m : \chi'_n \to 1_{\Sigma'}$ such that $M \upharpoonright_{\theta_m} = M_n$.



By the definition of G there exists $\delta' \in \Delta'_n$ such that $\neg\theta_m(\delta') \in p_{m+1}$. Since M is a generic model for p_{m+1}, by Corollary 3.6, $M \models_{\Sigma'} p_{m+1}$. We have $M \models_{\Sigma'} \neg\theta_m(\delta')$. Note that $\neg\theta_m(\delta') = \theta_m(\neg\delta')$, and by the satisfaction condition, $M_n \models_{\Sigma'_n} \neg\delta'$.

Finally, by Proposition 2.3 there is a generic $\mathcal{D}$-reachable model M for G and by the satisfaction condition $(M \upharpoonright_v)$ χ_n-omits Δ_n for all $n \in \mathbb{N}$. □

In the following we discuss the applicability of Theorem 3.9 by making an analysis of its underlying conditions in $\mathbf{FOL}'$.

Condition 4. We assumed that the symbols of any signature form a countable set.

Condition 5. Let $\Sigma \hookrightarrow \Sigma(X)$ and $\Sigma \hookrightarrow \Sigma(Y)$ be two signature inclusions, where $\Sigma = (S, F, P)$, and X, Y are finite sets of variables for Σ. Let $\theta : X \to T_{(S,F)}(Y)$ be a substitution, and e_2 a $\Sigma(Y)$-sentence. Assume a set of variables Z for $\Sigma(X)$ such that there exists a bijection $i : Y \to Z$. Then $(\exists\theta)e_2$ is semantically equivalent with the sentence $(\exists Z)(i(e_2) \wedge (\bigwedge_{x\in X} x = i(\theta(x))))$.

Condition 6. For all $\mathcal{D}$-extensions $\Sigma \hookrightarrow \Sigma(C)$, where $\Sigma = (S, F, P)$, and sorts $s \in S$, the set $(T_{(S,F\cup C)})_s$ is not empty. By Lemma 1.1, every set of atoms $E \subseteq \mathbb{S}en^{\mathbf{FOL}}(\Sigma(C))$ is basic.

Corollary 3.10. **FOL**, **POA**, **OSA** *and their infinitary variants* $\mathbf{FOL}_{\omega_1,\omega}$, $\mathbf{POA}_{\omega_1,\omega}$ *and* $\mathbf{OSA}_{\omega_1,\omega}$ *have the $\mathcal{D}$-OTP, where $\mathcal{D}$ consists of signature extensions with a finite number of variables.*

Proof. By the above discussion, $\mathbf{FOL}'$ falls into the framework of Theorem 3.9. Hence, $\mathbf{FOL}'$ has the $\mathcal{D}$-OTP which implies that **FOL** has the $\mathcal{D}$-OTP as well. Similarly, one can prove that **POA**, **OSA** and their infinitary variants have the $\mathcal{D}$-OTP. □

The partial algebras defining the sets of ground existential atoms as basic sets of sentences are not $\mathcal{D}$-reachable in the sense of Definition 1.6. In **HNK** not all sets of atoms are basic. Hence, Theorem 3.9 cannot be applied to partial algebra or higher-order logic.

4. Borrowing Omitting Types

In this section we borrow the OTP along institution mappings for more expressive logical systems which are encoded into the institutions of presentations of less refined institutions. Therefore we also need to lift the OTP from a base institution to the institution of its (countable) presentations.

The institution mappings used here for borrowing results is that of institution comorphisms [26] previously known as *plain maps* [34] or *representations* [46].

Definition 4.1. Let $\mathcal{I}' = (\mathbb{S}ig', \mathbb{S}en', \mathbb{M}od', \models')$ and $\mathcal{I} = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ be two institutions. An *institution comorphism* $(\phi, \alpha, \beta) : \mathcal{I} \to \mathcal{I}'$ consists of

- a functor $\phi : \mathbb{S}ig \to \mathbb{S}ig'$, and
- two natural transformations $\alpha : \mathbb{S}en \Rightarrow \phi; \mathbb{S}en'$ and $\beta : \phi^{op}; \mathbb{M}od' \Rightarrow \mathbb{M}od$ such that the following satisfaction condition for institution comorphisms holds:

$$M' \models'_{\phi(\Sigma)} \alpha_\Sigma(e) \text{ iff } \beta_\Sigma(M') \models_\Sigma e$$

  for every signature $\Sigma \in |\mathbb{S}ig|$, each $\phi(\Sigma)$-model M', and any Σ-sentence e.

