# Forcing and Calculi for Hybrid Logics

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The definition of *institution* formalizes the intuitive notion of logic in a category-based setting. Similarly, the concept of *stratified institution* provides an abstract approach to Kripke semantics. This includes hybrid logics, a type of modal logics expressive enough to allow references to the nodes/states/worlds of the models regarded as relational structures, or multi-graphs. Applications of hybrid logics involve many areas of research such as computational linguistics, transition systems, knowledge representation, artificial intelligence, biomedical informatics, semantic networks and ontologies. The present contribution sets a unified foundation for developing formal verification methodologies to reason about Kripke structures by defining proof calculi for a multitude of hybrid logics in the framework of *stratified institutions*. In order to prove completeness, the paper introduces a *forcing* technique for *stratified institutions with nominal and frame extraction* and studies a *forcing property* based on syntactic consistency. The proof calculus is shown to be complete and the significance of the general results is exhibited on a couple of benchmark examples of hybrid logical systems.

CCS Concepts: • **Theory of computation → Logic and verification**; **Proof theory**; **Modal and temporal logics**.

Additional Key Words and Phrases: institution, stratified institution, hybrid logic, forcing, proof theory, reconfiguration

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## 1 INTRODUCTION

This section provides an overview of the significance of the present contribution and its connection to different areas of research.

*Kripke semantics and hybrid logics.* Relational structures are ubiquitous: any diagram consisting of nodes, edges and labels can be regarded as a relational structure [8]. For example, in knowledge representation formalisms, role assertions describe relationships between individuals/objects grouped into classes determined by concepts; linguistic information can be represented by multi-graphs; other mathematical entities that can be viewed as relational structures are transition systems, derivation trees, semantic networks, etc. Therefore, it is useful to think of a Kripke model as a relational structure or a multi-relational graph:

(1) a frame consisting of a set of nodes together with a family of (typed) edge sets, and
(2) a mapping from the set of nodes to a class of local models that gives meaning to the nodes.

Modal logics provide a framework to reason formally about the properties of Kripke models. Hybrid logics increase the expressive power of ordinary modal logics by adding an additional sort of symbols called *nominals* such that each nominal is true relative to exactly one point. All model-theoretic and proof-theoretic results in the present contribution rely on this distinctive feature of hybrid logics. The history of hybrid logics goes back to Arthur Prior's work [62]. Further developments can be found in works such as [2, 8, 11, 59]. The research on hybrid logics received an additional boost due to the recent interest in the logical foundations of the *reconfiguration paradigm*.

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*Hybrid logics and their applications.* The recent developments in global connectivity and the demand for more flexibility to any software application triggered a paradigm shift from human-led engineering to reconfigurable software systems, i.e. applications that work in different operation modes, often called configurations, and are equipped with mechanisms for managing the evolution of their configurations in response to internal and external stimuli [68]. Most of the software-driven tools used in medicine are inherently reconfigurable since their execution depends not only on their internal state, but also on the interactions with the patient and with the environment and the context in which they coexist. Some examples are the medical imaging machines, pill cameras, doctor-on-a-chip products, artificial pacemakers, and various kinds of infusion pumps, like the insulin pump used in the treatment of diabetes.

Reconfigurable systems can be regarded as labelled transition structures, where the nodes represent configurations and the arrows correspond to transitions from one configuration to another. This suggests a two layered approach towards the design of systems with reconfigurable features: (a) *a local view*, which amounts to finding a suitable base logic for describing and reasoning about the configurations, (b) *a global perspective*, which refers to the overall dynamics of the system and describes the transitions from one configuration to another. This viewpoint is captured quite accurately by hybrid logics. The work reported in [8] argues successfully that hybrid logics specialize to temporal logics [45], description logics [4] and feature logics [65]. Therefore, the area of applications of hybrid logics is much larger and it involves knowledge representation, computational linguistics, artificial intelligence, biomedical informatics, semantic networks and ontologies. See [8] for more information on this topic.

*Institutions.* The concept of institution [31] formalizes the intuitive notion of logical system in a category-based setting, where the focus is on the relations between objects rather than their internal structure. The theory of institutions emerged in the context of a rapid multiplication of the logics in use in computer science. The original aim was to develop results as much as possible in a uniform way, independently of particular logical systems. This goal was achieved, as the definition of institution is at the core of the mathematical foundations underlying the development of algebraic specification methods and tools. In addition, the theory of institutions is considered a major trend in the so-called universal logic, a general study of logical structures with no commitment to any concrete logical system [6, 7]. There have been substantial developments towards an abstract institutional model theory and logic: interpolation and Beth definability were studied in [18, 33, 44, 61], ultra-products and saturated models in [17, 27], elementary chains in [43], Downward Löwenheim-Skolem Theorem in [34, 37], logic translation in [54], quasi-varieties and free models in [19, 69, 71], and proof-theory in [13, 21, 42]. See [22] for a monography dedicated to this topic.

*Hybridization.* The essence of the hybrid logic idea is independent of the internal structure of the logical formulae and models. The theory of institutions provides the necessary ingredients for defining a hybrid logic in an abstract way. For example, the hybridization process described in [26, 50] is a construction method which assumes an arbitrary base logic, formalized as an institution, and defines a hybrid logic on top of it. The hybridization development extends the previous work on institution-independent possible worlds semantics of [28] to nominals and multi-modalities. This approach has several benefits. Since the hybridization process is parameterized by an institution, it can be instantiated to many concrete examples of hybrid logics. Hybridization provides a general framework for developing results over hybrid logics and it has been used for this purpose in works such as [23, 26, 35, 49, 50, 57].

*Stratified institutions.* In the present work, a more "top-down" approach to the hybrid logic idea is chosen, in the spirit of universal logic and category theory. Though very general due to its

abstract parameter, hybridization is largely a "bottom-up" approach to logic, in the sense that the sentences, models and satisfaction relation are constructed. In the present contribution, the results are developed in a logical framework given by the definition of stratified institution [1] with nominal and frame extraction [24], which captures examples of hybrid logics that are not instances of the hybridization process. This is a very general approach to Kripke semantics, which supplements the definition of institution with an additional structure to extract (a) nominals and modalities from signatures, and (b) frames from models. The features of hybrid logics are described through dedicated functors instead of assuming an internal structure of the syntax or semantics, which is a top-down approach. In regard to the hybridization process, it is fair to say that both approaches — the one based on the definition of stratified institution and the other which rely on hybridization — provide different abstraction levels and the logic framework used for proving results should be determined based on the context.

*Forcing.* The present work introduces the forcing technique in the framework of stratified institutions and investigates a forcing property based on syntactic consistency, which leads to an elegant proof of completeness. The results are obtained by clean causality and are not hindered by the irrelevant details of concrete hybrid logics. Forcing is a method of constructing models based on consistency results. It was invented by Paul Cohen [14, 15] to prove the independence of the continuum hypothesis from the other axioms of Zermelo-Fraenkel set theory. Robinson [64] developed an analogous theory of forcing in model theory. Barwise [5] and Keisler [47] extended Robinson's theory to infinitary logic and used it to give a new proof of the Omitting Types Theorem. In institutional model theory, forcing was introduced by the author [42] to prove a Gödel Completeness Theorem. The result followed the investigation of a forcing property based on a notion of syntactic consistency defined in the context given by a system of proof rules for first-order institutions. The paper argued that the correspondence between the semantic truth and the syntactic provability is largely independent of the details of the concrete first-order logics. Another forcing property based on semantic consistency was studied in [34] in institution-independent model theory. This research led to the proof of an institution-independent generalization of two classical results from model theory: (a) the Downward Löwenheim-Skolem Theorem, which says that any consistent theory over a countable language has a countable model, and (b) the Omitting Types Theorem, which says that any non-principal type has a model which omits it.

*Layered approach.* The present study starts with an effort for extracting a minimal fragment from a given hybrid logic which captures its essence. This is referred to as the basic layer and it is separated from the rest of the features which are common to all hybrid logics and shape up the idea of hybrid logic. In first-order logics, the basic layer consists of the fragment obtained by restricting the logical formulae to ground atomic sentences. The semantics of the sentence building operators such as the Boolean connectives and quantifiers is similar in all first-order logics. For example, a model $M$ satisfies a sentence $\rho_1 \wedge \rho_2$, in symbols, $M \models \rho_1 \wedge \rho_2$, iff $M \models \rho_1$ and $M \models \rho_2$, which is independent from the internal structure of both the model $M$ and the sentences $\rho_i$. While in first-order logics the difference between the basic level and the first-order features is quite clear, in hybrid logics the distinction is more subtle. According to our experience, the essence of a concrete hybrid logic is given by the *atomic sentences* [1], and the "interaction" between local and global properties through the sentence building operator *retrieve* @. Therefore, the basic layer of a concrete hybrid logic is the fragment obtained by restricting the logical formulae to the sentences obtained by applying the operator retrieve to atomic sentences. The outmost layer is given by the sentences constructed from the basic layer by applying Boolean connectives, *retrieve* @, *possibility over binary modalities* $\langle\lambda\rangle$,

[1] We call atomic sentences, the sentences free of the sentence building operators.

*store* $\downarrow$ and the existential quantifier. This distinction is important as completeness is proved in two steps:

(1) The first step corresponds to the basic layer, and it is institution-dependent, which means that a system of proof rules is developed for the basic layer of each concrete hybrid logic, and then completeness is proved individually for each case.
(2) The second step corresponds to the outmost layer, and it is institution-independent, which means that a system of proof rules is defined for an abstract stratified institution, and then completeness is proved assuming the completeness of the basic layer.

The sentence building operator *retrieve* changes the point of evaluation in a formula and it appears in both (a) the basic layer, where it increases the expressivity of atomic sentences, and (b) the outmost layer, as the semantics of *retrieve* is common to all hybrid logics. It is worth mentioning that this contribution targets rigid quantification (as in [11]), where the possible worlds share a common domain and the variables are interpreted identically across the worlds. This approach is in contrast with the world-line semantics of [67], where the quantified variables may be interpreted differently across distinct worlds.

*Prerequisites.* We assume the reader is already acquainted with the notions of category, functor and natural transformation. A standard textbook on the topic is [48]. We are going to use the terminology from there, with a few exceptions that we point out below. We use both the terms "morphism" and "arrow" to refer to morphisms in a category. If $\mathsf{C}$ is a category then $|\mathsf{C}|$ denotes the class of objects of $\mathsf{C}$. Composition of morphisms and functors is denoted using the symbol ";" and is considered in diagrammatic order, $1_A$ denotes the identity at an object $A \in |\mathsf{C}|$, and $\mathsf{C}^{op}$ denotes the opposite category of $\mathsf{C}$.

*Structure of the paper.* The rest of the paper is organized as follows.

Section 2 recalls the definition of institution, which formalizes the intuitive idea of logical system, and stratified institution, which lays the foundation in which the results are proved.

Section 3 recalls the necessary fundamental concepts of institutional model theory such as *basic set of sentences*, *reachable model*, or *signature extension*. The essence of individual logical systems can be understood by looking into the internal structure of their models and atomic sentences. The definition of *basic set of sentences* captures some of the key features of atomic sentences, which are needed when reasoning about model-theoretic properties. The models of interest are constructed from elements obtained from syntactic compounds. The notion of *reachable model* gives a complete characterization of the quotient term models from concrete examples. Most of first-order completeness results require an infinite number of new constants as witnesses for the existentially quantified variables that might occur in formal deductions. The concept of *signature extension* provides a category-based description of the signature expansions with an infinite number of constants.

Section 4 is dedicated to the study of *forcing* in the abstract setting of stratified institutions, which acts like an interface for the development of further proof-theoretic results such as completeness presented in the present work, and model-theoretic results such as Downward Löwenheim-Skolem Theorem that will be explored in future contributions. The developments revolve around two key concepts: *generic set* and *generic model*. Roughly speaking, a *generic set* consists of an increasing chain of conditions, which can be used to build a reachable model. A *generic model* is a concrete realization of a generic set. The Generic Model Theorem [47] shows that each generic set has a generic model under some hypotheses which can be easily checked in concrete examples of logical systems.

Section 5 investigates a forcing property based on a notion of syntactic consistency defined in the context given by a system of proof rules for stratified institutions. Then the underlying

syntactic forcing property is used to prove a layered completeness result for stratified institutions. The definition of signature extension has a certain simplicity and clarity, which is explored to show that generic sets exist in the context given by our forcing property. This can be shown without the restriction to signatures composed of a countable number of symbols that is usually required for forcing. In this setting, a generic set "converges" to a maximally consistent set of sentences that have retrieve as top sentence building operator. It is worth noting that locality plays an essential role in the present contribution as retrieve is required at all development levels.

Section 6 instantiates the abstract results and provides complete entailment systems for a couple of benchmark examples of hybrid logics. It is worth noting that the area of applications is much larger, as pointed out in section 8.

Section 7 presents a case study, which introduces some ideas for developing a formal method based on the logical foundation introduced previously. This shows that the present contribution is directly connected to applications to formal methods design.

Section 8 concludes the paper and discusses future work.

## 2 PRELIMINARIES

In this section we define the category-based setting providing the framework for the remainder of the paper. We recall:

(1) the definition of *institution*, which is a formalization of the intuitive notion of logical system;
(2) the definition of *stratified institution*, which provides a general approach to Kripke semantics.

### 2.1 Institutions

The concept of institution was introduced by Goguen and Burstall in the seminal paper [31]. It provides a category-based definition of the informal notion of logical system, which allows one to reason about the properties of logics from a "higher" perspective.

*Definition 2.1 (Institution).* An *institution* $\mathtt{I} = (\mathsf{Sig}^{\mathtt{I}}, \mathsf{Sen}^{\mathtt{I}}, \mathsf{Mod}^{\mathtt{I}}, \models^{\mathtt{I}})$ consists of

(1) a category $\mathsf{Sig}^{\mathtt{I}}$, whose objects are called *signatures*,
(2) a functor $\mathsf{Sen}^{\mathtt{I}} : \mathsf{Sig}^{\mathtt{I}} \to \mathbb{S}\mathsf{et}$, providing for each signature $\Sigma$ a set whose elements are called *($\Sigma$-)sentences*,
(3) a functor $\mathsf{Mod}^{\mathtt{I}} : \mathsf{Sig}^{\mathtt{I}} \to \mathbb{C}\mathsf{at}^{op}$, providing for each signature $\Sigma$ a category whose objects are called *($\Sigma$-)models* and whose arrows are called *($\Sigma$-)homomorphisms*,
(4) a family of relations $\models^{\mathtt{I}} = \{\models^{\mathtt{I}}_{\Sigma}\}_{\Sigma \in |\mathsf{Sig}^{\mathtt{I}}|}$, where $\models^{\mathtt{I}}_{\Sigma} \subseteq |\mathsf{Mod}^{\mathtt{I}}(\Sigma)| \times \mathsf{Sen}^{\mathtt{I}}(\Sigma)$ is called *($\Sigma$-)satisfaction*, such that the following *satisfaction condition* holds:

$$M' \models^{\mathtt{I}}_{\Sigma'} \mathsf{Sen}^{\mathtt{I}}(\varphi)(e) \text{ iff } \mathsf{Mod}^{\mathtt{I}}(\varphi)(M') \models^{\mathtt{I}}_{\Sigma} e$$

for all $\Sigma \xrightarrow{\varphi} \Sigma' \in \mathsf{Sig}^{\mathtt{I}}$, $M' \in |\mathsf{Mod}^{\mathtt{I}}(\Sigma')|$ and $e \in \mathsf{Sen}^{\mathtt{I}}(\Sigma)$.

In concrete examples, the category of signatures Sig provides the vocabularies over which the sentences are built and the signature morphisms represent the change of notation. The sentences get translated in the same direction as the mappings between signatures, whereas models are translated in the opposite direction. More concretely, given a signature morphism $\varphi\colon \Sigma \to \Sigma'$ the sentences over the signature $\Sigma$ are translated to the signature $\Sigma'$ by the function $\mathsf{Sen}(\varphi) : \mathsf{Sen}(\Sigma) \to \mathsf{Sen}(\Sigma')$. The $\Sigma'$-models are "reduced" to the signature $\Sigma$ by the functor $\mathsf{Mod}(\varphi) : \mathsf{Mod}(\Sigma') \to \mathsf{Mod}(\Sigma)$. The satisfaction condition express that *the truth is invariant w.r.t. the change of notation.* We denote the *reduct* functor $\mathsf{Mod}(\varphi)$ by $\_\restriction_{\varphi}$ and the sentence translation $\mathsf{Sen}(\varphi)$ by $\varphi$. If $M = M'\restriction_{\varphi}$ we say that $M$ is the *$\varphi$-reduct* of $M'$, and $M'$ is a *$\varphi$-expansion* of $M$.

*2.1.1 Notations.* When there is no danger of confusion, we omit the superscript $\mathtt{I}$ from the notations of the institution components; for example $\mathsf{Sig}^{\mathtt{I}}$ may be simply denoted by Sig. For all signatures $\Sigma$, sets of $\Sigma$-sentences $\Gamma$ and $E$,

(1) for all $\Sigma$-models $M$, $(M \models E)$ iff $(M \models e$ for all $e \in E)$;
(2) $\Gamma \models E$ iff for all $\Sigma$-models $M$ we have $M \models \Gamma$ implies $M \models E$;
(3) $\Gamma \models\!\!\!\dashv E$ iff $\Gamma \models E$ and $E \models \Gamma$.

*2.1.2 Related concepts.* The *power* of a signature $\Sigma$ is the cardinality of the set $\mathsf{Sen}(\Sigma)$. A signature morphism $\varphi\colon \Sigma \to \Sigma'$ is *conservative* iff each $\Sigma$-model has a $\varphi$-expansion. In the logics given as examples in this paper, the injective signature morphisms are conservative. We say that an institution $\mathtt{I}$ is *compact* if for all sentences $\gamma$ and all sets of sentences $\Gamma$ over the same signature, the following property holds: $\Gamma \models \gamma$ implies $\Gamma_f \models \gamma$ for some finite subset $\Gamma_f \subseteq \Gamma$.

*2.1.3 Internal logic.* The following institutional notions dealing with the semantics of Boolean connectives and quantifiers were defined in [70].

*Definition 2.2 (Internal logic).* Given a signature $\Sigma$ in an institution, a $\Sigma$-sentence $\gamma$ is a semantic

(1) *negation* of a $\Sigma$-sentence $e$ when for each $\Sigma$-model $M$,
$M \models \gamma$ iff $M \not\models e$;
(2) *disjunction* of a (finite) set of $\Sigma$-sentences $E$ when for each $\Sigma$-model $M$,
$M \models \gamma$ iff $M \models e$ for some $e \in E$;
(3) *existential $\chi$-quantification* of a $\Sigma'$-sentence $e'$, where $\chi\colon \Sigma \to \Sigma'$, when for each $\Sigma$-model $M$,
$M \models \gamma$ iff $M' \models_{\Sigma'} e'$ for some $\chi$-expansion $M'$ of $M$.

Distinguished negation is usually denoted by $\neg\_$, distinguished disjunction by $\vee\_$, and distinguished existential $\chi$-quantification by $\exists\chi\cdot\_$.

The sentence building operators such as Boolean connectives and quantifiers are part of the metalanguage and they are used to construct sentences which belong to the internal language of individual institutions using the universal semantics presented above. One can also define $\wedge\_$, $\forall\chi\cdot\_$ using the classical definitions. For example, $\forall\chi\cdot e' := \neg\exists\chi\cdot\neg e'$ and $\bot := \vee\emptyset$.

In this paper, we consider disjunctions only over finite sets of sentences. The concept of quantification used here is very general, and in the particular case of classical model theory, it includes second order quantification. The existential quantifier $\exists X\cdot\_$ is regarded as an abbreviation for $\exists\chi\cdot\_$, where $\chi\colon \Sigma \hookrightarrow \Sigma[X]$ is an inclusion of signatures and $\Sigma[X]$ denotes the extension of $\Sigma$ with the variables from $X$ as constants. This internalization of the quantification does not use the ordinary concepts of open formulae and valuations. It considers the variables as part of an extended signature $\Sigma[X]$ (defined by the addition of the variables to the signature as constant symbols) and treats the valuations as model expansions along the signature extension (since each valuation of variables into a model can be regarded as an expansion of the model to the signature extended with the variables). Although this way of thinking about variables and quantification is well-known in conventional mathematical logic [46, 63], it is quite rare in the usual presentations of classical logic.

*2.1.4 Examples.* We give a few typical examples of institutions in algebraic specification literature.

*Example 2.3 (Propositional Logic* (PL)*).* The category of signatures $\mathsf{Sig}^{\mathsf{PL}}$ is $\mathbb{S}\mathrm{et}$. For any set of propositional symbols Prop, the set of Prop-sentences is generated by the following grammar:

$$e ::= \rho \mid \neg e \mid \vee E$$

where $\rho \in \mathsf{Prop}$ is a propositional symbol, and $E$ is a finite set of Prop-sentences. The category of models $\mathsf{Mod}^{\mathsf{PL}}(\mathsf{Prop})$ is $(\mathcal{P}(\mathsf{Prop}), \subseteq)$, where $\mathcal{P}(\mathsf{Prop})$ is the set of all subsets of Prop. For any

mapping $\varphi\colon \mathsf{Prop} \to \mathsf{Prop}'$, the function $\mathsf{Sen}^{\mathsf{PL}}(\varphi)$ replaces each element $\rho \in \mathsf{Prop}$ that occurs in a sentence of $\mathsf{Sen}^{\mathsf{PL}}(\mathsf{Prop})$ by $\varphi(\rho)$. For each model $M' \subseteq \mathsf{Prop}'$, we have $\mathsf{Mod}^{\mathsf{PL}}(\varphi)(M') = \varphi^{-1}(M')$. The satisfaction relation for propositional symbols is defined by $M \models \rho$ iff $\rho \in M$, where $M \subseteq \mathsf{Prop}$ and $\rho \in \mathsf{Prop}$. The satisfaction of complex sentences is as in Definition 2.2. The same remark holds for all examples of institutions in Section 2.1.4.

*Example 2.4 (First-order logic (*FOL*) [31]).*

*Signatures.* Signatures are of the form $(S, F, P)$, where $S$ is a set of sorts, $F = \{F_{\mathsf{ar}\to s}\}_{(\mathsf{ar},s)\in S^*\times S}$ is a ($S^* \times S$ -indexed) set of operation symbols, and $P = \{P_{\mathsf{ar}}\}_{\mathsf{ar}\in S^*}$ is a ($S^*$-indexed) set of relation symbols. [2] If $\mathsf{ar} = \varepsilon$ then an element of $F_{\mathsf{ar}\to s}$ is called a *constant symbol*. We overload the notation and let $F$ and $P$ also denote $\biguplus_{(\mathsf{ar},s)\in S^*\times S} F_{\mathsf{ar}\to s}$ and $\biguplus_{\mathsf{ar}\in S^*} P_{\mathsf{ar}}$, respectively. Therefore, we may write $\sigma \in F_{\mathsf{ar}\to s}$ or $(\sigma\colon \mathsf{ar} \to s) \in F$; both have the same meaning. A signature morphism $\varphi\colon \Sigma \to \Sigma'$, where $\Sigma = (S, F, P)$ and $\Sigma' = (S', F', P')$, is a triplet $\varphi = (\varphi^{st}, \varphi^{op}, \varphi^{rl})$ such that $\varphi^{st} : S \to S'$, $\varphi^{op} = \{\varphi^{op}_{\mathsf{ar}\to s} : F_{\mathsf{ar}\to s} \to F'_{\varphi^{st}(\mathsf{ar})\to\varphi^{st}(s)} \mid \mathsf{ar} \in S^*, s \in S\}$, $\varphi^{rl} = \{\varphi^{rl}_{\mathsf{ar}} : P_{\mathsf{ar}} \to P'_{\varphi^{st}(\mathsf{ar})} \mid \mathsf{ar} \in S^*\}$. When there is no danger of confusion, we may let $\varphi$ denote each of $\varphi^{st}$, $\varphi^{op}_{\mathsf{ar}\to s}$, $\varphi^{rl}_{\mathsf{ar}}$.

*Models.* Given a signature $\Sigma = (S, F, P)$, a $\Sigma$-model is a triple

$$M = (\{M_s\}_{s\in S}, \{M_\sigma\}_{(\mathsf{ar},s)\in S^*\times S, \sigma\in F_{\mathsf{ar}\to s}}, \{M_\pi\}_{\mathsf{ar}\in S^*, \pi\in P_{\mathsf{ar}}})$$

interpreting each sort $s$ as a non-empty set $M_s$, each operation symbol $\sigma \in F_{\mathsf{ar}\to s}$ as a function $M_\sigma : M_{\mathsf{ar}} \to M_s$ (where $M_{\mathsf{ar}}$ stands for $M_{s_1} \times \ldots \times M_{s_n}$ if $\mathsf{ar} = s_1 \ldots s_n$), and each relation symbol $\pi \in P_{\mathsf{ar}}$ as a relation $M_\pi \subseteq M_{\mathsf{ar}}$. Morphisms between models are the usual $\Sigma$-homomorphisms, i.e. $S$-sorted functions that preserve the structure.

For any signature morphism $\varphi\colon \Sigma \to \Sigma'$, where $\Sigma = (S, F, P)$ and $\Sigma' = (S', F', P')$, the model functor $\mathsf{Mod}(\varphi) : \mathsf{Mod}(\Sigma') \to \mathsf{Mod}(\Sigma)$ is defined as follows:

(1) the reduct $M' \restriction_\varphi$ of a $\Sigma'$-model $M'$ is a defined by $(M' \restriction_\varphi)_x = M'_{\varphi(x)}$ for each sort $x \in S$, or operation symbol $x \in F$, or relation symbol $x \in P$;
(2) rhe reduct $h' \restriction_\varphi$ of a homomorphism $h'$ is defined by $(h' \restriction_\varphi)_s = h'_{\varphi(s)}$ for all sorts $s \in S$.

*Sentences.* Given a signature $\Sigma = (S, F, P)$, the $S$-sorted set of $\Sigma$-terms is denoted by $T_\Sigma$. The set of $\Sigma$-sentences is given by:

$$e ::= t =_s t' \mid \pi(t_1, \ldots, t_n) \mid \neg e \mid \vee E \mid \exists X \cdot e'$$

where (1) $t =_s t'$ is an equation with $t, t' \in T_{\Sigma,s}$ and $s \in S$, (2) $\pi(t_1, \ldots, t_n)$ is a relational atom with $\pi \in P_{s_1\ldots s_n}$, $t_i \in T_{\Sigma,s_i}$ and $s_i \in S$, (3) $E$ is a finite set of $\Sigma$-sentences, (4) $X$ is a finite set of variables for $\Sigma$, [3] (5) $\exists X \cdot \_$ is just an abbreviation for $\exists \chi \cdot \_$ such that $\chi\colon \Sigma \hookrightarrow \Sigma[X]$ is an inclusion, $\Sigma[X] = (S, F[X], P)$, and $F[X]$ is the family of operation symbols obtained by adding the variables in $X$ as constants to $F$, (6) $e'$ is a $\Sigma[X]$-sentence. For any signature morphism $\varphi\colon \Sigma \to \Sigma'$ the function $\mathsf{Sen}(\varphi) : \mathsf{Sen}(\Sigma) \to \mathsf{Sen}(\Sigma')$ translates sentences symbolwise.

*Satisfaction relation.* Satisfaction is the usual first-order satisfaction and it is defined using the natural interpretations of ground terms $t$ as elements $M_t$ in models $M$. For example, $M \models t =_s t'$ iff $M_t = M_{t'}$.

*Non-void signatures.* A first-order signature $\Sigma$ is called *non-void* if all sorts in $\Sigma$ are inhabited by terms, that is $T_{\Sigma,s} \neq \emptyset$ for all sorts $s$ in $\Sigma$. If $\Sigma$ is a *non-void* signature then the set of $\Sigma$-terms $T_\Sigma$ can be regarded as a first-order model which interprets (a) any function symbol $(\sigma\colon \mathsf{ar} \to s) \in F$ as a function $T_{\Sigma,\sigma} : T_{\Sigma,\mathsf{ar}} \to T_{\Sigma,s}$ defined by $T_{\Sigma,\sigma}(t) = \sigma(t)$ for all $t \in T_{\Sigma,\mathsf{ar}}$, and (b) any relation symbol as the empty set.

[2] If $S$ is a set then we let $S^*$ denote the set of strings over the symbols in $S$, including the empty string $\varepsilon$.
[3] See section 3.1 for details about sets of variables for a given signature.

Note that the institution PL can be regarded as the fragment of FOL determined by the signatures with empty sets of sorts.

*Example 2.5 (*REL*).* The institution REL is the sub-institution of single-sorted first-order logic with signatures having only constants and relational symbols. Any REL signature $(\{\star\}, F, P)$ is simply denote by $(F, P)$, where $\star$ is a sort, $F$ is a set of constants and $P$ is a set of relation symbols indexed by arities. REL plays a key role in the present contribution, since it provides support for extracting (a) nominals and modalities from the signatures of hybrid logics, and (b) frames from Kripke models.

*Example 2.6 (First-Order Logic with Rigid symbols (*FOLR*)).* This institution is used in Example 2.16 to define a hybrid logic and it is not intended for any other application.

*Signatures.* The signatures $\Sigma^{\mathsf{r}} \subseteq \Sigma$ consist of FOL signatures $\Sigma = (S, F, P)$ enhanced with a subsignature $\Sigma^{\mathsf{r}} = (S^{\mathsf{r}}, F^{\mathsf{r}}, P^{\mathsf{r}})$ of "rigid" symbols. Given a signature $\Sigma^{\mathsf{r}} \subseteq \Sigma$, where $\Sigma^{\mathsf{r}} = (S^{\mathsf{r}}, F^{\mathsf{r}}, P^{\mathsf{r}})$ and $\Sigma = (S, F, P)$, we let $S^{\mathsf{f}} = S \setminus S^{\mathsf{r}}$, and $F^{\mathsf{f}}$ and $P^{\mathsf{f}}$ be the sub-families of $F$ and $P$, respectively, that consist of *flexible symbols* (obtained by removing rigid symbols). A signature morphism $\varphi\colon \Sigma^{\mathsf{r}} \subseteq \Sigma \to \Sigma_1^{\mathsf{r}} \subseteq \Sigma_1$ in FOLR is a signature morphism $\varphi\colon \Sigma \to \Sigma_1$ in FOL that maps rigid symbols to rigid symbols, i.e the following diagram is commutative

$$\begin{tikzcd} \Sigma \arrow[r, "\varphi"] & \Sigma_1 \\ \Sigma^{\mathsf{r}} \arrow[u, hook] \arrow[r, "\varphi^{r}"'] & \Sigma_1^{\mathsf{r}} \arrow[u, hook] \end{tikzcd}$$

where $\varphi^r$ is the restriction of $\varphi$ to rigid symbols.

*Models.* Given a signature $\Sigma^{\mathsf{r}} \subseteq \Sigma$ as above, a model over $\Sigma^{\mathsf{r}} \subseteq \Sigma$ is a triple

$$M = (\{M_s\}_{s \in S}, \{M_\sigma\}_{(\mathsf{ar}, s) \in S^* \times S, \sigma \in F_{\mathsf{ar} \to s}}, \{M_\pi\}_{\mathsf{ar} \in S^*, \pi \in P_{\mathsf{ar}}})$$

interpreting each sort $s \in S$ as a non-empty set $M_s$, each function symbol $(\sigma\colon \mathsf{ar} \to s) \in F$ as a function $M_\sigma : M_{\mathsf{ar}} \to M_s$, and each relation symbol $(\pi\colon \mathsf{ar}) \in P$ as a relation $M_\pi \subseteq M_{\mathsf{ar}}$. Morphisms between models are the usual $\Sigma$-homomorphisms, i.e. $S$-sorted functions that preserve the structure.

*Sentences.* The set of sentences $\mathsf{Sen}^{\mathsf{FOLR}}(\Sigma^{\mathsf{r}} \subseteq \Sigma)$ over the signature $\Sigma^{\mathsf{r}} \subseteq \Sigma$ consists of those sentences in $\mathsf{Sen}^{\mathsf{FOL}}(\Sigma)$ that contain only quantifiers over variables of rigid sorts.

*Satisfaction relation.* The satisfaction relation in FOLR is induced by the satisfaction relation in FOL, i.e. $(\models^{\mathsf{FOLR}}_{\Sigma^{\mathsf{r}} \subseteq \Sigma}) \ := \ (\models^{\mathsf{FOL}}_{\Sigma})$.

## 2.2 Stratified institutions

Stratified institutions can be regarded as an extension of the theory of institutions towards Kripke semantics.

*Definition 2.7 (Stratified institution with nominal and frame extraction [24]).* A six-tuple $\mathtt{SI} = (\mathsf{Sig}^{\mathtt{SI}}, \mathsf{F}^{\mathtt{SI}}, \mathsf{Sen}^{\mathtt{SI}}, \mathsf{Mod}^{\mathtt{SI}}, \mathsf{K}^{\mathtt{SI}}, \models^{\mathtt{SI}})$ is a *stratified institution with nominal and frame extraction* if

(1) $\mathsf{Sig}^{\mathtt{SI}}$ is a category of signatures;
(2) $\mathsf{F}^{\mathtt{SI}}\colon \mathsf{Sig}^{\mathtt{SI}} \to \mathsf{Sig}^{\mathsf{REL}}$ is a functor which extracts from each signature $\Delta$ its relational part $\mathsf{F}^{\mathtt{SI}}(\Delta) = (\mathsf{Nom}^{\Delta}, \Lambda^{\Delta})$, where
   (a) $\mathsf{Nom}^{\Delta}$ is a set of *nominals*, and
   (b) $\Lambda^{\Delta} = \{\Lambda^{\Delta}_n\}_{n \in \mathbb{N}}$ is a family of sets of *modalities*;
(3) $\mathsf{Sen}^{\mathtt{SI}}\colon \mathsf{Sig}^{\mathtt{SI}} \to \mathbb{S}\mathsf{et}$ is a sentence functor;
(4) $\mathsf{Mod}^{\mathtt{SI}}\colon \mathsf{Sig}^{\mathtt{SI}} \to \mathbb{C}\mathsf{at}^{op}$ is a model functor;

(5) $\mathsf{K}^{\mathsf{SI}}\colon \mathsf{Mod}^{\mathsf{SI}} \Rightarrow (\mathsf{F}^{\mathsf{SI}}; \mathsf{Mod}^{\mathsf{REL}})$ is a natural transformation providing for each signature $\Delta$ a frame functor $\mathsf{K}^{\mathsf{SI}}_{\Delta}\colon \mathsf{Mod}^{\mathsf{SI}}(\Delta) \to \mathsf{Mod}^{\mathsf{REL}}(\mathsf{Nom}^{\Delta}, \Lambda^{\Delta})$, which extracts from each model $M$ its *frame* $\mathsf{K}^{\mathsf{SI}}_{\Delta}(M)$, that is
(a) a set of *states/worlds* $|\mathsf{K}^{\mathsf{SI}}_{\Delta}(M)|$, and
(b) a family of *accessibility relations* $\mathsf{K}^{\mathsf{SI}}_{\Delta}(M)_{\lambda}$ indexed by modalities;

(6) $\models^{\mathsf{SI}} = \{M \models^{-}_{\Delta -}\}_{\Delta \in |\mathsf{Sig}^{\mathsf{SI}}|, M \in |\mathsf{Mod}^{\mathsf{SI}}(\Delta)|}$ is a *local satisfaction relation*, where $M \models^{-}_{\Delta -} \subseteq |\mathsf{K}^{\mathsf{SI}}_{\Delta}(M)| \times \mathsf{Sen}^{\mathsf{SI}}(\Delta)$ is a binary relation such that the following *local satisfaction condition* hols: $M' \models^{w'}_{\Delta'} \varphi(e)$ iff $M' \restriction_{\varphi} \models^{w'}_{\Delta} e$, for all signature morphisms $\varphi\colon \Delta \to \Delta'$, $\Delta'$-models $M'$, possible worlds $w'$ of $M'$, and $\Delta$-sentences $e$.

The notion of stratified institution was introduced in [1] as an abstract approach to parameterized satisfaction which can arise in different forms: (a) for example, in first-order logic, if the set of sentences is extended with open formulae then the satisfaction relation is parameterized by the valuations of the unbound variables into the models; (b) in modal logics, the satisfaction relation is parameterized by the states of the models. In [24], the natural transformation used to extract states from models was upgraded to extract frames, which makes stratified institutions a useful framework for reasoning about properties of modal logics from a "higher" perspective. By equipping stratified institutions with a functor to extract nominals and modalities from signatures, [24] provided a fully abstract approach to hybrid logics. Stratified institutions with nominal and frame extraction were used in [38] and [36] under the name of hybrid institutions. Since there is no danger of confusion, in the present contribution, we shall call stratified institutions with nominal and frame extraction, simply, stratified institutions.

Like for ordinary institutions, when appropriate we use simplified notations without superscripts or subscripts that are clear from the context. The well-definedness of the local satisfaction condition is ensured by the following result which says that the states of models are preserved by the reduct functors.

Lemma 2.8 ([36, 38]). *Let $\varphi\colon \Delta \to \Delta'$ be a signature morphism of a stratified institution. Any $\Delta'$-model $M'$ has exactly the same set of states as its reduct $M' \restriction_{\varphi}$.*

*2.2.1 Related concepts.* One can easily define a global satisfaction relation based on local satisfaction relation such that the satisfaction condition for ordinary institutions holds.

Remark 2.1. *Let $\Delta$ be a signature of a stratified institution $\mathsf{SI}$, $M$ a $\Delta$-model, and $\gamma$ a $\Delta$-sentence. We overload the notation and write $M \models^{\mathsf{SI}}_{\Delta} \gamma$ when $M \models^{w}_{\Delta} \gamma$ for all states $w$ of $M$. We call $\_\models^{\mathsf{SI}}_{\Delta}\_ \subseteq |\mathsf{Mod}^{\mathsf{SI}}(\Delta)| \times \mathsf{Sen}^{\mathsf{SI}}(\Delta)$ the global satisfaction relation of $\mathsf{SI}$.*

*Then $(\mathsf{Sig}^{\mathsf{SI}}, \mathsf{Sen}^{\mathsf{SI}}, \mathsf{Mod}^{\mathsf{SI}}, \{\_\models^{\mathsf{SI}}_{\Delta}\_\}_{\Delta \in |\mathsf{Sig}^{\mathsf{SI}}|})$ is an institution.*

Remark 2.1 says that a stratified institution is, in particular, an ordinary institution, and it can be used to import most of the definitions and notions given for ordinary institutions to stratified institutions. The following definition sets a pattern for defining the semantics of concrete stratified institutions starting from the model functor of some base institution, and can be used to justify some of the assumptions made for the abstract framework in which the results are proved.