Let $\Sigma \in |\mathbb{S}ig|$. We say that β_Σ is *conservative* if for all Σ-models M there exists a $\phi(\Sigma)$-model M' such that $\beta_\Sigma(M') = M$.

4.1. Main results

We present a general result of borrowing the OTP along an institution comorphism.

Theorem 4.1. *Let $(\phi, \alpha, \beta) : \mathcal{I} \to \mathcal{I}'$ be an institution comorphism as in Definition* 4.1 *such that*

- *$\mathcal{I}$ and $\mathcal{I}'$ have negations and $\alpha_\Sigma(\neg e) = \neg\alpha_\Sigma(e)$ for all Σ-sentences e, and*
- *for all signatures $\Sigma \in |\mathbb{S}ig|$*
  - *α_Σ is surjective modulo $\models\!\mid$ [4], i.e. for all $\phi(\Sigma)$-sentences ρ' there exists a Σ-sentence ρ such that $\alpha_\Sigma(\rho) \models\!\mid \rho'$, and*
  - *β_Σ is conservative.*

Then $\mathcal{I}$ has the $\mathcal{D}$-OTP if $\mathcal{I}'$ has the $\mathcal{D}'$-OTP for some broad subcategory $\mathcal{D}' \subseteq \mathbb{S}ig'$ of signature morphisms such that $\phi(\mathcal{D}) \subseteq \mathcal{D}'$.

Proof. Assume $\Gamma \subseteq \mathbb{S}en(\Sigma)$ locally χ_n-omits Δ_n for all $n \in \mathbb{N}$, where $(\Sigma \overset{\chi_n}{\to} \Sigma_n \in \mathcal{D})_{n\in\mathbb{N}}$ and $(\Delta_n \subseteq \mathbb{S}en(\Sigma_n))_{n\in\mathbb{N}}$.

We show that Γ' locally χ'_n-omits Δ'_n, where $\Gamma' = \alpha_\Sigma(\Gamma)$, $\chi'_n = \phi(\chi_n)$ and $\Delta'_n = \alpha_{\Sigma_n}(\Delta_n)$. Let $p' \subseteq \mathbb{S}en'(\phi(\Sigma_n))$ finite such that $\chi'_n(\Gamma') \cup p'$ is consistent'. Since α_{Σ_n} is surjective modulo $\models\!\mid$ there is $p \subseteq \mathbb{S}en(\Sigma_n)$ finite such that $\alpha_{\Sigma_n}(p) \models\!\mid p'$. It follows that $\chi_n(\Gamma) \cup p$ is consistent too. Since Γ locally χ_n-omits Δ_n, there is $\delta \in \Delta_n$ such that $\chi_n(\Gamma) \cup p \cup \{\neg\delta\}$ is consistent. Since β_{Σ_n} is conservative, $\chi'_n(\Gamma') \cup \alpha_{\Sigma_n}(p) \cup \{\neg\alpha_{\Sigma_n}(\delta)\}$ is consistent which implies $\chi'_n(\Gamma') \cup p' \cup \{\neg\delta'\}$ is consistent, where $\delta' = \alpha_{\Sigma_n}(\delta) \in \Delta'_n$.

$\phi(\mathcal{D}) \subseteq \mathcal{D}'$ implies $\chi'_n \in \mathcal{D}'$. Since $\mathcal{I}'$ has $\mathcal{D}'$-OTP, there is a $\phi(\Sigma)$-model M' of Γ' which χ'_n-omits Δ'_n, for all $n \in \mathbb{N}$. This implies that $\beta_\Sigma(M')$ satisfies Γ and χ_n-omits Δ_n, for all $n \in \mathbb{N}$. □

[4] In any institution $\mathcal{I} = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$, $E \models\!\mid E'$ when $E \models E'$ and $E' \models E$, for all sets of Σ-sentences E and E', where $\Sigma \in |\mathbb{S}ig|$.

We lift the OTP from a base institution to the institution of its countable presentations.