*Definition 2.9 (Kripke structures [26]).* Let $\mathsf{Mod}^{\mathsf{I}}\colon \mathsf{Sig}^{\mathsf{I}} \to \mathbb{C}\mathsf{at}^{op}$ be a *base* model functor. The *Kripke* functor $\mathsf{Mod}^{\mathsf{I}}_{\kappa}\colon \mathsf{Sig}^{\mathsf{REL}} \times \mathsf{Sig}^{\mathsf{I}} \to \mathbb{C}\mathsf{at}^{op}$ over $\mathsf{Mod}^{\mathsf{I}}$ is defined as follows:

(1) for each signature $(\mathsf{Nom}, \Lambda, \Sigma) \in |\mathsf{Sig}^{\mathsf{REL}} \times \mathsf{Sig}^{\mathsf{I}}|$, where $(\mathsf{Nom}, \Lambda) \in |\mathsf{Sig}^{\mathsf{REL}}|$ and $\Sigma \in |\mathsf{Sig}^{\mathsf{I}}|$, $\mathsf{Mod}^{\mathsf{I}}_{\kappa}(\mathsf{Nom}, \Lambda, \Sigma)$ is the category that consists of
   (a) Kripke models $(W, M)$, where $W$ is a $(\mathsf{Nom}, \Lambda)$-model and $M\colon |W| \to |\mathsf{Mod}^{\mathsf{I}}(\Sigma)|$, and
   (b) homomorphisms $h\colon (W_1, M_1) \to (W_2, M_2)$ of the form $(h^{\mathsf{REL}}, h^{mod})$, where

(i) $h^{\mathsf{REL}} \colon W_1 \longrightarrow W_2$ is a homomorphism in $\mathsf{REL}$, and
(ii) $h^{mod} = \{h^{mod}_w \colon (M_1)_w \longrightarrow (M_2)_{h^{\mathsf{REL}}(w)}\}_{w \in |W_1|}$ is a family of homomorphisms in $\mathsf{Mod}^{\mathtt{I}}(\Sigma)$, with $(M_1)_w$ and $(M_2)_{h^{\mathsf{REL}}(w)}$ denoting $M_1(w)$ and $M_2(h^{\mathsf{REL}}(w))$, respectively;

(2) for each arrow $(\mathsf{Nom}, \Lambda, \Sigma) \xrightarrow{\varphi} (\mathsf{Nom}', \Lambda', \Sigma') \in \mathsf{Sig}^{\mathsf{REL}} \times \mathsf{Sig}^{\mathtt{I}}$, where $\varphi = (\varphi^{\mathsf{REL}}, \varphi^{\mathtt{I}})$, the reduct functor $\mathsf{Mod}^{\mathtt{I}}_{\kappa}(\varphi) : \mathsf{Mod}^{\mathtt{I}}_{\kappa}(\mathsf{Nom}', \Lambda', \Sigma') \longrightarrow \mathsf{Mod}^{\mathtt{I}}_{\kappa}(\mathsf{Nom}, \Lambda, \Sigma)$ is defined by
(a) $\mathsf{Mod}^{\mathtt{I}}_{\kappa}(\varphi)(W', M') = (W, M)$ such that $\begin{cases} W = W' \restriction_{\varphi^{\mathsf{REL}}}, \text{ and} \\ M_w = M'_w \restriction_{\varphi^{\mathtt{I}}} \text{ for all } w \in |W|; \end{cases}$
(b) $\mathsf{Mod}^{\mathtt{I}}_{\kappa}(\varphi)(h') = h$ such that $\begin{cases} h^{\mathsf{REL}} = h'^{\mathsf{REL}} \restriction_{\varphi^{\mathsf{REL}}}, \text{ and} \\ h^{mod} = \{h'_w \restriction_{\varphi^{\mathtt{I}}}\}_{w \in |W|}. \end{cases}$

In our examples of stratified institutions, the model functor is a sub-functor of some Kripke functor $\mathsf{Mod}^{\mathtt{I}}_{\kappa} \colon \mathsf{Sig}^{\mathsf{REL}} \times \mathsf{Sig}^{\mathtt{I}} \longrightarrow \mathbb{C}\mathsf{at}^{op}$, the functor $\mathsf{F}^{\mathtt{SI}} \colon \mathsf{Sig}^{\mathsf{REL}} \times \mathsf{Sig}^{\mathtt{I}} \longrightarrow \mathsf{Sig}^{\mathsf{REL}}$ is the first projection, and for all signatures $\Delta$, the frame functor $\mathsf{K}_{\Delta}$ is the forgetful functor that maps each Kripke structure $(W, M)$ to $W$. Definition 2.9 provides a pattern for describing the semantics of stratified institutions but the results of this paper will be developed at the more abstract level provided by Definition 2.7, where the Kripke structures are implicitly assumed, not constructed. This approach corresponds to the universal logic ideas.

Framework 2.1 (Stratified institution with nominal variables). *Throughout this section, we work within a stratified institution* $\mathtt{SI}$ *with nominal variables,* [4] *which means that* $\mathtt{SI}$ *has the following properties: for each signature* $\Delta$ *and any nominal variable* $z$ *for* $\Delta$ *there exists a designated signature morphism* $\chi_z \colon \Delta \longrightarrow \Delta[z]$ *such that*

*(F1)* $\mathsf{F}(\chi_z) = \chi_z^{\mathsf{REL}}$*, where* $\chi_z^{\mathsf{REL}} \colon (\mathsf{Nom}^{\Delta}, \Lambda^{\Delta}) \hookrightarrow (\mathsf{Nom}^{\Delta}[z], \Lambda^{\Delta})$ *is an inclusion and* $\mathsf{Nom}^{\Delta}[z]$ *denotes the union* $\mathsf{Nom}^{\Delta} \cup \{z\}$*;*

*(F2) the following square is a pullback in* $\mathbb{C}\mathsf{lass}$*, the category of classes;* [5]

$$\begin{tikzcd}
|\mathsf{Mod}^{\mathtt{SI}}(\Delta[z])| \arrow[r, "\mathsf{K}_{\Delta[z]}"] \arrow[d, "\restriction_{\chi_z}"'] & |\mathsf{Mod}^{\mathsf{REL}}(\mathsf{Nom}^{\Delta}[z], \Lambda^{\Delta})| \arrow[d, "\restriction_{\chi_z^{\mathsf{REL}}}"] \\
|\mathsf{Mod}^{\mathtt{SI}}(\Delta)| \arrow[r, "\mathsf{K}_{\Delta}"'] & |\mathsf{Mod}^{\mathsf{REL}}(\mathsf{Nom}^{\Delta}, \Lambda^{\Delta})|
\end{tikzcd}$$

*this mean that for any* $\Delta$*-model* $M$ *and any possible world* $w$ *of* $M$*,*
*there exists a unique* $\chi_z$*-expansion* $M^{z \leftarrow w}$ *of* $M$ *such that* $\mathsf{K}_{\Delta[z]}(M^{z \leftarrow w}) = \mathsf{K}_{\Delta}(M)^{z \leftarrow w}$*,*
*where* $\mathsf{K}_{\Delta}(M)^{z \leftarrow w}$ *is the unique* $\chi_z^{\mathsf{REL}}$*-expansion of* $\mathsf{K}_{\Delta}(M)$ *which interprets* $z$ *as* $w$*;*

*(F3) for all nominals* $k \in \mathsf{Nom}^{\Delta}$*, there exists a unique signature morphism* $\varphi_{z \leftarrow k} \colon \Delta[z] \longrightarrow \Delta$ *such that* $\chi_z ; \varphi_{z \leftarrow k} = 1_{\Delta}$ *and* $\mathsf{F}(\varphi_{z \leftarrow k}) = \varphi^{\mathsf{REL}}_{z \leftarrow k}$*, where* $\varphi^{\mathsf{REL}}_{z \leftarrow k} \colon (\mathsf{Nom}^{\Delta}[z], \Lambda^{\Delta}) \longrightarrow (\mathsf{Nom}^{\Delta}, \Lambda^{\Delta})$ *is the* $\mathsf{REL}$ *signature morphism that preserves* $(\mathsf{Nom}^{\Delta}, \Lambda^{\Delta})$ *and maps* $k$ *to* $z$*.*

$$\begin{tikzcd}
\Delta[z] \arrow[r, dotted, "\mathsf{F}(\Delta[z])"] & (\mathsf{Nom}^{\Delta}[z], \Lambda^{\Delta}) \\
\Delta \arrow[u, bend left, "\chi_z"] \arrow[u, bend right, "\varphi_{z \leftarrow k}"'] \arrow[r, dotted, "\mathsf{F}(\Delta)"'] & (\mathsf{Nom}^{\Delta}, \Lambda^{\Delta}) \arrow[u, bend left, "\chi_z^{\mathsf{REL}}"] \arrow[u, bend right, "\varphi^{\mathsf{REL}}_{z \leftarrow k}"']
\end{tikzcd}$$

□

[4] A nominal variable $z$ for a signature $\Delta$ is a variable for the signature of nominals $(\mathsf{Nom}^{\Delta}, \Lambda^{\Delta})$. In other words, $z$ is a special constant different from the elements of $\mathsf{Nom}^{\Delta}$.

[5] This is the "extension" of Set having classes as objects, and it belongs, of course, to a higher set-theoretic universe (cf. Grothendieck universe).

The conditions above are easily justified by Definition 2.9. In concrete examples of stratified institutions,

(1) the signatures $\Delta$ are of the form $(\mathsf{Nom}, \Lambda, \Sigma)$, the signatures $\Delta[z]$ are of the form $(\mathsf{Nom}[z], \Lambda, \Sigma)$, and the signature morphisms $\chi_z : (\mathsf{Nom}, \Lambda, \Sigma) \hookrightarrow (\mathsf{Nom}[z], \Lambda, \Sigma)$ are inclusions;
(2) for any $(\mathsf{Nom}, \Lambda, \Sigma)$-model $(W, M)$ and any $\chi_z^{\mathsf{REL}}$ expansion $W'$ of $W$ there exists a unique $\chi_z$-expansion $(W', M)$ of $(W, M)$; this means that any $\chi_z$-expansion is obtained by giving an interpretation of $z$ into $|W|$;
(3) any mapping of $z$ to a nominal $k$ generates a unique signature morphism $\varphi_{z\leftarrow k} : (\mathsf{Nom}[z], \Lambda, \Sigma) \to (\mathsf{Nom}, \Lambda, \Sigma)$ that preserves $(\mathsf{Nom}, \Lambda, \Sigma)$.

Lemma 2.10 ([36]). *For every $\Delta$-model $M$, each nominal $k \in \mathsf{Nom}^\Delta$ and any nominal variable $z$, (1) $\mathsf{K}_\Delta(M) \restriction_{\varphi^{\mathsf{REL}}_{z\leftarrow k}} = \mathsf{K}_\Delta(M)^{z\leftarrow k}$, and (2) $M \restriction_{\varphi_{z\leftarrow k}} = M^{z\leftarrow w}$, where $w = K_\Delta(M)_k$.*

*2.2.2 Internal logic.* The semantics of the sentence building operators in stratified institutions were defined in [24] and [38].

*Definition 2.11 (Internal logic).* Let $\Delta$ be a signature in a stratified institution. A $\Delta$-sentence $\gamma$ is a semantic

(1) *nominal sentence* if there exists $k \in \mathsf{Nom}^\Delta$ such that for all $\Delta$-models $M$ and all states $w$ of $M$,

$$M \models^w \gamma \text{ iff } w = \mathsf{K}_\Delta(M)_k$$

We denote by $k$ a distinguished nominal sentence with the semantics above;
(2) *nominal relation* if there exist $n \in \mathbb{N}$, $\lambda \in \Lambda^\Delta_{n+1}$ and $k_1, \ldots, k_n \in \mathsf{Nom}^\Delta$ such that for all $\Delta$-models $M$ and all possible worlds $w$ of $M$,

$$M \models^w \gamma \text{ iff } (w, \mathsf{K}_\Delta(M)_{k_1}, \ldots, \mathsf{K}_\Delta(M)_{k_n}) \in \mathsf{K}_\Delta(M)_\lambda$$

We denote by $\underline{\lambda}(k_1, \ldots, k_n)$ a distinguished nominal relation with the semantics above;
(3) *retrieve at* $k$ of a $\Delta$-sentence $e$, where $k \in \mathsf{Nom}^\Delta$, if for all $\Delta$-models $M$ and all states $w$ of $M$,

$$M \models^w \gamma \text{ iff } M \models^{w'} e, \text{ where } w' = \mathsf{K}_\Delta(M)_k$$

We denote by $@_k\, e$ a distinguished sentence with the semantics above;
(4) *disjunction* of a set of $\Delta$-sentences $E$ if for all $\Delta$-models $M$ and all states $w$ of $M$,

$$M \models^w \gamma \text{ iff } M \models^w e \text{ for some } e \in E$$

We denote by $\vee E$ a distinguished sentence with the semantics above;
(5) *negation* of a $\Delta$-sentence $e$ if for all $\Delta$-models $M$ and all states $w$ of $M$,

$$M \models^w \neg e \text{ iff } M \not\models^w e$$

We denote by $\neg e$ a distinguished sentence with the semantics above;
(6) *possibility* of $e$ over the binary modality $\lambda \in \Lambda_2$, if for all $\Delta$-models $M$ and all states $w$ of $M$,

$$M \models^w \gamma \text{ iff } M \models^{w'} e \text{ for some } w' \in |\mathsf{K}_\Delta(M)| \text{ such that } (w, w') \in \mathsf{K}_\Delta(M)_\lambda$$

We denote by $\langle\lambda\rangle e$ a distinguished sentence with the semantics above;
(7) *store quantification* of a $\Delta[z]$-sentence $e'$, where $z$ is a nominal variable, if for all $\Delta$-models $M$ and all states $w$ of $M$,

$$M \models^w \gamma \text{ iff } M^{z\leftarrow w} \models^w e'$$

We denote by $\downarrow z \cdot e'$ a distinguished sentence with the semantics above;

(8) *existential $\chi$-quantification* of a $\Delta'$-sentence $e'$, where $\chi\colon \Delta \to \Delta'$, if for all $\Delta$-models $M$ and all states $w$ of $M$,

$$M \models^w \gamma \text{ iff } M' \models^w e' \text{ for some } \chi\text{-expansion } M' \text{ of } M$$

We denote by $\exists\chi \cdot e'$ a distinguished sentence with the semantics above.

Notice that none of the sentence building operators defined above need to exist in a stratified institution. See Section 2.2.3 for examples of stratified institutions. For the sake of simplicity, throughout this paper, we consider the possibility only over binary modalities.

Remark 2.2. *For any binary modality $\lambda$, the sentence $\underline{\lambda}(k)$ is semantically equivalent to $\langle\lambda\rangle k$. Therefore, $\underline{\lambda}(k)$ can be regarded as a copy of $\langle\lambda\rangle k$ that has the advantage of being atomic.*

The operator @ is called *retrieve* because it changes the point of evaluation in a sentence. The operator $\downarrow$ is called *store* because it allows one to give a name to the current state that can be referred later on in sentences. As in the case of ordinary institutions, the semantics of other sentence building operators can be defined using the above constructors for sentences. For example, given a binary modality $\lambda$, the semantics of the necessity can be defined as follows: $M \models^w [\lambda]e$ iff $M \models^w \neg\langle\lambda\rangle\neg e$. We adopt the following convention about the precedence of the logical operators to avoid the need to write parentheses in some cases: (a) $\neg$, @ and $\Diamond$ have the same precedence and bind stronger than $\vee$, (b) $\vee$ binds stronger than quantifiers, and (c) $\downarrow$ and $\exists$ have the same precedence. We say that a stratified institution is closed under retrieve if $@_k\, e$ is a $\Delta$-sentence, for each signature $\Delta$, any nominal $k \in \mathsf{Nom}^\Delta$ and all $\Delta$-sentences $e$. The closure under other sentence building operators can be defined in a similar manner.

*2.2.3 Examples.* We present several examples of stratified institutions including some unconventional ones linked to pragmatic applications in computer science.

*Example 2.12 (Hybrid Propositional Logic (*HPL*)).* This is the multi-modal variant of the most common form of hybrid logic (e.g. [2]).

*Models.* $\mathsf{Mod}^{\mathsf{HPL}}$ is $\mathsf{Mod}^{\mathsf{PL}}_\kappa$, where $\mathsf{Mod}^{\mathsf{PL}}_\kappa$ is obtained by applying Definition 2.9 to $\mathsf{Mod}^{\mathsf{PL}}$.

*Sentences.* For any signature $\Delta = (\mathsf{Nom}, \Lambda, \mathsf{Prop})$, the set of $\Delta$-sentences is given by the following grammar:

$$e ::= \rho \mid k \mid \underline{\lambda}(k) \mid \neg e \mid \vee E \mid @_k\, e \mid \langle\lambda\rangle e$$

where (a) $\rho$ is a propositional symbol, (b) $k$ is a nominal, (c) $\lambda$ is a binary modality, and (d) $E$ is a finite set of $\Delta$-sentences.

*Satisfaction relation.* The satisfaction relation for propositional symbols is defined as follows: $(W, M) \models^w_\Delta \rho$ iff $\rho \in M_w$ for all $\Delta$-models $(W, M)$, possible worlds $w \in |W|$ and propositional symbols $\rho \in \mathsf{Prop}$. The satisfaction relation for complex sentences is based on Definition 2.11. The same remark holds for all stratified institutions presented in Section 2.2.3.

*Non-void signatures.* A signature $\Delta = (\mathsf{Nom}, \Lambda, \mathsf{Prop})$ is called non-void if $(\mathsf{Nom}, \Lambda)$ is a non-void REL signature. [6]

*Related logics.* Hybrid propositional logic defined, for example, in [8] is obtained by eliminating nominal relations. The "standard" hybrid propositional logic is obtained from HPL by eliminating nominal relations and by allowing one single binary modality for the signatures, i.e. for all signatures $(\mathsf{Nom}, \Lambda, \mathsf{Prop})$, $\Lambda_2 = \{\lambda\}$ and $\Lambda_n = \emptyset$ for all $n \neq 2$; in this case, $\langle\lambda\rangle\rho$ is denoted simply by $\Diamond\rho$.

[6] Since REL is a fragment of FOL, one can import the notion of non-void signature from FOL to REL; thus, $(\mathsf{Nom}, \Lambda)$ is non-void iff $\mathsf{Nom} \neq \emptyset$.

*Example 2.13 (Hybrid Propositional Logic with Quantification (*HPLQ*)).* This institution is obtained from HPL by adding to the grammar of HPL

$$e ::= \exists X \cdot e'$$

where $X$ is a finite set of nominal variables and $e'$ is a sentence over $\Delta[X]$. Notice that HPLQ is semantically closed under store, since $\downarrow z \cdot e''$ can be defined as an abbreviation for $\forall z \cdot z \Rightarrow e''$. Slight variations of HPLQ have been studied in [24, 36, 38].

*Example 2.14 (First-order hybrid logic (*HFOL*)).* This institution extends first-order logic with modalities and explicit syntax for the possible worlds in the same way hybrid propositional logic extends propositional logic. HFOL is built on the ideas used to define first-order modal logic [30].

*Models.* $\mathsf{Mod}^{\mathsf{HFOL}}$ is a sub-functor of the Kripke functor $\mathsf{Mod}^{\mathsf{FOL}}_{\kappa} \colon \mathsf{Sig}^{\mathsf{REL}} \times \mathsf{Sig}^{\mathsf{FOL}} \to \mathbb{C}\mathsf{at}^{op}$ such that for all signatures $\Delta = (\mathsf{Nom}, \Lambda, \Sigma)$, the following sharing conditions hold:

(1) $M_{w_1,x} = M_{w_2,x}$ for all $\Delta$-models $(W, M)$, all possible worlds $w_1, w_2 \in |W|$, and all sorts or constants $x$ in $\Sigma$;
(2) $h_{w_1,s} = h_{w_2,s}$ for all $\Delta$-homomorphisms $h : (W, M) \to (W', M')$, all possible worlds $w_1, w_2 \in |W|$ and all sorts $s \in S$.

*Sentences.* Given a signature $\Delta = (\mathsf{Nom}, \Lambda, \Sigma)$, the set of $\Delta$-sentences is given by the following grammar:

$$e ::= \rho \mid k \mid \underline{\lambda}(k) \mid \neg e \mid \vee E \mid @_k\, e \mid \langle\lambda\rangle e \mid \downarrow z \cdot e'$$

where (a) $\rho$ is a first-order atomic $\Sigma$-sentence, (b) $k$ is a nominal, (c) $\lambda$ is a binary modality, (d) $E$ is a finite set of $\Delta$-sentences, (e) $z$ is a nominal variable and $e'$ is a sentence over $\Delta[z]$.

*Satisfaction relation.* The satisfaction of first-order atomic sentences is defined by $(W, M) \models^{w}_{\Delta} \rho$ iff $M_w \models^{\mathsf{FOL}}_{\Sigma} \rho$ for all signatures $\Delta = (\mathsf{Nom}, \Lambda, \Sigma)$, first-order atomic $\Sigma$-sentences $\rho$, $\Delta$-models $(W, M)$ and possible worlds $w \in |W|$.

*Related logics.* Notice that HFOL is developed from many-sorted first-order logic with equality using multi-modalities; quantification over nominals is allowed only in a weak form through possibility over binary modalities. The present approach is different from [23] or [24], where HFOL is developed from first-order modal logic by allowing quantification over nominals and defining $\downarrow z \cdot e'$ as an abbreviation for $\forall z \cdot z \Rightarrow e'$. This presentation of HFOL corresponds to the hybridization ideas used, for example, in [10].

*Example 2.15 (*HREL*).* This is a sub-institution of HFOL obtained by restricting the signatures $(\mathsf{Nom}, \Lambda, \Sigma)$ such that $\Sigma$ is a REL signature $(F, P)$.

*Example 2.16 (Hybrid First-Order Logic with user-defined Sharing (*HFOLS*)).* This institution is obtained by generalizing the construction of HFOL such that the shared symbols are gathered in a "rigid" subsignature.