Proposition 4.2 (Lifting OTP to Presentations). *For any institution $\mathcal{I}$ which has $\mathcal{D}$-OTP, the institution $\mathcal{I}^{cpres}$ of countable presentations over $\mathcal{I}$ has $\mathcal{D}^{cpres}$-OTP, where $\mathcal{D}^{cpres}$ consists of presentation morphisms of the form $\chi : (\Sigma, E) \to (\Sigma', E')$ such that $\chi \in \mathcal{D}$ and $E' \models\!\!\mid \chi(E)$.*

Proof. For every signature morphism $(\Sigma, E) \xrightarrow{\chi} (\Sigma', E') \in \mathcal{D}^{cpres}$, each countable set of Σ-sentences Γ, and any countable set Δ of Σ'-sentences, the following are equivalent:

1. In $\mathcal{I}^{cpres}$, Γ locally χ-omits Δ, where $\chi : (\Sigma, E) \to (\Sigma', E')$.
2. In $\mathcal{I}$, $\Gamma \cup E$ locally χ-omits Δ, where $\chi : \Sigma \to \Sigma'$.

Let $\Gamma \subseteq \mathbb{S}en(\Sigma)$ countable, $((\Sigma, E) \xrightarrow{\chi_n} (\Sigma_n, E_n) \in \mathcal{D}^{cpres})_{n\in\mathbb{N}}$ and $(\Delta_n \subseteq \mathbb{S}en(\Sigma_n))_{n\in\mathbb{N}}$ such that Δ_n is countable. Assume Γ χ_n-omits Δ_n in $\mathcal{I}^{cpres}$. It follows that $\Gamma \cup E$ χ_n-omits Δ_n in $\mathcal{I}$. Since $\mathcal{I}$ has $\mathcal{D}$-OTP, there exists a Σ-model M of $\Gamma \cup E$ such that M χ_n-omits Δ_n, where $\chi_n : \Sigma \to \Sigma_n$ is regarded as a signature morphism in $\mathcal{D}$. Then in $\mathcal{I}^{cpres}$, M is a (Σ, E)-model of Γ which χ_n-omits Δ_n, where $\chi_n : (\Sigma, E) \to (\Sigma_n, E_n)$ is regarded as a signature morphism in $\mathcal{D}^{cpres}$. □

4.2. Omitting Types in PA

In order to establish that **PA** has $\mathcal{D}$-OTP, where $\mathcal{D}$ consists of signature extensions with finite number of total variables, we need to set the parameters of Theorem 4.1. We recall the definition of a comorphism $(\phi, \alpha, \beta) : \mathbf{PA} \to \mathbf{FOL}^{cpres}$ which can be found, for example, in [40] or [37].

- Each **PA**-signature (S, TF, PF) is mapped to the **FOL** presentation $((S, TF, \overline{PF}), E_{(S,TF,PF)})$, where $\overline{PF}_{ws} = PF_{w\to s}$ for all $w \in S^*$ and $s \in S$, and $E_{(S,TF,PF)} = \{(\forall X \uplus \{y, z\})\sigma(X, y) \wedge \sigma(X, z) \Rightarrow (y = z) \mid \sigma \in PF\}$.
- For all **PA**-signatures (S, TF, PF)
  - $\alpha_{(S,TF,PF)}(t \stackrel{e}{=} t') = (\exists X \uplus \{x_0\})bind(t, x_0) \wedge bind(t', x_0)$, where for each term $t \in T_{(S,TF\cup PF)}$ and variable x, $bind(t, x)$ is a finite conjunction of atoms defined as follows:

$$bind(\sigma(t_1, \ldots, t_n), x) = \bigwedge_{1\leq i\leq n} bind(t_i, x_i) \wedge \begin{cases} \sigma(x_1, \ldots, x_n) = x & \text{if } \sigma \in TF \\ \sigma(x_1, \ldots, x_n, x) & \text{if } \sigma \in PF \end{cases}$$

    where X is the set of new constant symbols introduced by $bind(t, x_0)$ and $bind(t', x_0)$.
  - commutes with the first-order constructors on sentences.
- For each $((S, TF, \overline{PF}), E_{(S,TF,PF)})$-model M, the algebra $\beta_{(S,TF,PF)}(M)$ is defined as follows:
  - $\beta_{(S,TF,PF)}(M)_x = M_x$ for all $x \in S$ or $x \in TF$,
  - $\beta_{(S,TF,PF)}(M)_\sigma(m) = n$ for all $\sigma \in PF$ such that $(m, n) \in M_\sigma$.

Corollary 4.3. **PA** *has the $\mathcal{D}$-OTP, where $\mathcal{D}$ consists of signature extensions with finite number of total variables.*

Proof. By Proposition 4.2, we lift the OTP from **FOL** to $\mathbf{FOL}^{cpres}$. Then we apply Theorem 4.1 to the above comorphism. Note that for all **PA**-signatures (S, TF, PF) we have

- $\beta_{(S,TF,PF)}$ is conservative because it is an isomorphism, and
- $\alpha_{(S,TF,PF)}$ is surjective modulo $\models\!\!\mid$ because it is surjective modulo $\models\!\!\mid$ on atoms and it commutes with the first-order constructors on sentences.