Notice that the functor $\mathsf{Mod}^{\mathsf{FOLR}}_{\kappa} \colon \mathsf{Sig}^{\mathsf{REL}} \times \mathsf{Sig}^{\mathsf{FOLR}} \to \mathbb{C}\mathsf{at}^{op}$ is obtained by applying Definition 2.9 to the functor $\mathsf{Mod}^{\mathsf{FOLR}}$ defined in Example 2.6. The functor $\mathsf{Mod}^{\mathsf{HFOLS}}$ is a sub-functor of $\mathsf{Mod}^{\mathsf{FOLR}}_{\kappa}$ which restricts the models and the homomorphisms of $\mathsf{Mod}^{\mathsf{FOLR}}_{\kappa}$ such that the rigid symbols have the same interpretation across the worlds; that is:

(1) for all signatures $\Delta = (\mathsf{Nom}, \Lambda, \Sigma^{\mathsf{r}} \subseteq \Sigma) \in |\mathsf{Sig}^{\mathsf{HFOLS}}|$ and all models $(W, M) \in |\mathsf{Mod}^{\mathsf{FOLR}}_{\kappa}(\Delta)|$,

$$(W, M) \in |\mathsf{Mod}^{\mathsf{HFOLS}}(\Delta)| \text{ iff } M_{w_1} \restriction_{\Sigma^{\mathsf{r}}} = M_{w_2} \restriction_{\Sigma^{\mathsf{r}}} \text{ for all possible worlds } w_1, w_2 \in |W|;$$

(2) for all signatures $\Delta = (\mathsf{Nom}, \Lambda, \Sigma^{\mathsf{r}} \subseteq \Sigma) \in |\mathsf{Sig}^{\mathsf{HFOLS}}|$ and all homomorphisms $h \in \mathsf{Mod}^{\mathsf{FOLR}}_{\kappa}(\Delta)$,

$$h \in \mathsf{Mod}^{\mathsf{HFOLS}}(\Delta) \text{ iff } h_{w_1} \restriction_{\Sigma^{\mathsf{r}}} = h_{w_2} \restriction_{\Sigma^{\mathsf{r}}} \text{ for all possible worlds } w_1, w_2 \in |W|.$$

The set of $\Delta$-sentences is given by the following grammar:

$$e ::= \rho \mid k \mid \underline{\lambda}(k_1, \ldots, k_n) \mid \neg e \mid \vee E \mid @_k\, e \mid \exists X, Y \cdot e'$$

where (a) $\rho$ is an atomic sentence in FOLR, (b) $k$ is a nominal, (c) $\underline{\lambda}(k_1, \ldots, k_n)$ is a nominal relation, (d) $E$ is a finite set of $\Delta$-sentences, (e) $X$ is a finite set of nominal variables, $Y$ is a finite set of variables of rigid sorts, $\exists X, Y \cdot \_$ is an abbreviation for $\exists \chi \cdot \_$ with $\chi \colon \Delta \hookrightarrow \Delta[X, Y]$, $\Delta[X, Y] = (\mathsf{Nom}[X], \Lambda, \Sigma[Y])$ and $\Sigma[Y] = (S^{\mathsf{r}}, F^{\mathsf{r}}[Y], P^{\mathsf{r}}) \subseteq (S, F[Y], P)$, and (f) $e'$ is a sentence over the signature $\Delta[X, Y]$.

The satisfaction of atoms is defined by: $(W, M) \models^{w} \rho$ iff $M_w \models^{\mathsf{FOLR}} \rho$, for all signatures $\Delta = (\mathsf{Nom}, \Lambda, \Sigma)$, atomic sentences $\rho \in \mathsf{Sen}^{\mathsf{FOLR}}(\Sigma)$, $\Delta$-models $(W, M)$ and possible worlds $w \in |W|$.

Notice that HFOLS is semantically closed under possibility and store, since $\langle \lambda \rangle e$ and $\downarrow z \cdot e''$ can be defined as abbreviations for $\exists z \cdot \underline{\lambda}(z) \wedge @_z\, \chi_z(e)$ and $\forall z \cdot z \Rightarrow e''$, respectively. Variants of HFOLS have been used in works such as [23, 26, 50].

*Example 2.17 (Hybrid First-Order Logic with rigid symbols (*HFOLR*)).* The key point in defining HFOLR is to use retrieve to rigidify not merely sentences but other types of symbols as well.

*Models.* This institution has the same model functor as HFOLS, i.e. $\mathsf{Mod}^{\mathsf{HFOLR}} := \mathsf{Mod}^{\mathsf{HFOLS}}$.

*Hybrid terms.* Let $\Delta = (\mathsf{Nom}, \Lambda, \Sigma^{\mathsf{r}} \subseteq \Sigma)$ be a HFOLR signature, where $\Sigma^{\mathsf{r}} = (S^{\mathsf{r}}, F^{\mathsf{r}}, P^{\mathsf{r}})$ and $\Sigma = (S, F, P)$. As in case of FOLR, we let $S^{\mathsf{f}} = S \setminus S^{\mathsf{r}}$, and $F^{\mathsf{f}}$ and $P^{\mathsf{f}}$ be the sub-families of $F$ and $P$ that consist of *flexible symbols* (obtained by removing rigid symbols). The *rigidification* of $\Sigma$ with respect to Nom is the signature $@_{\mathsf{Nom}}\, \Sigma = (@_{\mathsf{Nom}}\, S, @_{\mathsf{Nom}}\, F, @_{\mathsf{Nom}}\, P)$, where

(1) $@_{\mathsf{Nom}}\, S = \{@_k s \mid k \in \mathsf{Nom} \text{ and } s \in S\}$,
(2) $@_{\mathsf{Nom}}\, F = \{@_k \sigma \colon @_k \mathsf{ar} \to @_k s \mid k \in \mathsf{Nom} \text{ and } (\sigma \colon \mathsf{ar} \to s) \in F\}$, [7] and
(3) $@_{\mathsf{Nom}}\, P = \{@_k \pi \colon @_k \mathsf{ar} \mid k \in \mathsf{Nom} \text{ and } (\pi \colon \mathsf{ar}) \in P\}$.

Since the rigid symbols have the same interpretation across the worlds, we further define $@_k x = x$ for all nominals $k \in \mathsf{Nom}$ and all symbols $x$ in $\Sigma^{\mathsf{r}}$. The subscript Nom may be dropped from the above notations when there is no danger of confusion. The set of *rigid* $\Delta$*-terms* is $T_{@\Sigma}$, while the set of *open* $\Delta$*-terms* is $T_{\Sigma}$. The set of *hybrid* $\Delta$*-terms* is $T_{\overline{\Sigma}}$, where $\overline{\Sigma} = (\overline{S}, \overline{F}, \overline{P})$, $\overline{S} = S \cup @S^{\mathsf{f}}$, $\overline{F} = F \cup @F^{\mathsf{f}}$, and $\overline{P} = P \cup @P^{\mathsf{f}}$.

*Sentences.* Given a signature $\Delta = (\mathsf{Nom}, \Lambda, \Sigma)$, the proper atomic $\Delta$-sentences consist of

(S1) hybrid equations $t =_s t'$, where $t, t' \in T_{\overline{\Sigma}, s}$, and $s \in \overline{S}$, and
(S2) hybrid relations $\pi(t)$, where $(\pi \colon \mathsf{ar}) \in \overline{P}$ and $t \in T_{\overline{\Sigma}, \mathsf{ar}}$.

The set of $\Delta$-sentences is given by the following grammar:

$$e ::= t_1 = t_2 \mid \pi(t) \mid k \mid \underline{\lambda}(k_1, \ldots, k_n) \mid \neg e \mid \vee E \mid @_k\, e \mid \exists X, Y \cdot e'$$

where (a) $t_1 = t_2$ is a hybrid equation, (b) $\pi(t)$ is a hybrid relation, (c) $k$ is a nominal, (d) $\underline{\lambda}(k_1, \ldots, k_n)$ is a nominal relation, (e) $E$ is a finite set of $\Delta$-sentences, (f) $X$ is a finite set of nominal variables, $Y$ is a finite set of variables of rigid sorts, and $e'$ is a sentence over the signature $\Delta[X, Y]$.

*Satisfaction relation.* The satisfaction of proper atomic sentences is based on the interpretation of the hybrid terms into Kripke structures. For any $\Delta$-model $(W, M)$, any possible world $w \in |W|$, and any hybrid $\Delta$-term $t$,

(1) $M_{w, \sigma(t)} = (M_{w, \sigma})(M_{w, t})$, where $(\sigma \colon \mathsf{ar} \to s) \in F$; [8]
(2) $M_{w, (@_k \sigma)(t)} = (M_{w', \sigma})(M_{w, t})$, where $(@_k \sigma \colon @_k \mathsf{ar} \to @_k s) \in @F^{\mathsf{f}}$ and $w' = W_k$.

The satisfaction of proper atomic sentences is defined by

[7] $@_k(s_1 \ldots s_n) = @_k s_1 \ldots @_k s_n$ for all arities $s_1 \ldots s_n$.
[8] $M_{w, (t_1, \ldots, t_2)} = M_{w, t_1}, \ldots, M_{w, t_n}$ for all lists of hybrid terms $t_1, \ldots, t_n$.

(S1) $M \models^w t_1 = t_2$ iff $M_{w,t_1} = M_{w,t_2}$;
(S2) $M \models^w \pi(t)$ if $M_{w,t} \in M_{w,\pi}$.

*Non-void signatures.* A signature $\Delta = (\mathsf{Nom}, \Lambda, \Sigma^{\mathsf{r}} \subseteq \Sigma)$ is called *non-void* if $(\mathsf{Nom}, \Lambda)$ is a non-void REL signature and $\Sigma$ is a non-void FOL signature.

Lemma 2.18. *If $\Delta$ is non-void then there exists an initial model of terms $(W^\Delta, M^\Delta)$ defined as follows: (1) $W^\Delta = \mathsf{Nom}$, and (2) $M^\Delta \colon \mathsf{Nom} \to |\mathsf{Mod}^{\mathsf{FOLR}}(\Sigma^{\mathsf{r}} \subseteq \Sigma)|$, where $M^\Delta_k$ is a* FOLR *model such that (a) for all $s \in S$, $M^\Delta_{k,s} = T_{@\Sigma,@_k s}$, (b) for all $(\sigma \colon \mathsf{ar} \to s) \in F$, $M^\Delta_{k,\sigma} \colon T_{@\Sigma,@_k \mathsf{ar}} \to T_{@\Sigma,@_k s}$ is defined by $M^\Delta_{k,\sigma}(t) = (@_k\sigma)(t)$ for all lists of hybrid terms $t \in T_{@\Sigma,@_k \mathsf{ar}}$, and (c) for all $(\pi \colon \mathsf{ar}) \in P$, $M^\Delta_{k,\pi}$ is the empty set.*

The proof of Lemma 2.18 is based on the unique interpretation of terms into models, and it is straightforward. Therefore, we leave it as an exercise for the interested reader.

*Related logics.* This institution is defined using ideas from [9] to define Rigid First-Order Hybrid Logic and [36, 38] to define Hybrid First-Order Logic with user-defined Sharing and Annotation. Rigid first-order hybrid logic [9] is single-sorted, and its unique sort is rigid, while all function and relation symbols (except variables) are flexible; HFOLR is many-sorted, and it has flexible sorts exactly as hybrid first-order logic with user-defined sharing and annotation [36, 38]. HFOLR relies on hybrid terms exactly as rigid first-order hybrid logic; this approach is different from [36, 38], which relies only on rigid terms to construct sentences.

Fact 2.1. HFOLS *is a fragment of* HFOLR, *as* $\mathsf{Sen}^{\mathsf{HFOLS}}(\Delta) \subseteq \mathsf{Sen}^{\mathsf{HFOLR}}(\Delta)$ *for all* HFOLS *signatures $\Delta$. Similarly, rigid first-order hybrid logic [9] and hybrid first-order logic with user-defined sharing and annotation [36, 38] are fragments of* HFOLR.

Despite Fact 2.1, HFOLR has the same expressivity power as HFOLS and hybrid first-order logic with user-defined sharing and annotation. However, the atomic layer of HFOLR is more expressive, which is very useful in applications.

*2.2.4* HFOLR *vs.* HFOLS. We show that HFOLS has at least the same expressive power as HFOLR: for any HFOLR sentence $e$ there exists a HFOLS sentence $\gamma_e$ such that $(W, M) \models^w e$ iff $(W, M) \models^w \gamma_e$ for all Kripke structures $(W, M)$ and all possible worlds $w \in |W|$. Notice that it suffices to prove this property for each proper atomic sentence $\rho$ in HFOLR.

Let $\Delta = (\mathsf{Nom}, \Lambda, \Sigma^{\mathsf{r}} \subseteq \Sigma)$ be a HFOLS signature, where $\Sigma^{\mathsf{r}} = (S^{\mathsf{r}}, F^{\mathsf{r}}, P^{\mathsf{r}})$ and $\Sigma = (S, F, P)$. Let $X_\Delta$ be a set of rigid variables for $\Delta$ such that $\mathsf{X}_{\Delta,s}$ is infinite and countable for all rigid sorts $s \in S^{\mathsf{r}}$. For any hybrid term $t \in T_{\overline{\Sigma}}$ we denote by $\mathsf{Sub}_{\mathsf{r}}(t)$ the set of all subterms of $t$ that have a rigid sort and are maximal w.r.t. the subterm relation among the subterms with rigid sort; this means that if $t_1, t_2 \in \mathsf{Sub}_{\mathsf{r}}(t)$ and $t_1$ is a subterm of $t_2$ then $t_1 = t_2$; in particular, if $t$ is a term of rigid sort then $\mathsf{Sub}_{\mathsf{r}}(t) = \{t\}$. Given a context where a finite set of variables from $\mathsf{X}_\Delta$ is used, we assume that for each finite set of hybrid terms of rigid sort $\{t_1, \ldots, t_n\}$, we have an algorithmic way to choose $n$ new distinct variables $x_{t_1}, \ldots x_{t_n}$ from $\mathsf{X}_\Delta$ such that $x_{t_i}$ has the same sort as $t_i$ for all $i \in \{1, \ldots, n\}$.

Let $\upsilon_\Delta \colon T_{\overline{\Sigma}} \to T_{\overline{\Sigma}}(\mathsf{X}_\Delta)$ be the function which maps a term $t$ to a term obtained from $t$ by substituting $x_{t'}$ for $t'$, for each $t' \in \mathsf{Sub}_{\mathsf{r}}(t)$. In particular, if $t$ is a term of rigid sort then $\upsilon_\Delta(t) = x_t$.

Let $\delta_\Delta \colon T_{\overline{\Sigma}} \to T_\Sigma$ defined by (a) $\delta_\Delta(\sigma(t)) = \sigma(\delta_\Delta(t))$ for all $(\sigma \colon \mathsf{ar} \to s) \in F$ and $t \in T_{\overline{\Sigma},\mathsf{ar}}$, and (b) $\delta_\Delta((@_k\sigma)(t)) = \sigma(\delta_\Delta(t))$ for all $(\sigma \colon \mathsf{ar} \to s) \in F^{\mathsf{f}}$, $k \in \mathsf{Nom}$ and $t \in T_{\overline{\Sigma},@_k\mathsf{ar}}$.

We define $\phi_\Delta \colon \mathsf{Sen}^{\mathsf{HFOLR}_0}(\Delta) \to \mathsf{Sen}^{\mathsf{HFOLS}}(\Delta)$ for all atomic $\Delta$-sentences by induction on the structure of atomic sentences, simultaneously for all signatures $\Delta$:

(1) For hybrid equations:
[ $s \in S^{\mathsf{r}}$ *and* $c, c' \in F^{\mathsf{r}}_{\to s}$ ] $\phi_\Delta(c = c') := (c = c')$.

[ $s \in S^{\mathsf{r}}, c \in F^{\mathsf{r}}_{\to s}$ *and* $\sigma \in F^{\mathsf{r}}$ *is not a constant* ] $\phi_\Delta(c = \sigma(t_1, \ldots, t_n)) :=$
$\exists x_{t_1}, \ldots, x_{t_n} \cdot \bigwedge_{i=1}^{n} \phi_{\Delta[Y]}(x_{t_i} = t_i) \wedge c = \sigma(x_{t_1}, \ldots, x_{t_n})$,
where $Y = \{x_{t_1}, \ldots, x_{t_n}\}$.

[ $s \in S^{\mathsf{r}}, c \in F^{\mathsf{r}}_{\to s}$ *and* $\sigma \in F^{\mathsf{f}}$ ] $\phi_\Delta(\, c = \sigma(t_1, \ldots, t_n)\,) :=$
$\exists x_{t'_1}, \ldots, x_{t'_m} \cdot \bigwedge_{i=1}^{m} \phi_{\Delta[Y]}(x_{t'_i} = t'_i) \wedge c = \sigma(\upsilon_\Delta(t_1), \ldots, \upsilon_\Delta(t_n))$,
where $\mathsf{Sub}_{\mathsf{r}}(\{t_1, \ldots, t_n\}) = \{t'_1, \ldots, t'_m\}$ and $Y = \{x_{t'_1}, \ldots, x_{t'_m}\}$.

[ $s \in S^{\mathsf{r}}, c \in F^{\mathsf{r}}_{\to s}$ *and* $@_k\sigma \in @F^{\mathsf{f}}$ ] $\phi_\Delta(\, c = (@_k\sigma)(t_1, \ldots, t_n)\,) :=$
$\exists x_{t'_1}, \ldots, x_{t'_m} \cdot \bigwedge_{i=1}^{m} \phi_{\Delta[Y]}(x_{t'_i} = t'_i) \wedge @_k\,(\, c = \sigma(\,\delta_{\Delta[Y]}(\upsilon_\Delta(t_1)), \ldots, \delta_{\Delta[Y]}(\upsilon_\Delta(t_n)))\,)$,
where $\mathsf{Sub}_{\mathsf{r}}(\{t_1, \ldots, t_n\}) = \{t'_1, \ldots, t'_m\}$ and $Y = \{x_{t'_1}, \ldots, x_{t'_m}\}$.

[ $s \in S^{\mathsf{r}}$ *and* $t, t' \in T_{\overline{\Sigma}, s}$ *are terms different from constants* ] $\phi_\Delta(t = t') :=$
$\exists x_t, x_{t'} \cdot \phi_{\Delta[Y]}(x_t = t) \wedge \phi_{\Delta[Y]}(x_{t'} = t') \wedge x_t = x_{t'}$,
where $Y = \{x_t, x_{t'}\}$.

[ $s \in S^{\mathsf{f}}$ *and* $t, t' \in T_{\overline{\Sigma}, s}$ ] $\phi_\Delta(t = t') :=$
$\exists x_{t_1}, \ldots x_{t_n} \cdot \bigwedge_{i=1}^{n} \phi_{\Delta[Y]}(x_{t_i} = t_i) \wedge \upsilon_\Delta(t) = \upsilon_\Delta(t')$,
where $\mathsf{Sub}_{\mathsf{r}}(\{t, t'\}) = \{t_1, \ldots, t_n\}$ and $Y = \{x_{t_1}, \ldots, x_{t_n}\}$.

[ $@_k s \in @S^{\mathsf{f}}$ *and* $t, t' \in T_{\overline{\Sigma}, @_k s}$ ] $\phi_\Delta(t = t') :=$
$\exists x_{t_1}, \ldots x_{t_n} \cdot \bigwedge_{i=1}^{n} \phi_{\Delta[Y]}(x_{t_i} = t_i) \wedge @_k\,(\,\delta_{\Delta[Y]}(\upsilon_\Delta(t)) = \delta_{\Delta[Y]}(\upsilon_\Delta(t'))\,)$
where $\mathsf{Sub}_{\mathsf{r}}(\{t, t'\}) = \{t_1, \ldots, t_n\}$ and $Y = \{x_{t_1}, \ldots, x_{t_n}\}$.

(2) For hybrid relations:
[ $\pi \in P^{\mathsf{r}}$ ] $\phi_\Delta(\pi(t_1, \ldots, t_n)) :=$
$\exists x_{t_1}, \ldots x_{t_n} \cdot \bigwedge_{i=1}^{n} \phi_{\Delta[Y]}(x_{t_i} = t_i) \wedge \pi(x_{t_1}, \ldots, x_{t_n})$,
where $Y = \{x_{t_1}, \ldots, x_{t_n}\}$.

[ $\pi \in P^{\mathsf{f}}$ ] $\phi_\Delta(\pi(t_1, \ldots, t_n)) :=$
$\exists x_{t'_1}, \ldots x_{t'_m} \cdot \bigwedge_{i=1}^{m} \phi_{\Delta[Y]}(x_{t'_i} = t'_i) \wedge \pi(\,\upsilon_\Delta(t_1), \ldots, \upsilon_\Delta(t_n)\,)$,
where $\mathsf{Sub}_{\mathsf{r}}(\{t_1, \ldots, t_n\}) = \{t'_1, \ldots, t'_m\}$ and $Y = \{x_{t'_1}, \ldots, x_{t'_m}\}$.

[ $@_k\pi \in @P^{\mathsf{f}}$ ] $\phi_\Delta(\,(@_k\pi)(t_1, \ldots, t_n)\,) :=$
$\exists x_{t'_1}, \ldots x_{t'_m} \cdot \bigwedge_{i=1}^{m} \phi_{\Delta[Y]}(x_{t'_i} = t'_i) \wedge @_k\,(\,\pi(\,\delta_{\Delta[Y]}(\upsilon_\Delta(t_1)), \ldots, \delta_{\Delta[Y]}(\upsilon_\Delta(t_n)))\,)$,
$\mathsf{Sub}_{\mathsf{r}}(\{t_1, \ldots, t_n\}) = \{t'_1, \ldots, t'_m\}$ and $Y = \{x_{t'_1}, \ldots, x_{t'_m}\}$.

(3) For nominals and nominal relations:
$\phi_\Delta$ is the identity on nominals and nominal relations.

As usual, we drop the subscript $\Delta$ from notations when it is clear from the context.

*Example 2.19.* Let $\Delta$ be the signature defined as follows: (a) $\mathsf{Nom} = \{k_1, k_2\}$, (b) $S^{\mathsf{r}} = \{s_1\}$, $S = \{s_1, s_2\}$, (c) $F = \{(c_1\colon \to s_1), (c_2\colon \to s_2), (f\colon s_1 \to s_2)\}$, and (d) the rest of the signature components do not contain any symbols.

Note that $\phi_\Delta(\,(@_{k_2} f)(@_{k_1} c_1) = @_{k_2} c_2\,)$ is $\exists y \cdot @_{k_1}\,(y = c_1) \wedge @_{k_2}\,(\,f(y) = c_2\,)$, where $y = x_{(@_{k_1} c_1)}$, and both sentences are satisfied by the same class of Kripke structures.

Lemma 2.20. *For all atomic sentences* $\rho \in \mathsf{Sen}^{\mathsf{HFOLR}_0}(\Delta)$, *Kripke structures* $(W, M)$ *and possible worlds* $w \in |W|$, *we have* $(W, M) \models^w \rho$ *iff* $(W, M) \models^w \phi_\Delta(\rho)$.

Proof. We focus on hybrid equations as the case corresponding to hybrid relations is similar.

Firstly, we show that $(W, M) \models^w c = t$ iff $(W, M) \models^w \phi(c = t)$ for all Kripke structures $(W, M)$ and all hybrid equations $c = t$ such that $c$ is a rigid constant. We prove the statement by induction

on the height of $t$: we assume that the property holds for all hybrid terms of rigid sorts of height strictly less than the height of $t$, and then we prove it for $t$. The most interesting case is when $t = (@_k\sigma)(t_1, \ldots, t_n)$, where $k \in \mathsf{Nom}$, $\sigma \in F^{\mathsf{f}}$, and $t_i \in T_{\overline{\Sigma}}$.

Assume $(W, M) \models^w c = t$ and prove $(W, M) \models^w \phi(c = t)$:

| | | |
|---|---|---|
| 1 | let $\mathsf{Sub}_{\mathsf{r}}(\{t_1, \ldots, t_n\}) = \{t'_1, \ldots, t'_m\}$ and $Y = \{x_{t'_1}, \ldots, x_{t'_m}\}$ | |
| 2 | let $(W, M')$ be the expansion of $(W, M)$ to $\Delta[Y]$ such that $M'_{w, x_{t'_i}} = M_{w, t'_i}$ for all $i \in \{1, \ldots, m\}$ | |
| 3 | $M'_{W_k, c} = M'_{w, c}$ | since $c$ is rigid |
| 4 | $M'_{w, c} = M'_{w, t}$ | by the satisfaction condition from $(W, M) \models^w c = t$ |
| 5 | $M'_{w, t} = M'_{w, v(t)}$ | by the definition of $v$ |
| 6 | $M'_{w, v(t)} = M'_{W_k, \delta(v(t))}$ | since $v(t)$ is a rigid term and $k$ is the only nominal occurring in it |
| 7 | $M'_{W_k, c} = M'_{W_k, \delta(v(t))}$ | by 3 - 6 |
| 8 | $(W, M') \models^w_{\Delta[Y]} @_k\,(\,c = \delta(v(t))\,)$ | by the definition of $\models^{\mathsf{HFOLS}}$ |
| 9 | $(W, M') \models^w_{\Delta[Y]} x_{t'_i} = t'_i$ iff $(W, M) \models^w_{\Delta[Y]} \phi(x_{t'_i} = t'_i)$ | by the induction hypothesis |
| 10 | $(W, M) \models^w \phi(c = t)$ | by the definition of $\phi$ from 8 and 9 |

Assume that $(W, M) \models^w \phi(c = t)$, where $w \in |W|$. One can prove $(W, M) \models^w c = t$ by applying backwards the arguments used above.

Secondly, we prove that $(W, M) \models^w t = t'$ iff $(W, M) \models^w \phi(t = t')$ for all hybrid equations $t = t'$ and all possible worlds $w \in |W|$. By the definition of $\phi$, there are three cases which are similar to the case presented above. □

## 3 INSTITUTION-INDEPENDENT CONCEPTS

In this section, we investigate some concepts necessary to prove our abstract results:

(1) *quantification space*, which describes the properties of the signature morphisms used for quantification,
(2) *substitution*, an abstract description of mapping of variables to terms,
(3) *signature extension*, a category-based definition of signature extensions with an infinite number of constants,
(4) *reachable model*, a complete abstract characterization of the models with elements constructed from syntactic compounds,
(5) *basic set of sentences*, an institution-independent description of the sets of atomic sentences of some major classes of institutions.

### 3.1 Quantification space

A variable can be regarded as a special constant, and in a category-based logical setting, one can capture variables for a signature through signature morphisms. Quantification comes with some subtle issues related to the translation of quantified sentences along signature morphisms that will be discussed in this section.

*Definition 3.1 (Quantification space [23]).* Given a category Sig, a subclass of arrows $\mathsf{Q} \subseteq \mathsf{Sig}$ is called a *quantification space* if it satisfies the following properties:

(1) for any signature morphisms $\Sigma \xrightarrow{\chi} \Sigma' \in \mathsf{Q}$ and $\Sigma \xrightarrow{\varphi} \Sigma_1 \in \mathsf{Sig}$ there is a designated pushout such as the one depicted in the left side of Figure 1 such that $\chi(\varphi) \in \mathsf{Q}$;
(2) the horizontal composition of such designated pushouts is again a designated pushout:
  (a) $\chi(1_\Sigma) = \chi$, $1_\Sigma[\chi] = 1_{\Sigma'}$, and

$$\begin{tikzcd} \Sigma' \arrow[r, "\varphi[\chi]"] & \Sigma'_1 \\ \Sigma \arrow[u, "\chi"] \arrow[r, "\varphi"'] & \Sigma_1 \arrow[u, "\chi(\varphi)"'] \end{tikzcd} \qquad \begin{tikzcd} \Sigma' \arrow[r, "\varphi[\chi]"] & \Sigma'_1 \arrow[r, "\psi[\chi(\varphi)]"] & \Sigma'_2 \\ \Sigma \arrow[u, "\chi"] \arrow[r, "\varphi"'] & \Sigma_1 \arrow[u, "\chi(\varphi)"'] \arrow[r, "\psi"'] & \Sigma_2 \arrow[u, "\chi(\varphi)(\psi)"'] \end{tikzcd}$$

Fig. 1. Quantification pushouts

(b) for all pushouts such as the ones in the right side of Figure 1 we have $\varphi[\chi];\psi[\chi(\varphi)] = (\varphi;\psi)[\chi]$ and $\chi(\varphi)(\psi) = \chi(\varphi;\psi)$.

In concrete examples of institutions, the quantification space is fixed and the translation of a quantified sentence $\exists\chi \cdot e' \in \mathsf{Sen}(\Sigma)$ along $\varphi$ is $\exists\chi(\varphi) \cdot \varphi[\chi](e')$. The second condition in Definition 3.1 is required by the functoriality of the translations.

*Definition 3.2 (Commutativity of signature morphisms with sentence building operators).* Let $\mathtt{SI}$ be a stratified institution with nominal variables equipped with a quantification space $\mathsf{Q}^{\mathtt{SI}}$; this means that for all signatures $\Delta$ and all nominal variables $z$ for $\Delta$, we have $\chi_z \in \mathsf{Q}^{\mathtt{SI}}$.

A signature morphism $\varphi\colon \Delta \to \Delta_1$ commutes with the sentence building operators if:

(1) the translation of a nominal variable $z$ along a signature morphism $\varphi\colon \Delta \to \Delta_1$ is again a nominal variable, i.e. $\chi_z(\varphi) = \chi_{z_1}$ and $\mathsf{F}^{\mathtt{SI}}(\varphi[\chi_z])(z) = z_1$, for some nominal variable $z_1$;
(2) if $\neg e \in \mathsf{Sen}^{\mathtt{SI}}(\Delta)$, meaning that the negation of $e$ is a sentence of $\mathtt{SI}$, then $\varphi(\neg e) = \neg\varphi(e)$;
(3) if $\vee E \in \mathsf{Sen}^{\mathtt{SI}}(\Delta)$ then $\varphi(\vee E) = \vee\varphi(E)$;
(4) if $@_k\, e \in \mathsf{Sen}^{\mathtt{SI}}(\Delta)$ then $\varphi(@_k\, e) = @_{k_1}\, \varphi(e)$, where $k_1 = \mathsf{F}(\varphi)(k)$;
(5) if $\langle\lambda\rangle e \in \mathsf{Sen}^{\mathtt{SI}}(\Delta)$ then $\varphi(\langle\lambda\rangle e) = \langle\lambda'\rangle\varphi(e)$, where $\lambda' = \mathsf{F}(\varphi)(\lambda)$;
(6) if $\downarrow z \cdot e' \in \mathsf{Sen}^{\mathtt{SI}}(\Delta)$ then $\varphi(\downarrow z \cdot e') = \downarrow z_1 \cdot \varphi[\chi_z](e')$, where $z_1$ is as described above at (1);
(7) if $\exists\chi \cdot e' \in \mathsf{Sen}^{\mathtt{SI}}(\Delta)$ then $\chi \in \mathsf{Q}^{\mathtt{SI}}$ and $\varphi(\exists\chi \cdot e') = \exists\chi(\varphi) \cdot \varphi[\chi](e')$.

Each stratified institution presented in this paper is equipped with a quantification space such that its signature morphisms commute with the sentence building operators.

*3.1.1 Examples.* We give some examples of quantification spaces for the institutions defined above. To this end, we assume a countably infinite set $\{\mathfrak{x}_i \mid i \in \mathbb{N}\}$ of variables names.

*Example 3.3 ($\mathsf{Q}^{\mathsf{FOL}}$).* A first-order variable for a signature $\Sigma = (S, F, P)$ is a triple $(\mathfrak{x}_i, s, \Sigma)$, where $i \in \mathbb{N}$, $\mathfrak{x}_i$ is the name of the variable, and $s \in S$ is the sort of the variable. In FOL, the quantification space $\mathsf{Q}^{\mathsf{FOL}}$ consists of signature extensions with a finite number of variables of the form $\chi\colon \Sigma \hookrightarrow \Sigma[X]$, where $\Sigma$ is a first-order signature and $X = \{X_s\}_{s\in S}$ is a finite set of variables for $\Sigma$. Given a signature morphism $\varphi\colon \Sigma \to \Sigma_1$ in FOL, where $\Sigma_1 = (S_1, F_1, P_1)$, then

- $\chi(\varphi)\colon \Sigma_1 \hookrightarrow \Sigma_1[X^\varphi]$ is an inclusion, where $X^\varphi = \{(\mathfrak{x}_i, \varphi(s), \Sigma_1) \mid (\mathfrak{x}_i, s, \Sigma) \in X\}$,
- $\varphi[\chi]$ is the extension of $\varphi$ that maps each $(\mathfrak{x}_i, s, \Sigma)$ to $(\mathfrak{x}_i, \varphi(s), \Sigma_1)$.

*Example 3.4 ($\mathsf{Q}^{\mathsf{HPL}}$ and $\mathsf{Q}^{\mathsf{HPLQ}}$).* Notice that HPL allows a weak form of quantification through possibility over binary modalities. Therefore, HPL has a non-trivial quantification space different from the class of identity signature morphisms. Let $\Delta = (\mathsf{Nom}, \Lambda, \mathsf{Prop})$ be a signature in HPL. A nominal variable for $\Delta$ is a variable $(\mathfrak{x}_i, \mathsf{Nom}, \Lambda)$ for the REL signature $(\mathsf{Nom}, \Lambda)$. The quantification space $\mathsf{Q}^{\mathsf{HPL}}$ for HPL consists of signature extensions with a finite number of nominal variables of the form $\Delta \hookrightarrow \Delta[X]$, where (1) $\Delta = (\mathsf{Nom}, \Lambda, \mathsf{Prop})$ is a signature in HPL, (2) $X$ is a finite set of nominal variables, and (3) $\Delta[X] = (\mathsf{Nom}[X], \Lambda, \mathsf{Prop})$. If $\varphi\colon \Delta \to \Delta_1$ is a signature morphism in HPL, where $\Delta = (\mathsf{Nom}, \Lambda, \mathsf{Prop})$ and $\Delta_1 = (\mathsf{Nom}_1, \Lambda_1, \mathsf{Prop}_1)$, then

- $\chi(\varphi)\colon \Delta_1 \hookrightarrow \Delta_1[X^\varphi]$ is an inclusion, where $X^\varphi = \{(\mathfrak{x}_i, \mathsf{Nom}_1, \Lambda_1) \mid (\mathfrak{x}_i, \mathsf{Nom}, \Lambda) \in X\}$, and

- $\varphi[\chi]\colon \Delta[X] \to \Delta_1[X^\varphi]$ is the extension of $\varphi$ that maps each nominal variable $(\mathfrak{x}_i, \mathsf{Nom}, \Lambda) \in X$ to $(\mathfrak{x}_i, \mathsf{Nom}_1, \Lambda_1) \in X^\varphi$.

The quantification space $\mathsf{Q}^{\mathsf{HPLQ}}$ for HPLQ is $\mathsf{Q}^{\mathsf{HPL}}$.

*Example 3.5 ($\mathsf{Q}^{\mathsf{HFOLR}}$ and $\mathsf{Q}^{\mathsf{HFOLS}}$).* In HFOLR, the quantification space $\mathsf{Q}^{\mathsf{HFOLR}}$ consists of signature extensions with a finite number of nominal and rigid variables of the form $\Delta \hookrightarrow \Delta[X, Y]$, where (1) $\Delta = (\mathsf{Nom}, \Lambda, \Sigma^{\mathsf{r}} \subseteq \Sigma)$ is a signature in HFOLR, (2) $X$ is a finite set of nominal variables, and (3) $Y$ is a finite set of rigid variables. If $\varphi\colon \Delta \to \Delta_1$ is a signature morphism in HFOLR, where $\Delta = (\mathsf{Nom}, \Lambda, \Sigma^{\mathsf{r}} \subseteq \Sigma)$ and $\Delta_1 = (\mathsf{Nom}_1, \Lambda_1, \Sigma_1^{\mathsf{r}} \subseteq \Sigma_1)$, then

- $\chi(\varphi)\colon \Delta_1 \hookrightarrow \Delta_1[X^\varphi, Y^\varphi]$ is an inclusion, where $X^\varphi = \{(\mathfrak{x}_i, \mathsf{Nom}_1, \Lambda_1) \mid (\mathfrak{x}_i, \mathsf{Nom}, \Lambda) \in X\}$ and $Y^\varphi = \{(\mathfrak{x}_j, \varphi(s), \Sigma_1^{\mathsf{r}} \subseteq \Sigma_1) \mid (\mathfrak{x}_j, s, \Sigma^{\mathsf{r}} \subseteq \Sigma) \in Y\}$, and
- $\varphi[\chi]\colon \Delta[X, Y] \to \Delta_1[X^\varphi, Y^\varphi]$ is the extension of $\varphi$ that maps each nominal variable $(\mathfrak{x}_i, \mathsf{Nom}, \Lambda) \in X$ to $(\mathfrak{x}_i, \mathsf{Nom}_1, \Lambda_1) \in X^\varphi$ and each rigid variable $(\mathfrak{x}_j, s, \Sigma^{\mathsf{r}} \subseteq \Sigma) \in Y$ to $(\mathfrak{x}_j, \varphi(s), \Sigma_1^{\mathsf{r}} \subseteq \Sigma_1) \in Y^\varphi$.

The quantification space $\mathsf{Q}^{\mathsf{HFOLS}}$ for HFOLS is $\mathsf{Q}^{\mathsf{HFOLR}}$.

When quantified sentences get translated along signature morphisms using the present approach, one avoids clashing of variables with the constants from the target signature. All examples of quantification spaces given above are not categories, as they are not stable under composition. Since all sorts are interpreted by models as non-empty sets, the signature morphisms used for quantification in this paper are conservative.

### 3.2 Substitutions

Since variables are represented by signature morphisms in institution theory, substitutions are defined between signature morphisms. We recall the notion of substitution in institutions.

*Definition 3.6 (Substitution [20]).* For any signature morphisms $\chi_1\colon \Sigma \to \Sigma_1$ and $\chi_2\colon \Sigma \to \Sigma_2$ in an institution, a $\Sigma$-*substitution* $\theta\colon \chi_1 \to \chi_2$ consists of a pair $(\mathsf{Sen}(\theta), \mathsf{Mod}(\theta))$, where $\mathsf{Sen}(\theta)\colon \mathsf{Sen}(\Sigma_1) \to \mathsf{Sen}(\Sigma_2)$ is a function and $\mathsf{Mod}(\theta)\colon \mathsf{Mod}(\Sigma_2) \to \mathsf{Mod}(\Sigma_1)$ is a functor, such that $\Sigma$ is preserved, i.e. the following diagrams commute,

$$\begin{tikzcd}
\mathsf{Sen}(\Sigma_1) \arrow[rr, "\mathsf{Sen}(\theta)"] & & \mathsf{Sen}(\Sigma_2) \\
& \mathsf{Sen}(\Sigma) \arrow[ul, "\mathsf{Sen}(\chi_1)"] \arrow[ur, "\mathsf{Sen}(\chi_2)"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{Mod}(\Sigma_1) \arrow[dr, "\mathsf{Mod}(\chi_1)"'] & & \mathsf{Mod}(\Sigma_2) \arrow[ll, "\mathsf{Mod}(\theta)"'] \arrow[dl, "\mathsf{Mod}(\chi_2)"] \\
& \mathsf{Mod}(\Sigma) &
\end{tikzcd}$$

and the following *satisfaction condition* holds:

$$\mathsf{Mod}(\theta)(M_2) \models e_1 \text{ iff } M_2 \models \mathsf{Sen}(e_1)$$

for each $\Sigma_2$-model $M_2$ and any $\Sigma_1$-sentence $e_1$.

The concept of substitution adopted in this paper accommodates also second-order substitutions, which replace function symbols with derived operations [22]. But a detailed discussion is beyond the scope of the present contribution. Note that a substitution $\theta\colon \chi_1 \to \chi_2$ is uniquely identified by its domain $\chi_1$, codomain $\chi_2$ and the pair $(\mathsf{Sen}(\theta), \mathsf{Mod}(\theta))$. When there is no danger of confusion, we let $\_\restriction_\theta$ denote the functor $\mathsf{Mod}(\theta)$, and $\theta$ denote the sentence translation $\mathsf{Sen}(\theta)$.