Therefore the conditions of Theorem 4.1 are fulfilled and we infer that **PA** has the OTP. □

Similarly one can define a comorphism $\mathbf{PA}_{\omega_1,\omega} \to \mathbf{FOL}^{cpres}_{\omega_1,\omega}$ and establish in a similar manner as above that $\mathbf{PA}_{\omega_1,\omega}$ has the OTP.

4.3. Omitting Types in HNK

We borrow the OTP for **HNK** along a comorphism $(\phi, \alpha, \beta) : \mathbf{HNK} \to \mathbf{FOEQL}^{cpres}$ using the ideas from [36].

- Each **HNK**-signature (S, F) is mapped to the presentation $((\overrightarrow{S}, \overrightarrow{F}), E_{(S,F)})$ where
  - $\overrightarrow{S}$ is the set of all types over S,
  - $\overrightarrow{F}_s = F_s$ for each $s \in \overrightarrow{S}$, $\overrightarrow{F}_{[(s\to s')s]\to s'} = \{app_{s,s'}\}$ for all $s, s' \in \overrightarrow{S}$ and $\overrightarrow{F}_{w\to s} = \emptyset$ otherwise.
  - $E_{(S,F)} = \{(\forall f, g)((\forall x)app_{s,s'}(f, x) = app_{s,s'}(g, x)) \Rightarrow (f = g) \mid s, s' \in \overrightarrow{S}\}$
- α is defined as the canonical extension of the mapping on the terms α^{term} defined by $\alpha^{term}(tt') = apply(\alpha^{term}(t), \alpha^{term}(t'))$.
- $\beta_{(S,F)}(M) = \overline{M}$, where $\overline{M}_s = \{\overline{m} \mid m \in M_s\}$ for all types $s \in \overrightarrow{S}$, such that
  - for each $s \in S$ and $m \in M_s$, we have $\overline{m} = m$, and
  - for each $s \to s' \in \overrightarrow{S}$ and $m \in M_{s\to s'}$, the function $\overline{m} : \overline{M}_s \to \overline{M}_{s'}$ is defined as follows: $\overline{m}(\overline{x}) = \overline{M_{apply_{s,s'}}(m, x)}$, for all $x \in M_s$.

The reader may complete the details of this definition (such as the definitions of ϕ on the signature morphisms and of the $\beta_{(S,F)}$ on the model morphisms) by herself/himself or may look into [9]. Note that $\beta_{(S,F)}$ is an isomorphism of categories for all higher-order signatures (S, F). Now we have established all the necessary conditions for the application of Theorem 4.1 to $(\phi, \alpha, \beta) : \mathbf{HNK} \to \mathbf{FOEQL}^{cpres}$.

Corollary 4.4. HNK *has the $\mathcal{D}$-OTP, where $\mathcal{D}$ consists of signature extensions with finite numbers of variables of any type.*

5. Conclusions and Future Work

We have lifted the OTP from the conventional model theory to the institution independent framework and we have developed two ways of obtaining the OTP

1. within an arbitrary institution by using the forcing technique introduced in [27]; as instances of our abstract results we have obtained the OTP for first-order logic, preorder algebra, order-sorted algebra, and also their infinitary variants.
2. by transporting it (backwards) along the institution comorphisms; we have illustrated the applicability power of our method by deriving the OTP to partial algebra and higher-order logic with Henkin semantics; as in the previous case, the abstract results can be applied to infinitary variants of the institutions we have just mentioned.

Our work is justified by the institution-independent status of the results, and the multitude of instances of the abstract theorems. Due to the use of forcing, our work covers uniformly both finitary and infinitary case. We obtained also an abstract version of the famous Downward Löwenheim-Skolem Theorem. The interested reader may complete the details for borrowing this result along institution comorphisms to partial algebra and higher-order logic with Henkin semantics.

In the future we are planning to apply our results to other logics such as institutions with predefined types [13]. We expect our methods to be applicable to most of the multitude of combinations between the logics discussed here, such as order-sorted algebra with transitions. An interesting topic for the future work would be forcing and OTP for modal logics.