*Example 3.7 (FOL substitutions [20]).* Consider two signature inclusions $\chi_1\colon \Sigma \hookrightarrow \Sigma[C_1]$ and $\chi_2\colon \Sigma \hookrightarrow \Sigma[C_2]$, where $\Sigma = (S, F, P)$ is a first-order signature, and $C_i$ is a set of constant symbols different from the the constants in $F$. A function $\theta\colon C_1 \to T_{\Sigma[C_2]}$ represents a substitution between $\chi_1$ and $\chi_2$:

(1) On the syntactic side, $\theta$ can be canonically extended to a function $\mathsf{Sen}(\theta)\colon \mathsf{Sen}(\Sigma[C_1]) \to \mathsf{Sen}(\Sigma[C_2])$ that substitutes terms in $T_{\Sigma[C_2]}$ for constants in $C_1$ according to $\theta$.
(2) On the semantics side, $\theta$ determines a functor $\mathsf{Mod}(\theta)\colon \mathsf{Mod}(\Sigma[C_2]) \to \mathsf{Mod}(\Sigma[C_1])$ such that for all $\Sigma[C_2]$-models $M$ we have:
  (a) $\mathsf{Mod}(\theta)(M)_x = M_x$, for each sort $x \in S$, or operation symbol $x \in F$, or relation symbol $x \in P$;
  (b) $\mathsf{Mod}(\theta)(M)_{c_1} = M_{\theta(c_1)}$ for each $c_1 \in C_1$, where $M_{\theta(c_1)}$ is the interpretation of $\theta(c_1)$ in $M$.

*3.2.1 Substitution functors.* Given a signature $\Sigma$ in an institution $\mathtt{I}$, $\Sigma$-substitutions form a category $\mathsf{Sb}^{\mathtt{I}}(\Sigma)$, where

(1) *the objects* are signature morphisms $\chi\colon \Sigma \to \Sigma' \in \mathsf{Sig}^{\mathtt{I}}$, and
(2) *the arrows* are substitutions $\theta\colon \chi_1 \to \chi_2$ described in Definition 3.6, where the composition is performed componentwise.

For any signature morphism $\varphi\colon \Sigma_0 \to \Sigma$ one can easily define a reduct functor $\mathsf{Sb}^{\mathtt{I}}(\varphi)\colon \mathsf{Sb}^{\mathtt{I}}(\Sigma) \to \mathsf{Sb}^{\mathtt{I}}(\Sigma_0)$ which maps a $\Sigma$-substitution $\theta\colon \chi_1 \to \chi_2$ to the $\Sigma_0$-substitution $\mathsf{Sb}^{\mathtt{I}}(\varphi)(\theta)\colon \varphi;\chi_1 \to \varphi;\chi_2$ such that $\mathsf{Sen}^{\mathtt{I}}(\mathsf{Sb}^{\mathtt{I}}(\varphi)(\theta)) = \mathsf{Sen}^{\mathtt{I}}(\theta)$ and $\mathsf{Mod}^{\mathtt{I}}(\mathsf{Sb}^{\mathtt{I}}(\varphi)(\theta)) = \mathsf{Mod}^{\mathtt{I}}(\theta)$.

Lemma 3.8 ([36, 38]). *$\mathsf{Sb}^{\mathtt{I}}\colon \mathsf{Sig}^{\mathtt{I}} \to \mathbb{C}\mathsf{at}^{op}$ is a functor.*

In applications, not all substitutions are of interest, and it is often assumed a *substitution sub-functor* to work with.

*Definition 3.9 (Substitution functor).* A sub-functor $\mathsf{St}^{\mathtt{I}}\colon \mathsf{D}^{\mathtt{I}} \to \mathbb{C}\mathsf{at}^{op}$ of $\mathsf{Sb}^{\mathtt{I}}\colon \mathsf{Sig}^{\mathtt{I}} \to \mathbb{C}\mathsf{at}^{op}$ is a *substitution functor* if it satisfies the following properties:

(1) $\mathsf{D}^{\mathtt{I}} \subseteq \mathsf{Sig}^{\mathtt{I}}$ is a broad subcategory of signature morphisms, which means $|\mathsf{Sig}^{\mathtt{I}}| = |\mathsf{D}^{\mathtt{I}}|$;
(2) for any signature $\Sigma$,
  (a) the objects of $\mathsf{St}^{\mathtt{I}}(\Sigma)$ are all signature morphisms in $\mathsf{D}^{\mathtt{I}}$ with the domain $\Sigma$;
  (b) the arrows of $\mathsf{St}^{\mathtt{I}}(\Sigma)$ include all substitutions induced by the signature morphisms, i.e. if $\chi_1\colon \Sigma \to \Sigma_1 \in \mathsf{D}^{\mathtt{I}}$, $\chi_2\colon \Sigma \to \Sigma_2 \in \mathsf{D}^{\mathtt{I}}$ and $\varphi\colon \Sigma_1 \to \Sigma_2 \in \mathsf{Sig}^{\mathtt{I}}$ such that $\chi_1;\varphi = \chi_2$ then $(\mathsf{Sen}(\varphi), \mathsf{Mod}(\varphi)) \in \mathsf{St}^{\mathtt{I}}(\Sigma)(\chi_1, \chi_2)$.

When there is no danger of confusion, we may drop the superscript $\mathtt{I}$ from notations.

*Example 3.10 (*FOL *substitution functor [37]).* Let $\mathsf{D}^{\mathsf{FOL}} \subseteq \mathsf{Sig}^{\mathsf{FOL}}$ be the broad subcategory of signature extensions with constants. First-order substitutions are represented by functions $\theta\colon C_1 \to T_{\Sigma[C_2]}$, where $\Sigma$ is a first-order signature and $C_i$ are finite sets of new constants for $\Sigma$. Let $\mathsf{St}^{\mathsf{FOL}}\colon \mathsf{D}^{\mathsf{FOL}} \to \mathbb{C}\mathsf{at}^{op}$ denote the substitution functor which maps each signature $\Sigma$ to the subcategory of $\Sigma$-substitutions represented by functions $\theta\colon C_1 \to T_{\Sigma[C_2]}$ as in Example 3.7. Given a first-order signature $\Sigma$, it is straightforward to check that $\mathsf{St}^{\mathsf{FOL}}(\Sigma)$ satisfies the condition 2 in Definition 3.9:

(2a) the objects of $\mathsf{St}^{\mathsf{FOL}}(\Sigma)$ are all signature extensions with constants;
(2b) given $\chi_1\colon \Sigma \hookrightarrow \Sigma[C_1]$, $\chi_2\colon \Sigma \hookrightarrow \Sigma[C_3]$ and $\varphi\colon \Sigma[C_1] \to \Sigma[C_2]$ such that $\chi_1;\varphi = \chi_2$ then $\theta_\varphi\colon C_1 \to T_{\Sigma[C_2]}$ defined as the restriction of $\varphi$ to $C_1$ is the substitution induced by $\varphi$.

*3.2.2 Stratified substitutions.* We recall the notion of stratified substitution, which upgrades the concept of substitution to stratified institutions.

*Definition 3.11 (Stratified substitution [38]).* Given a stratified institution $\mathtt{SI}$, a *stratified substitution* between the signature morphisms $\chi_1\colon \Delta \to \Delta_1$ and $\chi_2\colon \Delta \to \Delta_2$ is a substitution $\theta\colon \chi_1 \to \chi_2$ such that the following *local satisfaction condition* holds:

$$M_2 \models^w \theta(e_1) \text{ iff } M_2 \restriction_\theta \models^w e_1$$

for each $\Delta_1$-sentence $e_1$, any $\Delta_2$-model $M_2$, and all possible worlds $w$ of $M_2$.

The following result shows that the local satisfaction condition for substitutions is well-defined, as substitutions preserve the possible worlds.

Lemma 3.12 ([38]). *$M_2$ and $M_2 \restriction_\theta$ from Definition 3.11 share the same possible worlds.*

Not only in first-order logics, but also in hybrid logics, we restrict the substitutions to mappings of constants to terms.

*Example 3.13 (*HFOLR *substitutions).* Consider two signature extensions with nominals and rigid constants $\chi_1 \colon \Delta \hookrightarrow \Delta[C_1, D_1]$ and $\chi_2 \colon \Delta \hookrightarrow \Delta[C_2, D_2]$, where $\Delta = (\mathsf{Nom}, \Lambda, \Sigma^{\mathsf{r}} \subseteq \Sigma)$ is a signature, $C_i$ is a set of nominals, and $D_i$ is a set of rigid constants. A pair of functions

$$\theta = \langle \theta_a \colon C_1 \to \mathsf{Nom}[C_2], \theta_b \colon D_1 \to T_{@\Sigma}(D_2) \rangle$$

represents a substitution between $\chi_1$ and $\chi_2$:

(1) On the syntactic side, $\theta$ determines a sentence translation
$\mathsf{Sen}^{\mathsf{HFOLR}}(\theta) \colon \mathsf{Sen}^{\mathsf{HFOLR}}(\Delta[C_1, D_1]) \to \mathsf{Sen}^{\mathsf{HFOLR}}(\Delta[C_2, D_2])$,
which preserves $\Delta$ and substitutes (a) nominals in $\mathsf{Nom}[C_2]$ for nominals in $C_1$ corresponding to $\theta_a$, and (b) $\Delta[C_2, D_2]$-terms for constants in $D_1$ according to $\theta_b$.
(2) On the semantics side, $\theta$ determines a model functor
$\mathsf{Mod}^{\mathsf{HFOLR}}(\theta) \colon \mathsf{Mod}^{\mathsf{HFOLR}}(\Delta[C_2, D_2]) \to \mathsf{Mod}^{\mathsf{HFOLR}}(\Delta[C_1, D_1])$
such that for all $\Delta[C_2, D_2]$-models $(W, M)$, the model $(W, M) \restriction_\theta$ interprets (a) each symbol $x$ in $\Delta$ exactly as $(W, M)$, (b) each nominal $c_1 \in C_1$ as $W_{c_2}$, where $c_2 = \theta_a(c_1)$, and (c) any rigid constant $d_1 \in D_1$ as $M_{w, t_2}$, where $t_2 = \theta_b(d_1)$ and $w$ is any possible world in $|W|$.

Notice that $\theta \restriction_{\_}$ is well-defined, since $t_2 = \theta_b(d_1)$ is a rigid term, which implies $M_{w, t_2} = M_{w', t_2}$ for all $\Delta[C_2, D_2]$-models $(W, M)$ and all possible worlds $w, w' \in |W|$. The proof of the local satisfaction condition for HFOLR substitutions is conceptually the same as the proof of [38, Corollary 39].

*3.2.3 Stratified substitution functors.* For any signature $\Delta$ in a stratified institution $\mathtt{SI}$, the stratified $\Delta$-substitutions form a subcategory $\mathsf{SSb}^{\mathsf{SI}}(\Delta)$ of $\mathsf{Sb}^{\mathsf{SI}}(\Delta)$.

Lemma 3.14 ([38]). $\mathsf{SSb}^{\mathsf{SI}} \colon \mathsf{Sig}^{\mathsf{SI}} \to \mathbb{C}\mathsf{at}^{op}$ *is a sub-functor of* $\mathsf{Sb}^{\mathsf{SI}}$.

In applications, we work with a substitution sub-functor $\mathsf{SSt}^{\mathsf{SI}} \colon \mathsf{D}^{\mathsf{SI}} \to \mathbb{C}\mathsf{at}^{op}$ of $\mathsf{SSb}^{\mathsf{SI}}$.

*Example 3.15 (*HPL *substitution functor).* Let $\mathsf{D}^{\mathsf{HPL}}$ be the broad subcategory of signature extensions with nominals. Let $\mathsf{SSt}^{\mathsf{HPL}} \colon \mathsf{D}^{\mathsf{HPL}} \to \mathbb{C}\mathsf{at}^{op}$ denote the substitution functor which maps each signature $\Delta$ to the category of stratified substitutions represented by functions $\theta \colon C_1 \to \mathsf{Nom}^{\Delta}[C_2]$, where $C_i$ is a set of nominals different from the elements of $\mathsf{Nom}^{\Delta}$.

*Example 3.16 (*HFOLR *substitution functor).* In the present contribution, only substitutions represented by pairs of functions $\langle \theta_a \colon C_1 \to \mathsf{Nom}[C_2], \theta_b \colon D_1 \to T_{@\Sigma}(D_2) \rangle$ as described in Example 3.13 are considered for HFOLR. Let $\mathsf{D}^{\mathsf{HFOLR}} \subseteq \mathsf{Sig}^{\mathsf{HFOLR}}$ be the broad subcategory of signature extensions with nominals and rigid constants. Let $\mathsf{SSt}^{\mathsf{HFOLR}} \colon \mathsf{D}^{\mathsf{HFOLR}} \to \mathbb{C}\mathsf{at}^{op}$ denote the substitution functor which maps each signature $\Delta$ to the category of stratified substitutions represented by pairs of functions $\langle \theta_a \colon C_1 \to \mathsf{Nom}[C_2], \theta_b \colon D_1 \to T_{@\Sigma}(D_2) \rangle$ as described in Example 3.13.

### 3.3 Extensions

In many cases, model-theoretic results such as Downward Löwenheim-Skolem Theorem or proof-theoretic results such as completeness, are developed in an extension of the underlying signature with an infinite number of new constants ("Henkin witness constants"). In a category-based framework such signature extensions can be described by signature colimits.

We start by giving some preliminary definitions. Given a set $C$ and a cardinal $\alpha$, we denote by $\mathcal{P}_\alpha(C)$ the set of all subsets of $C$ of cardinality strictly less than $\alpha$. We say that a diagram of signatures $\mathcal{V}\colon J \to \mathrm{Sig}^{\mathtt{I}}$ in an institution $\mathtt{I}$ is conservative if $\mathcal{V}(f)$ is conservative for all arrows $f \in J$. We say that a colimit of signatures $\vartheta\colon \mathcal{V} \Rightarrow \Sigma$ in an institution $\mathtt{I}$ is conservative if $\vartheta_i$ is conservative for all $i \in |J|$.

In the following, we define a category-based notion of signature extension, which captures the essential properties of signature extensions with an infinite number of constants.

*Definition 3.17 (Signature extension).* Let $\Sigma$ be a signature of power $\alpha$ in an institution $\mathtt{I}$ equipped with a quantification space $\mathrm{Q}^{\mathtt{I}}$. A pair $\langle \mathcal{V}, \vartheta \rangle$ is an extension of $\Sigma$ if:

(1) $\mathcal{V}\colon (\mathcal{P}_\alpha(C), \subseteq) \to \mathrm{Sig}^{\mathtt{I}}$ is a conservative diagram such that $C$ is a set of cardinality $\alpha$ and $\mathcal{V}(\emptyset) = \Sigma$,
(2) $\vartheta\colon \mathcal{V} \Rightarrow \Sigma_C$ is a conservative colimit such that
  (a) the power of $\Sigma_C$ is $\alpha$;
  (b) for all $\Sigma_C$-sentences $\gamma$ there exist a set $C' \in \mathcal{P}_\alpha(C)$ and a sentence $e \in \mathrm{Sen}^{\mathtt{I}}(\Sigma_{C'})$ such that $\vartheta_{C'}(e) = \gamma$, where $\Sigma_{C'}$ denotes $\mathcal{V}(C')$;
  (c) for all $C' \in \mathcal{P}_\alpha(C)$ and all $\chi\colon \Sigma_{C'} \to \Sigma' \in \mathrm{Q}^{\mathtt{I}}$ there exist $(C' \subseteq C'') \in (\mathcal{P}_\alpha(C), \subseteq)$ and $\varphi\colon \Sigma' \to \mathcal{V}(C'')$ conservative such that $\chi;\varphi = \mathcal{V}(C' \subseteq C'')$.

Variations of Definition 3.17 can be found in [42], [34] or [37]. In concrete examples of logics,

- for each $C' \in \mathcal{P}_\alpha(C)$, the signature $\Sigma_{C'}$ is $\Sigma[C']$, the extension of $\Sigma$ with constants from $C'$,
- the vertex $\Sigma_C$ of the colimit $\vartheta$ is $\Sigma[C]$, the extension of $\Sigma$ with constants from $C$, and
- the extension $\langle \mathcal{V}, \vartheta \rangle$ is identified with the signature morphism $\vartheta_\emptyset\colon \Sigma \to \Sigma[C]$.

Proposition 3.18 (Signature extensions in FOL). *In* FOL*, each signature has an extension.*

Proof. Let $\Sigma = (S, F, P)$ be a first-order signature. We denote by $\alpha$ the power of $\Sigma$. Let $C = \{C_s\}_{s \in S}$ be a set of new constant symbols such that $\mathrm{card}(C_s) = \alpha$ for all sorts $s \in S$. Note that the power of $\Sigma[C]$ is $\alpha$. The diagram $\mathcal{V}\colon (\mathcal{P}_\alpha(C), \subseteq) \to \mathrm{Sig}^{\mathsf{FOL}}$ is defined as follows:

(1) $\Sigma_{C'}$ is $\Sigma[C']$, the extension of $\Sigma$ with constants from $C'$, for all $C' \in \mathcal{P}_\alpha(C)$, and
(2) $\mathcal{V}(C' \subseteq C'')\colon \Sigma[C'] \hookrightarrow \Sigma[C'']$ is an inclusion for all $(C' \subseteq C'') \in (\mathcal{P}_\alpha(C), \subseteq)$.

Notice that $\vartheta\colon \mathcal{V} \Rightarrow \Sigma[C]$ is a colimit, where $\vartheta_{C'}\colon \Sigma[C'] \hookrightarrow \Sigma[C]$ is an inclusion for all $C' \in \mathcal{P}_\alpha(C)$.

Let $\gamma$ be any $\Delta[C]$-sentence. Let $C'$ be the finite set of constants from $C$ that occur in $\gamma$. It follows that $\gamma = \vartheta_{C'}(e)$ for some $e \in \mathrm{Sen}^{\mathsf{FOL}}(\Sigma[C'])$. Note that $e$ is obtained from $\gamma$ by changing the qualification of variables to fit the signature $\Sigma[C']$.

Now, let $C' \in \mathcal{P}_\alpha(C)$ and $\chi\colon \Sigma[C'] \hookrightarrow \Sigma[C' \cup X] \in \mathrm{Q}^{\mathsf{FOL}}$. Since $\mathrm{card}(C') < \mathrm{card}(C)$ and the set of variables $X$ is finite, there exists an injective function $f\colon X \to C \setminus C'$. Notice that $C'' = C' \cup f(X) \in \mathcal{P}_\alpha(C)$. Let $\varphi\colon \Sigma[C' \cup X] \to \Sigma[C'']$ be the signature morphism that preserves $\Sigma[C']$ and maps each $x \in X$ to $f(x)$. Since $\varphi$ is the identity on $\Sigma[C']$, we have $\chi;\varphi = \mathcal{V}(C' \subseteq C'')$. Since $f$ is injective, $\varphi$ is injective too. Therefore, $\varphi$ is conservative. □

The proof above is similar to the one given in [37] for the existence of first-order extensions. We generalize the construction of signature extensions to HFOLR.

Proposition 3.19 (Signature extensions in HFOLR). *Each* HFOLR *signature has an extension.*

Proof. Let $(\mathsf{Nom}, \Lambda, \Sigma^{\mathsf{r}} \subseteq \Sigma)$ be a HFOLR signature, where $\Sigma^{\mathsf{r}} = (S^{\mathsf{r}}, F^{\mathsf{r}}, P^{\mathsf{r}})$ and $\Sigma = (S, F, P)$. Let $S^{\mathsf{e}} = S \cup \{\mathsf{n}\}$, the extended set of sorts, where $\mathsf{n}$ stands for the sort of nominals. Let $C$ be an $S^{\mathsf{e}}$-sorted set of new constant symbols such that $\mathrm{card}(C_s) = \alpha$ for all $s \in S^{\mathsf{e}}$. Let $\Delta[C] = (\mathsf{Nom}[C_{\mathsf{n}}], \Sigma^{\mathsf{r}}[C_{S^{\mathsf{r}}}] \subseteq \Sigma[C_S])$, where $C_{S^{\mathsf{r}}} = \{C_s\}_{s \in S^{\mathsf{r}}}$ and $C_S = \{C_s\}_{s \in S}$. Note that the power of $\Delta[C]$ is $\alpha$. The diagram $\mathcal{V}\colon (\mathcal{P}_\alpha(C), \subseteq) \to \mathrm{Sig}^{\mathsf{HFOLR}}$ is defined as follows:

(1) $\Sigma_{C'}$ is $\Delta[C'] = (\mathsf{Nom}[C'_{\mathsf{n}}], \Sigma^{\mathsf{r}}[C'_{S^{\mathsf{r}}}] \subseteq \Sigma[C'_S])$ for all $C' \in \mathcal{P}_\alpha(C)$, and
(2) $\mathcal{V}(C' \subseteq C'')\colon \Delta[C'] \hookrightarrow \Delta[C'']$ is an inclusion for all $(C' \subseteq C'') \in (\mathcal{P}_\alpha(C), \subseteq)$.