6. Exiled proofs

Section 2:

Lemma 2.1. 1. $p \Vdash^w e$ iff $p \Vdash \neg\neg e$ iff for each $q \geq p$, $q \nVdash \neg e$ iff for each $q \geq p$, there exists $r \geq q$ such that $r \Vdash e$.

2. By induction on e.

   For $e \in \mathbb{S}en_0(\Sigma)$*:* The conclusion follows from $f(p) \subseteq f(q)$.

   For $\neg e \in \mathcal{L}$*:* We have $p \Vdash \neg e$. Suppose towards a contradiction $q \nVdash \neg e$, then by definition of forcing there is $q' \geq q$ such that $q' \Vdash e$. Therefore there is $q' \geq p$ such that $q' \Vdash e$, thus $p \nVdash \neg e$, which is a contradiction.

   For $\bigvee E \in \mathcal{L}$*:* $p \Vdash e$ for some $e \in E$. By induction $q \Vdash e$ which implies $q \Vdash \bigvee E$.

   For $(\exists\chi)e \in \mathcal{L}$*:* Since $p \Vdash (\exists\chi)e$ then $p \Vdash \theta(e)$ for some substitution $\theta : \chi \to 1_\Sigma$. By induction $q \Vdash \theta(e)$, and by the definition of forcing relation $q \Vdash (\exists\chi)e$.
3. It follows easily from 1 and 2.
4. Obvious.

□

Lemma 2.2. The proof of this lemma is similar to the one in [32]. Since $\mathcal{L}$ is countable let $\{e_n \mid n < \omega\}$ be an enumeration of $\mathcal{L}$. We form a chain of conditions $p_0 \leq p_1 \leq \ldots$ in P as follows. Let $p_0 = p$. If $p_n \Vdash \neg e_n$, let $p_{n+1} = p_n$, otherwise choose $p_{n+1} \geq p_n$ such that $p_{n+1} \Vdash e_n$. The set $G = \{q \in P \mid q \leq p_n \text{ for some } n < \omega\}$ is generic and contains p. $\square$

Proposition 2.3. Let T be the set of all sentences of $\mathcal{L}$ which are forced by G. Let $B = \mathbb{S}en_0(\Sigma) \cap T$. We prove by induction on $e \in \mathcal{L}$ that $M_B \models e$ iff $e \in T$ by induction on e.

For $e \in \mathbb{S}en_0(\Sigma)$: Suppose $M_B \models e$ then we have $B \models e$ and by the hypothesis there is $B' \subseteq B$ finite such that $B' \models e$. Since G is generic there exists $p \in G$ such that $B' \subseteq f(p)$. Suppose towards a contradiction that $e \notin T$ which because G is generic leads to $\neg e \in T$. Then there is $q \in G$ such that $q \Vdash \neg e$. Since G is generic there is $r \in G$ such that $r \geq p$ and $r \geq q$. We have $B' \subseteq f(r)$ and using Lemma 2.1(2) we obtain $r \Vdash \neg e$. Since $B' \subseteq f(r)$ and $B' \models e$, there exists $s \geq r$ such that $s \Vdash e$. By Lemma 2.1(2), $s \Vdash \neg e$ which is a contradiction. If $e \in T$ then $e \in B$ and $M_B \models e$.

For $\neg e \in \mathcal{L}$: Exactly one of e and $\neg e$ is in T. Since G is generic there is $p \in G$ such that either $p \Vdash e$ or $p \Vdash \neg e$. Therefore $e \in T$ or $\neg e \in T$. Suppose towards a contradiction that $\{e, \neg e\} \subseteq T$, then there exists $p, q \in G$ such that $p \Vdash e$ and $q \Vdash \neg e$. By the definition of generic sets there is $r \in G$ such that $r \geq p$ and $r \geq q$. By Lemma 1(2) $r \Vdash e$ and $r \Vdash \neg e$ which is a contradiction.

Let $\neg e \in T$. Suppose that $M_B \models e$, then by induction we have $e \in T$, which is a contradiction. Therefore $M_B \models \neg e$. Now if $M_B \models \neg e$, then $e \notin T$. Hence, $\neg e \in T$.

For $\bigvee E \in \mathcal{L}$: If $M_B \models \bigvee E$ then $M_B \models e$ for some $e \in E$. By induction hypothesis, $e \in T$. We have $p \Vdash e$ for some $p \in G$ and we obtain $p \Vdash \bigvee E$. Thus, $\bigvee E \in T$. Now if $\bigvee E \in T$ then $e \in T$, for some $e \in E$. Therefore, by induction, $M_B \models e$ and thus $M_B \models \bigvee E$.

For $(\exists \chi)e \in \mathcal{L}$: Assume that $M_B \models (\exists \chi)e$ where $\chi : \Sigma \to \Sigma'$. There exists a χ-expansion N of M_B such that $N \models e$. Because M_B is $\mathcal{D}$-reachable there exists a substitution $\theta : \chi \to 1_\Sigma$ such that $M_B \upharpoonright_\theta = N$. By the satisfaction condition $M_B = N \upharpoonright_\chi \models \theta(e)$. By induction $\theta(e) \in T$ which implies $(\exists \chi)e \in T$. For the converse implication assume that $p \Vdash (\exists \chi)e$ for some $p \in G$. We have that $p \Vdash \theta(e)$ for some substitution $\theta : \chi \to 1_\Sigma$. By induction $M_B \models \theta(e)$ which implies $M_B \upharpoonright_\theta \models e$. Since $(M_B \upharpoonright_\theta) \upharpoonright_\chi = M_B$ we obtain $M_B \models (\exists \chi)e$. $\square$

Theorem 2.3. By Lemma 2.2 there is a generic set $G \subseteq P$ such that $p \in G$ and by Proposition 2.3 there is a $\mathcal{D}$-reachable model M for G. $\square$

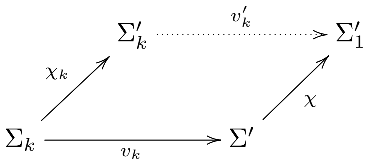
Corollary 2.5. Suppose $p \Vdash^w e$ and M is a generic model for p which is also $\mathcal{D}$-reachable. We have $p \Vdash \neg\neg e$ which implies $M \models \neg\neg e$ and $M \models e$. For the converse implication, suppose that $p \nVdash^w e$. There is $q \Vdash \neg e$ for some $q \geq p$. By Proposition 2.3 there is a generic model M for q which is also $\mathcal{D}$-reachable; this implies $M \models \neg e$. But M is also a generic model for p. $\square$

Section 3:

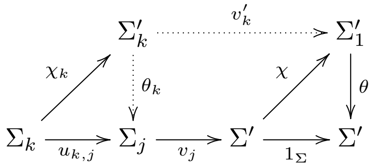
Lemma 3.1. We show $\mathbb{S}en_0(\Sigma') \subseteq \bigcup_{i \in J} v_i(\mathbb{S}en_0(\Sigma_i))$. Let $e \in \mathbb{S}en_0(\Sigma')$. Since e is finitary it can be written as $\varphi(e_f)$ where $\varphi : \Sigma_f \to \Sigma'$ is a signature morphism such that Σ_f is finitely presented in the category $\mathbb{S}ig$. By finiteness of Σ_f there exists a signature morphism $\varphi_i : \Sigma_f \to \Sigma_i$ such that $\varphi_i; v_i = \varphi$. We have that $e = v_i(\varphi_i(e_f))$. Therefore $\mathbb{S}en_0(\Sigma') = \bigcup_{i \in J} v_i(\mathbb{S}en_0(\Sigma_i))$. □

Proposition 3.2. By Lemma 3.1 we have that $\mathbb{S}en_0(\Sigma) \subseteq \mathcal{L}_v$.

The most difficult closer property to prove is the closure of $\mathcal{L}_v$ to substitutions. The remaining cases are straightforward. Let $(\exists \chi)e \in \mathcal{L}_v$, where $\chi : \Sigma' \to \Sigma'_1$, and a substitution $\theta : \chi \to 1_{\Sigma'}$. By the definition of $\mathcal{L}_v$, $(\exists \chi)e = v_k((\exists \chi_k)e_k)$ for some $(\exists \chi_k)e_k \in \mathbb{S}en(\Sigma_k)$, where $\chi_k : \Sigma_k \to \Sigma'_k$. By our assumptions, there exists $v'_k : \Sigma'_k \to \Sigma'_1$ such that



is a pushout. Since χ_k is finitary and $(u_{k,i} \xrightarrow{v_i} v_k)_{(k \leq i) \in (J, \leq)}$ is a directed colimit in the category $\Sigma_k / \mathbb{S}ig$, there exists $\theta_k : \chi_k \to u_{k,j}$, where $k \leq j$ such that $\theta_k; v_j = v'_k; \theta$.



Therefore $\theta(e) = \theta(v'_k(e_k)) = v_j(\theta_k(e_k)) \in \mathcal{L}_v$. □

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