Notice that $\vartheta\colon \mathcal{V} \Rightarrow \Delta[C]$ is a colimit, where $\vartheta_{C'}\colon \Delta[C'] \hookrightarrow \Delta[C]$ is an inclusion for all $C' \in \mathcal{P}_\alpha(C)$. One can show that $\langle \mathcal{V}, \vartheta \rangle$ is an extension of $\Delta$ applying the same arguments used in the proof of Proposition 3.18. □

### 3.4 Reachable models

In this section we give an abstract description of the models which consist of elements that are denotations of terms. The concept of reachable model appeared in a proof-theoretic setting in [60], and it has been used successfully for developing several institution-independent results in the area of proof-theory [36, 41, 42] as well as model-theroy [16, 33, 34, 37, 38, 40].

*Definition 3.20 (Reachable model).* Let $\mathsf{St}^{\mathtt{I}}\colon \mathsf{D}^{\mathtt{I}} \to \mathbb{C}\mathsf{at}^{op}$ be a (stratified) substitution functor for a (stratified) institution $\mathtt{I}$. A $\Sigma$-model $M$ is $\mathsf{St}^{\mathtt{I}}$*-reachable*, where $\Sigma \in |\mathsf{Sig}^{\mathtt{I}}|$, if for each signature morphism $\Sigma \xrightarrow{\chi} \Sigma' \in \mathsf{D}^{\mathtt{I}}$ and any $\chi$-expansion $M'$ of $M$ there exists a substitution $\theta\colon \chi \to 1_\Sigma \in \mathsf{St}^{\mathtt{I}}(\Sigma)$ such that $M\restriction_\theta = M'$.

As in case of extensions, the parameter $\mathsf{St}^{\mathtt{I}}$ may be omitted when it is implicitly fixed. For example, we may call $\mathsf{St}^{\mathtt{I}}$-reachable models, simply, reachable. The notion of reachability has been studied extensively in the algebraic specification literature as one can see below.

Proposition 3.21 (Reachable models in FOL [37, 41]). *In* FOL*, a model is* $\mathsf{St}^{\mathsf{FOL}}$*-reachable iff its elements are denotations of terms.*

In hybrid propositional logic, the states of reachable models consist of interpretation of nominals.

Proposition 3.22 (Reachable models in HPL [38]). *In* HPL*, a model is* $\mathsf{SSt}^{\mathsf{HPL}}$*-reachable iff its states are denotations of nominals.*

In HFOLR, the elements of reachable models consist of interpretation of nominals or rigid terms.

Proposition 3.23 (Reachable models in HFOLR). *In* HFOLR*, a model is* $\mathsf{SSt}^{\mathsf{HFOLR}}$*-reachable iff (1) its set of states consists of denotations of nominals, and (2) its carrier sets for the rigid sorts consist of denotations of rigid terms.*

The proof of Proposition 3.23 is conceptually the same as the proof of [38, Proposition 49]. The expansions of reachable models along signature morphisms used for quantification generate substitutions in HFOLR, since retrieve is used to rigidify not merely sentences but also sorts and function symbols. For this reason, the abstract framework developed in this paper can be applied to HFOLR. On the other hand, it is difficult to use the same ideas for HFOLS, since it lacks such rigidification.

### 3.5 Basic sentences

We have introduced the semantics of Boolean connectives, quantifiers and other sentence building operators at an abstract level. The case of atomic sentences is a little bit different since their essence depends on the actual institutions. Therefore, the concept of atomic sentences can only be approximated in the institution-independent setting. This is achieved by the definition of basic sentences which is given below.

*Definition 3.24 (Basic set of sentences [17]).* Given an institution $\mathtt{I}$, a set of sentences $B \subseteq \mathsf{Sen}^{\mathtt{I}}(\Sigma)$ is *basic* if there exists a $\Sigma$-model $M^B$ such that

$$M \models B \text{ iff there exists a homomorphism } M^B \to M$$

for all $\Sigma$-models $M$. We say that $M^B$ is a *basic model* of $B$. If in addition the homomorphism $M^B \to M$ is unique then the set $B$ is called *epi-basic*.

Note that any epi-basic set of sentences has an initial model which is the basic model. We show that the sets of atomic sentences of the (ordinary) institutions presented above are basic.

Proposition 3.25 (Basic sets of sentences in FOL). *Let $\Sigma$ be a* FOL *signature, and let $B$ be a set of atomic sentences over $\Sigma$. Then $B$ is basic. Moreover, if $\Sigma$ is non-void then $B$ is epi-basic and it has a basic model which is reachable.*

Proof. Let $C$ be any set of new constants for $\Sigma$ such that $\Sigma[C]$ is non-void. [9] Let $B' = \chi(B)$, where $\chi\colon \Sigma \hookrightarrow \Sigma[C]$. By [22, Fact 5.20], $B'$ is a epi-basic set of sentences, and it has a basic model $M^{B'}$ which is reachable. We show that $B$ is a basic set of sentences, and $M^B = M^{B'}\restriction_\Sigma$ is a basic model for $B$:

[ *Let $M$ be a $\Sigma$-model such that $M \models B$* ] Then:

| | | |
|---|---|---|
| 1 | let $M'$ be a $\chi$-expansion of $M$ | since $\chi$ is consevative |
| 2 | $M' \models B'$ | by the satisfaction condition |
| 3 | there exists a unique homomorphism $h' : M^{B'} \to M'$ | since $B'$ is epi-basic |
| 4 | $h'\restriction_\Sigma$ is an arrow from $M^B$ to $M$ | |

[ *Let $h : M^B \to M$ be a $\Sigma$-homomorphism* ] Then:

| | | |
|---|---|---|
| 1 | Let $h' : M^{B'} \to M'$ be the $\chi$-expansion of $h$ such that $M'$ is the $\chi$-expansion of $M$ which interprets each $c \in C$ as $h(M_c^{B'})$ | |
| 2 | $M' \models B'$ | since $B'$ is epi-basic |
| 3 | $M \models B$ | by the satisfaction condition |

If $\Sigma$ is non-void then let $C = \emptyset$; hence, $B$ is epi-basic and $M^B$ is reachable. □

Note that a similar result as above holds in POA too. In PL, all sets of atomic sentences are epi-basic, as PL is a fragment of FOL with all signatures non-void. One important property of basic sentences is the preservation of their satisfaction along homomorphisms: given a set of basic sentences $B$ and a homomorphism $h\colon M \to N$, if $M \models B$ then $N \models B$. In hybrid logics, this property does not hold, in general.

*Example 3.26.* Consider the following HPL signature $\Delta = (\mathsf{Nom}, \Lambda, \mathsf{Prop})$ such that $\mathsf{Nom} = \{k\}$, $\Lambda_2 = \{\lambda\}$, $\Lambda_n = \emptyset$ for all $n \in \mathbb{N} \setminus \{2\}$, and $\mathsf{Prop} = \{\rho\}$. Let $h\colon (W, M) \hookrightarrow (W', M')$ be the inclusion homomorphism defined by:

(1) $|W| = \{k\}$, $W_\lambda = \{(k, k)\}$ , $\rho$ is true in $M_k$, and
(2) $|W'| = \{k, w\}$, $W'_\lambda = \{(k, k)\}$, $\rho$ is true in $M'_k$, $\rho$ is not true in $M'_w$.

Example 3.26 points out a significant difference between ordinary logics and hybrid logics. Note that $(W, M) \models^{\mathsf{HPL}} k$, $(W, M) \models^{\mathsf{HPL}} \underline{\lambda}(k)$ and $(W, M) \models^{\mathsf{HPL}} \rho$. Since $(W', M') \not\models^w k$, $(W', M') \not\models^w \underline{\lambda}(k)$ and $(W', M') \not\models^w \rho$ we have $(W', M') \not\models^{\mathsf{HPL}} k$, $(W', M') \not\models^{\mathsf{HPL}} \underline{\lambda}(k)$ and $(W', M') \not\models^{\mathsf{HPL}} \rho$. It follows that the homomorphisms do not preserve the satisfaction of atomic sentences. Hence, they are not basic. However, in hybrid logics, homomorphisms preserve local satisfaction of atomic sentences.

*Definition 3.27.* Let $\Delta$ be a signature in a stratified institution $\mathtt{SI}$ such that $\mathsf{F}(\Delta) = (\mathsf{Nom}, \Lambda)$.

[9] For example, let $C$ be a set which consists of a new constant for each sort $s \in S$ that in not inhabited by the $\Sigma$-terms.

- $@_{\mathsf{Nom}}\, e \coloneqq \begin{cases} \{e\} \text{ if } e \models\!\!\!\dashv @_k\, e \text{ for all } k \in \mathsf{Nom} \text{ such that } @_k\, e \in \mathsf{Sen}^{\mathtt{SI}}(\Delta), \\ \{@_k\, e \mid k \in \mathsf{Nom} \text{ and } @_k\, e \in \mathsf{Sen}^{\mathtt{SI}}(\Delta)\}, \text{otherwise,} \end{cases}$
  for all sentences $e$ over $\Delta$.
- $@_{\mathsf{Nom}}\, E \coloneqq \bigcup_{e \in E} @_{\mathsf{Nom}}\, e$ for all sets of sentences $E$ over $\Delta$.

As usual, the subscript Nom may be dropped from the notations above when it is clear from the context. Notice that $@_{\mathsf{Nom}}\, e$ is $\{e\}$ if for any nominal $k$, $@_k\, e$ is not a sentence in SI (regardless of the semantic relationship between $e$ and $@_k\, e$, where $k$ is any nominal, and $@_k\, e$ is a sentence in the metalanguage). Therefore, the notation $@_{\mathsf{Nom}}\, e$ is particularly useful when SI is closed under retrieve.

*Definition 3.28 (Locally basic set of sentences).* Let $\Delta$ be a signature in a stratified institution. A set of $\Delta$-sentences $B$ is *locally (epi-)basic* if $@B$ is (epi-)basic.

Definition 3.28 is a step forward in understanding the essence of hybrid logics and it is essential for developing our abstract results. The following result presents two situations where the notion of locally basic set of sentences coincides with the notion of basic set of sentences.

Lemma 3.29. *Let* SI *be a stratified institution closed under retrieve, and let $B$ be a locally basic set of sentences.*

*(1) $M \models B$ iff $M \models @B$, for all models $M$ with the set of states consisting of denotations of nominals.*
*(2) If all sentences in $B$ are semantically equivalent to a sentence of the form $@_k\, \gamma$ then $B$ is semantically equivalent to $@B$, in symbols $B \models\!\!\!\dashv @B$, which implies that $B$ is basic.*

We show that in concrete examples of hybrid logics, the sets of atoms are locally basic.

Proposition 3.30 (Locally basic sets of sentences in HPL). *In* HPL*, any set of sentences $B$ constructed from the atomic sentences over a non-void signature $\Delta$ by applying at most one time retrieve is locally epi-basic, and it has a basic model which is reachable.*

Proof. By [38, Theorem 55], $B_r = \{@_k\, e \mid e \in B \text{ and } k \in \mathsf{Nom}^{\Delta}\}$ is epi-basic, and it has a basic model which is reachable. Since $B_r \models\!\!\!\dashv @B$, the set $@B$ is epi-basic. Hence, $B$ is locally epi-basic. □

A similar result holds for HFOLR, too.

*Definition 3.31 (Rigidification in* HFOLR*).* Let $\Delta = (\mathsf{Nom}, \Lambda, \Sigma^{\mathsf{r}} \subseteq \Sigma)$ be a HFOLR signature. The *rigidification function* $\mathsf{at}_k\, \_ \colon T_{\overline{\Sigma}} \to T_{@\Sigma}$, where $k \in \mathsf{Nom}$, is recursively defined by

- $\mathsf{at}_k\, \sigma(t) \coloneqq \begin{cases} (@_k\sigma)(\mathsf{at}_k\, t) & \text{if } (\sigma \colon \mathsf{ar} \to s) \in F^{\mathsf{f}}, \\ \sigma(\mathsf{at}_k\, t) & \text{if } (\sigma \colon \mathsf{ar} \to s) \in F^{\mathsf{r}} \cup @F^{\mathsf{f}}. \end{cases}$

Its extension $\mathsf{at}_k\, \_ \colon \mathsf{Sen}^{\mathsf{HFOLR}}(\Delta) \to \mathsf{Sen}^{\mathsf{HFOLR}}(\Delta)$ is recursively defined by:

- $\mathsf{at}_k\, k' \coloneqq @_k\, k'$;
- $\mathsf{at}_k\, \underline{\lambda}(k_1, \ldots, k_n) \coloneqq @_k\, \underline{\lambda}(k_1, \ldots, k_n)$;
- $\mathsf{at}_k\, (t_1 = t_2) \coloneqq (\mathsf{at}_k\, t_1 = \mathsf{at}_k\, t_2)$;
- $\mathsf{at}_k\, \pi(t) \coloneqq \begin{cases} (@_k\pi)(\mathsf{at}_k\, t) & \text{if } \pi \in P^{\mathsf{f}} \\ \pi(\mathsf{at}_k\, t) & \text{if } \pi \in P^{\mathsf{r}} \cup @P^{\mathsf{f}} \end{cases}$;
- $\mathsf{at}_k\, \neg e \coloneqq \neg \mathsf{at}_k\, e$;
- $\mathsf{at}_k\, \vee E \coloneqq \vee \mathsf{at}_k\, E$;
- $\mathsf{at}_k\, @_{k'}\, e \coloneqq \mathsf{at}_{k'}\, e$;
- $\mathsf{at}_k\, \exists X, Y \cdot e'' \coloneqq \exists X, Y \cdot \mathsf{at}_k\, e''$.

Any sentence in the image of some $\mathsf{at}_k$ , where $k \in \mathsf{Nom}$, is called a *rigid sentence*.

The proof of the following lemma is straightforward and we leave it as an exercise for the readers.

Lemma 3.32. *In* HFOLR*, any sentence* $@_k\, e$ *is semantically equivalent to* $\mathsf{at}_k\, e$.

Proposition 3.33 (Locally basic sets of sentences in HFOLR). *Consider a* HFOLR *signature* $\Delta = (\mathsf{Nom}, \Lambda, \Sigma^{\mathsf{r}} \subseteq \Sigma)$*, and let* $B$ *be a set of* $\Delta$*-sentences constructed from atomic sentences by applying at most one time retrieve.*

*(1) If* $(\mathsf{Nom}, \Lambda)$ *is non-void then* $B$ *is locally basic, and it has a basic model which is reachable.*
*(2) If* $\Delta$ *is non-void then* $B$ *is locally epi-basic, and it has a basic model which is reachable.*

Proof. Assume that $(\mathsf{Nom}, \Lambda)$ is non-void, i.e. $\mathsf{Nom} \neq \emptyset$. Let $D$ be a set of first-order constants such that $\Delta[D]$ is non-void, where $\Delta[D] = (\mathsf{Nom}, \Lambda, \Sigma^{\mathsf{r}}[D_{S^{\mathsf{r}}}] \subseteq \Sigma[D])$ and $D_{S^{\mathsf{r}}} = \{D_s\}_{s \in S^{\mathsf{r}}}$. Let $B' = \chi(B)$, where $\chi \colon \Delta \hookrightarrow \Delta[D]$ is an inclusion. By Lemma 3.32, $@_{\mathsf{Nom}}\, B' \mathrel{|\!\!\!\models\!\!\!|} \mathsf{at}_{\mathsf{Nom}}\, B'$, where $\mathsf{at}_{\mathsf{Nom}}\, B' = \{\mathsf{at}_k\, e \mid k \in \mathsf{Nom} \text{ and } e \in B'\}$. By [38, Theorem 56], $\mathsf{at}_{\mathsf{Nom}}\, B'$ is epi-basic and it has a basic model $(W^{B'}, M^{B'})$, which is reachable. Since $\mathsf{at}_{\mathsf{Nom}}\, B' \mathrel{|\!\!\!\models\!\!\!|} @_{\mathsf{Nom}}\, B'$, the set $@_{\mathsf{Nom}}\, B'$ is epi-basic, too. We show that $B$ is locally basic and $(W^B, M^B) = (W^{B'}, M^{B'}) \restriction_\chi$ is a basic model for $B$:

[ *Let* $(W, M)$ *be a Kripke structure such that* $(W, M) \models @_{\mathsf{Nom}}\, B$ ] Then:

| | | |
|---|---|---|
| 1 | let $(W, M')$ be a $\chi$-expansion of $(W, M)$ | |
| 2 | $\chi(@_{\mathsf{Nom}}\, B) = @_{\mathsf{Nom}}\, B'$ | since $\chi$ is the identity on nominals |
| 3 | $(W, M') \models @_{\mathsf{Nom}}\, B'$ | by the satisfaction condition |
| 4 | there exists a homomorphism $h' : (W^{B'}, M^{B'}) \to (W, M')$ | since $@_{\mathsf{Nom}}\, B'$ is epi-basic |
| 5 | $h' \restriction_\chi$ is an arrow from $(W^B, M^B)$ to $(W, M)$ | |

[ *Let* $h : (W^B, M^B) \to (W, M)$ *be a homomorphism* ] Then:

| | | |
|---|---|---|
| 1 | let $h' : (W^{B'}, M^{B'}) \to (W, M')$ be the $\chi$-expansion of $h$ such that $(W, M')$ is the $\chi$-expansion of $(W, M)$ which interprets each constant $d \in D$ at $w \in \lvert W^{B'} \rvert$ as $h(M^{B'}_{w,d})$ | $h'$ is well-defined, as $\lvert W^{B'} \rvert \neq \emptyset$ and $M^{B'}_{w,d} \in M^B_w$ for all possible worlds $w \in \lvert W^{B'} \rvert$ and all constants $d \in D$ |
| 2 | $(W, M') \models @_{\mathsf{Nom}}\, B'$ | since $@_{\mathsf{Nom}}\, B'$ is epi-basic |
| 3 | $(W, M) \models @_{\mathsf{Nom}}\, B$ | by the satisfaction condition and the fact that $\chi(@_{\mathsf{Nom}}\, B) = @_{\mathsf{Nom}}\, B'$ |

Assume that $\Delta$ is non-void. Take $D = \emptyset$, and by the same argument used above, $@B$ is epi-basic. Hence, $B$ is locally epi-basic. □

## 4 FORCING

Forcing is a method of constructing models satisfying some properties forced by some conditions. In this section, we develop a model-theoretic notion of forcing for stratified institutions. This is a generalization of the forcing defined in [42] for ordinary institutions, and it includes the following concepts:

(1) *forcing property*, which is composed of a set of abstract conditions compatible with the local satisfaction of atomic sentences,
(2) *forcing relation*, which is generated automatically by a forcing property and it simulates the satisfaction relation for reachable models,
(3) *generic set*, which is, roughly speaking, an increasing chain of conditions, which can be used to build a reachable model, and
(4) *generic model*, a concrete realization of a generic set.

The framework in which the results of this section will be developed is defined as follows.

Framework 4.1 (Stratified institution with structured syntax). *Throughout this section, we work within a* stratified institution SI with a structured syntax*, which means that:*

*(F1)* SI *is a stratified institution with nominal variables according to Framework 2.1;*

*(F2)* $\mathtt{SI}$ *is equipped with a substitution functor* $\mathsf{SSt}^{\mathtt{SI}}\colon \mathsf{D}^{\mathtt{SI}} \to \mathbb{C}\mathsf{at}^{op}$ *such that if* $\theta\colon \chi_z \to 1_\Delta$ *is a substitution in* $\mathsf{SSt}^{\mathtt{SI}}(\Delta)$*, where* $\Delta$ *is a signature and* $z$ *is a nominal variable, then there exists* $k \in \mathsf{Nom}^\Delta$ *such that* $\theta$ *is induced by the signature morphism* $\varphi_{z\leftarrow k}$ *(defined in Framework 2.1);*

*(F3)* $\mathtt{SI}$ *is equipped with a quantification space* $\mathsf{Q}^{\mathtt{SI}}$ *such that (a)* $\mathsf{Q}^{\mathtt{SI}} \subseteq \mathsf{D}^{\mathtt{SI}}$*, and (b) the signature morphisms commute with the sentence building operators according to Definition 3.2;*

*(F4) there exists a sub-functor* $\mathsf{Sen}_0\colon \mathsf{Sig}^{\mathtt{SI}} \to \mathbb{S}\mathsf{et}$ *of* $\mathsf{Sen}^{\mathtt{SI}}$ *such that all sentences of* $\mathtt{SI}$ *are constructed from the sentences of* $\mathtt{SI}_0 = (\mathsf{Sig}^{\mathtt{SI}}, \mathsf{F}^{\mathtt{SI}}, \mathsf{Sen}_0, \mathsf{Mod}^{\mathtt{SI}}, \mathsf{K}^{\mathtt{SI}}, \models^{\mathtt{SI}})$ *by applying Boolean connectives, retrieve, possibility over binary modalities, store and existential quantification over the signature morphisms in* $\mathsf{Q}^{\mathtt{SI}}$*;*

*(F5)* $\mathtt{SI}$ *is closed under negation and retrieve;*

*(F6)* $\mathtt{SI}$ *is weakly closed under nominal relations, i.e. if* $\langle\lambda\rangle e \in \mathsf{Sen}^{\mathtt{SI}}(\Delta)$ *then* $\underline{\lambda}(k) \in \mathsf{Sen}_0(\Delta)$ *for all* $k \in \mathsf{Nom}^\Delta$*.* □

A concrete example of $\mathtt{SI}$ is HPL, where $\mathsf{HPL}_0$ is obtained from HPL by restricting the syntax to atomic sentences, i.e. nominals, nominal relations and propositional symbols. Notice that HPL is quantifier-free. Another example of $\mathtt{SI}$ is HFOLR, where $\mathsf{HFOLR}_0$ is obtained from HFOLR by restricting the syntax to atomic sentences, i.e. nominals, nominal relations, hybrid equations and hybrid relations.

Framework 4.1 (F2) says that all substitutions from $\chi_z\colon \Delta \to \Delta[z]$ to $1_\Delta\colon \Delta \to \Delta$ are induced by the signature morphisms which map $z$ to nominals and are identities on the symbols in $\Delta$. A direct consequence is the following important result.

Lemma 4.1. *Any state of a reachable model is the denotation of some nominal.*

Proof. Let $M$ be a reachable $\Delta$-model, and $w$ a possible world of $M$. Let $z$ be a nominal variable. Since $M$ is reachable and $M^{z\leftarrow w}$ is a $\chi_z$-expansion of $M$, $M\restriction_\theta = M^{z\leftarrow w}$ for some substitution $\theta\colon \chi_z \to 1_\Delta$. By Framework 4.1 (F2), $\theta$ is induced by $\varphi_{z\leftarrow k}$, for some $k \in \mathsf{Nom}^\Delta$. It is straightforward to show that $w$ is the denotation of $k$. □

Fact 4.1. *We denote by* $\mathsf{Sen}_b$ *the sub-functor of* $\mathsf{Sen}^{\mathtt{SI}}$ *which maps each signature to the set of sentences obtained by applying at most one time retrieve to the sentences in* $\mathtt{SI}_0$*. Hence,* $\mathtt{SI}_b = (\mathsf{Sig}^{\mathtt{SI}}, \mathsf{F}^{\mathtt{SI}}, \mathsf{Sen}_b, \mathsf{Mod}^{\mathtt{SI}}, \mathsf{K}^{\mathtt{SI}}, \models^{\mathtt{SI}})$ *is a stratified institution.*

The starting point of our developments consists of a basic level given by the institution $\mathtt{SI}_b$. This has the advantage of being semantically closed under retrieve, i.e. for all $\mathtt{SI}_b$ sentences $e$ and all nominals $k$ over the same signature $\Delta$, there exists a sentence $\gamma$ in $\mathtt{SI}_b$, which is semantically equivalent to $@_k\, e$, in symbols, $\gamma \mathrel{|\!\!\!\!=\!\!\!\!|} @_k\, e$. This property is necessary as the local satisfaction relation sits at the core of stratified institution definition and the operator retrieve makes it possible to switch the point of evaluation in a sentence. The following remark concerns the soundness of structural induction on sentences.

Fact 4.2. *For any* $n \in \mathbb{N}$*, we denote by* $\mathsf{Sen}_n$ *the sub-functor of* $\mathsf{Sen}^{\mathtt{SI}}$ *which maps each signature to the set of sentences obtained by applying at most n-times the sentence building operators enumerated in Framework 4.1. Hence,* $\mathtt{SI}_n = (\mathsf{Sig}^{\mathtt{SI}}, \mathsf{F}^{\mathtt{SI}}, \mathsf{Sen}_n, \mathsf{Mod}^{\mathtt{SI}}, \mathsf{K}^{\mathtt{SI}}, \models^{\mathtt{SI}})$ *is a stratified institution.*

There are two important basic features with deep ramifications in all further developments that distinguish the present contribution from the results in [34, 42]:

(1) the forcing relation is indexed by nominals, which means that "locality" plays an essential role in the present contribution;
(2) the basic level is semantically closed under retrieve; in both [42] and [34], the atomic level is the basic level.

In classical model theory, the definition of forcing is based on the notion of congruence [5, 64]. While the notion of congruence for first-order structures is a straightforward generalization of the notion of equivalence, the definition of congruence for Kripke structures with shared domains is a non-trivial matter if it implies an equivalence on the set of possible worlds. The satisfaction relation at the basic level, which is given by $\mathtt{SI}_b$ in our framework, encapsulates the essential features of congruences such as reflexivity, symmetry, transitivity, compatibility with the operations and with the state/world identities. This is the key for defining a notion of forcing in an institution-independent setting. The same idea is explored in Section 6 to define proof rules for the fragments of concrete hybrid logics obtained by restricting the formulae to basic sentences.

*Definition 4.2 (Forcing property).* Given a signature $\Delta$, a forcing property over $\Delta$ is a triple $\mathbb{P} = \langle \mathsf{P}, \leq, f \rangle$ such that:

(1) $(\mathsf{P}, \leq)$ is a partially ordered set with a least element 0.
The elements of $p$ are traditionally called *conditions*.
(2) $f : \mathsf{P} \to \mathcal{P}(\mathsf{Sen}_b(\Delta))$ is a function,
(3) if $p \leq q$ then $f(p) \subseteq f(q)$, and
(4) if $f(p) \models^{\mathtt{SI}} @_k\, e$ then $@_k\, e \in f(q)$ for some $q \geq p$,

where $p \in \mathsf{P}$, $q \in \mathsf{P}$ and $@_k\, e \in \mathsf{Sen}_b(\Delta)$.

Since ordinary institutions can be regarded as stratified institutions such that the sets of possible worlds of all models consist of singletons, the definition of forcing for ordinary institutions [34, 42] can be obtained from Definition 4.2 by rewriting the fourth condition without the sentence building operator retrieve. As for ordinary institutions, a forcing property generates a forcing relation.

*Definition 4.3 (Forcing relation).* Let $\mathbb{P} = \langle \mathsf{P}, \leq, f \rangle$ be a forcing property over $\Delta$. The family of relations $\Vdash = \{\Vdash^k\}_{k \in \mathsf{Nom}}$, where $\Vdash^k \subseteq \mathsf{P} \times \mathsf{Sen}^{\mathtt{SI}}(\Delta)$, is defined by induction on the structure of sentences:

(1) *For* $e \in \mathsf{Sen}_0(\Delta)$*:* $p \Vdash^k e$ iff $@_k\, e \in f(p)$;
(2) *For* $\neg e$*:* $p \Vdash^k \neg e$ iff there is no $q \geq p$ such that $q \Vdash^k e$;
(3) *For* $\vee E$*:* $p \Vdash^k \vee E$ iff $p \Vdash^k e$ for some $e \in E$;
(4) *For* $@_{k_1}\, e$*:* $p \Vdash^k @_{k_1}\, e$ iff $p \Vdash^{k_1} e$;
(5) *For* $\langle \lambda \rangle e$*:* $p \Vdash^k \langle \lambda \rangle e$ iff $p \Vdash^k \underline{\lambda}(k_1)$ and $p \Vdash^{k_1} e$ for some nominal $k_1 \in \mathsf{Nom}^{\Delta}$;
(6) *For* $\downarrow z \cdot e$*:* $p \Vdash^k \downarrow z \cdot e$ iff $p \Vdash^k \varphi_{z \leftarrow k}(e)$.
(7) *For* $\exists \chi \cdot e$*:* $p \Vdash^k \exists \chi \cdot e$ iff $p \Vdash^k \theta(e)$ for some substitution $\theta : \chi \to 1_{\Delta} \in \mathsf{St}(\Delta)$.

The notation $p \Vdash^k e$ is read *p forces e at k*. The definition of $\Vdash^k$ is based on substitutions given by the substitution functor $\mathsf{SSt}^{\mathtt{SI}}$. See statement 7 of Definition 4.3. The decision of choosing a class of substitutions and not all substitutions for the formalization of forcing is motivated by the fact that in concrete examples of hybrid logics, the class of substitutions is implicitly restricted, e.g. to first-order substitutions that match variables to terms. Framework 4.1 (F6) ensures that $p \Vdash^k \langle \lambda \rangle e$ is well-defined.

The connection with the results on forcing in ordinary institutions can be established by defining a global forcing relation: $p \Vdash e$ iff $p \Vdash^k$ for all nominals $k$. For example, in [42] and [34], there exists only a global forcing relation.

Lemma 4.4. *Let* $\mathbb{P} = \langle \mathsf{P}, \leq, f \rangle$ *be a forcing property as in Definition 4.2. Then:*

*(1)* $p \Vdash^k \neg\neg e$ *iff for each* $q \geq p$ *there is* $r \geq q$ *such that* $r \Vdash^k e$.
*(2) If* $q \geq p$ *and* $p \Vdash^k e$ *then* $q \Vdash^k e$.
*(3) If* $p \Vdash^k e$ *then* $p \Vdash^k \neg\neg e$.

*(4) We cannot have both $p \Vdash^k e$ and $p \Vdash^k \neg e$.*

Proof. Notice that the statements 1 and 3 are well-defined as SI is closed under negation.

(1) $p \Vdash^k \neg\neg e$ iff for each $q \geq p$ we have $q \nVdash^k \neg e$ iff
for each $q \geq p$ there is $r \geq q$ such that $r \Vdash^k e$.

(2) By induction on the structure of sentences:

[ *For* $e \in \mathsf{Sen}_0(\Delta)$ ] The conclusion follows easily from $f(p) \subseteq f(q)$.

[ *For* $\neg e$ ] We have $p \Vdash^k \neg e$. This means $r \nVdash^k e$ for all $r \geq p$. In particular, $r \nVdash^k e$ for all $r \geq q$. Hence, $q \Vdash^k \neg e$.

[ *For* $\vee E$ ] $p \Vdash^k e$ for some $e \in E$. by the induction hypothesis, $q \Vdash^k e$ which implies $q \Vdash^k \vee E$.

[ *For* $@_{k_1} e$ ] We have $p \Vdash^k @_{k_1} e$ iff $p \Vdash^{k_1} e$. By the induction hypothesis, $q \Vdash^{k_1} e$. Hence, $p \Vdash^k @_{k_1} e$.

[ *For* $\langle\lambda\rangle e$ ] We have $p \Vdash^k \langle\lambda\rangle e$ iff $p \Vdash^k \underline{\lambda}(k_1)$ and $p \Vdash^{k_1} e$ for some $k_1 \in \mathsf{Nom}$. By the induction hypothesis, $q \Vdash^k \underline{\lambda}(k_1)$ and $q \Vdash^{k_1} e$. Hence, $q \Vdash^k \langle\lambda\rangle e$.

[ *For* $\downarrow z \cdot e$ ] We have $p \Vdash^k \downarrow z \cdot e$ iff $p \Vdash^k \varphi_{z \leftarrow k}(e)$. By the induction hypothesis, $q \Vdash^k \varphi_{z \leftarrow k}(e)$, which is equivalent to $q \Vdash^k \downarrow z \cdot e$.

[ *For* $\exists\chi \cdot e$ ] Since $p \Vdash^k \exists\chi \cdot e$ then we have $p \Vdash^k \theta(e)$ for some substitution $\chi \xrightarrow{\theta} 1_\Sigma \in \mathsf{St}(\Delta)$. By the induction hypothesis, $q \Vdash^k \theta(e)$. Hence, $q \Vdash^k \exists\chi \cdot e$.

(3) It follows from 1 and 2.

(4) By the reflexivity of $(\mathsf{P}, \leq)$. □

We formalize the notion of generic set in stratified institutions, which plays a central role in developing results based on forcing.

*Definition 4.5 (Generic set).* Let $\mathbb{P} = \langle \mathsf{P}, \leq, f \rangle$ be a forcing property over a signature $\Delta$. A subset $G \subseteq \mathsf{P}$ is generic if it has the following properties:

(1) $r \in G$ if $r \leq p$ and $p \in G$;
(2) there exists $r \in G$ such that $r \geq p$ and $r \geq q$, for all $p, q \in G$;
(3) there exists $r \in G$ such that $r \Vdash^k e$ or $r \Vdash^k \neg e$, for all $\Delta$-sentences $@_k\, e$.

Note that $G$ in Definition 4.5 is well-defined, as SI is closed under negation. We write $G \Vdash^k e$ whenever $p \Vdash^k e$ for some $p \in G$. The following lemma ensures the existence of generic sets. The result is based on the assumption that signatures consist of a countable number of symbols.

Lemma 4.6 (Existence of generic sets). *Let $\mathbb{P} = \langle \mathsf{P}, \leq, f \rangle$ be a forcing property over a signature $\Delta$. If $\mathsf{Sen}^{\mathsf{SI}}(\Delta)$ is countable then every $p$ belongs to a generic set.*

Proof. Since $\mathsf{Sen}^{\mathsf{SI}}(\Delta)$ is countable, let $\{@_{k_n} e_n \mid n < \mathbb{N}\}$ be an enumeration of the $\Delta$-sentences which have retrieve as the top operator. We form a chain of conditions $p_0 \leq p_1 \leq \ldots$ in $\mathsf{P}$ as follows:

- let $p_0 = p$;
- if $p_n \Vdash^{k_n} \neg e_n$ then let $p_{n+1} = p_n$
  else choose $p_{n+1} \geq p_n$ such that $p_{n+1} \Vdash^{k_n} e_n$.

Note that the chain $p_0 \leq p_1 \leq \ldots$ is well-defined as SI is closed under negation. The set $G = \{q \in \mathsf{P} \mid q \leq p_n \text{ for some } n < \mathbb{N}\}$ is generic and contains $p$. □

For concrete forcing properties, we will develop separately results about the existence of generic sets which are not based on the assumption that signatures consist of countable symbols. Notice that the definition of forcing relation and the definition of generic set are based on syntactic compounds. The following definition gives the semantics/meaning to these concepts.

*Definition 4.7 (Generic model).* Let $\mathbb{P} = \langle \mathsf{P}, \leq, f \rangle$ be a forcing property over a signature $\Delta$.

- $M$ is a model for a generic set $G \subseteq \mathsf{P}$ when $M \models^{\mathtt{SI}} @_k\, e$ iff $G \Vdash^k e$, for all $\Delta$-sentences $@_k\, e$.
- $M$ is a model for a condition $p \in \mathsf{P}$ if there is a generic set $G \subseteq \mathsf{P}$ such that $p \in G$ and $M$ is a model for $G$.

The models $M$ from Definition 4.7 are called, traditionally, *generic models*. The following result ensures the existence of generic models.

Theorem 4.8 (Generic Model Theorem). *Assume that $\mathtt{SI}_b$ is compact. Let $\Delta$ be a signature such that each subset of $\mathsf{Sen}_b(\Delta)$ is locally basic and it has a basic model which is reachable. Let $\mathbb{P} = \langle \mathsf{P}, \leq, f \rangle$ be a forcing property over $\Delta$. Then each generic set $G$ of $\mathbb{P}$ has a generic model which is reachable.*

Proof. Let $G$ be a generic set. We define $T = \{@_k\, e \in \mathsf{Sen}^{\mathtt{SI}}(\Delta) \mid G \Vdash^k e\}$ and $B = T \cap \mathsf{Sen}_b(\Delta)$. We show that $M^B \models^{\mathtt{SI}} @_k\, e$ iff $G \Vdash^k e$, for all $\Delta$-sentences $@_k\, e$, where $M^B$ is a basic model of $B$ that is reachable. Let $W^B = \mathsf{K}_\Delta(M^B)$. We proceed by induction on the structure of $e$:

[ *For* $e \in \mathsf{Sen}_0(\Delta)$ ] Assume that $M^B \models^{\mathtt{SI}} @_k\, e$.

| | | |
|---|---|---|
| 1 | $B$ and $\{@_k\, e\}$ are basic | by Lemma 3.29 (2) |
| 2 | there exists an arrow $M^{\{@_k\, e\}} \to M^B$ | since $\{@_k\, e\}$ is basic |
| 3 | $B \models @_k\, e$ | since both $B$ and $\{@_k\, e\}$ are basic |
| 4 | there exists $B_f \subseteq B$ finite such that $B_f \models @_k\, e$ | since $\mathtt{SI}_b$ is compact |
| 5 | $B_f = \{@_{k_1} e_1, \ldots, @_{k_n} e_n\}$ for some $e_i \in \mathsf{Sen}_0(\Delta)$ and some $k_i \in \mathsf{Nom}^\Delta$ | by the definition of $B$ |
| 6 | there exists $p_i \in G$ such that $p_i \Vdash^{k_i} e_i$ | by the definition of $B$ |
| 7 | there exists $p \in G$ such that $p \geq p_i$ for all $i \in \{1, \ldots, n\}$ | since $B_f$ is finite and $G$ is generic |
| 8 | $B_f \subseteq f(p)$ | since $B_f \subseteq \mathsf{Sen}_b(\Delta)$ |
| 9 | $q \Vdash^k e$ or $q \Vdash^k \neg e$ for some $q \in G$ | since $G$ is generic |
| 10 | suppose towards a contradiction that $q \Vdash^k \neg e$ | |
| 10.1 | $r \geq p$ and $r \geq q$ for some $r \in G$ | since $G$ is generic |
| 10.2 | $r \Vdash^k \neg e$ | by Lemma 4.4 (2) from $r \geq q$ and $q \Vdash^k \neg e$ |
| 10.3 | $B_f \subseteq f(r)$ | since $B_f \subseteq f(p)$ and $r \geq p$ |
| 10.4 | there exists $s \geq r$ such that $@_k\, e \in f(s)$ | since $B_f \models @_k\, e$, we have $f(r) \models @_k\, e$ |
| 10.5 | $s \Vdash^k e$ | by Definition 4.3 from 10.4 |
| 10.6 | $s \Vdash^k \neg e$ | by Lemma 4.4 (2) from 10.2 |
| 10.7 | contradiction | by Lemma 4.4 (4) from 10.5 and 10.6 |
| 11 | $q \Vdash^k e$ | by 9 and 10 |
| 12 | $G \Vdash^k e$ | since $q \in G$ |

Now assume that $G \Vdash^k e$. By the definition of $B$, we have $@_k\, e \in B$. It follows that $B \models @_k\, e$. Hence, $M^B \models^{\mathtt{SI}} @_k\, e$.

[ *For* $\neg e$ ] The following are equivalent:

1 $M^B \models^{\mathtt{SI}} @_k \neg e$

| | | |
|---|---|---|
| 2 | $M^B \not\models^{\mathsf{SI}} @_k\, e$ | by the semantics of negation |
| 3 | $G \not\Vdash^k e$ | by the induction hypothesis |
| 4 | $p \not\Vdash^k e$ for all $p \in G$ | by the definition of $\Vdash$ |
| 5 | $p \Vdash^k \neg e$ for some $p \in G$ | since $G$ is generic |
| 6 | $G \Vdash^k \neg e$ | |

[ *For* $\vee E$ ] The following are equivalent:

| | | |
|---|---|---|
| 1 | $M^B \models^{\mathsf{SI}} @_k\, \vee E$ | |
| 2 | $M^B \models^{\mathsf{SI}} @_k\, e$ for some $e \in E$ | by the semantics of disjunction |
| 3 | $G \Vdash^k e$ for some $e \in E$ | by the induction hypothesis |
| 4 | $G \Vdash^k \vee E$ | by the definition of $\Vdash$ |

[ *For* $@_{k_1}\, e$ ] This case is straightforward since $@_k\, @_{k_1}\, e$ is semantically equivalent to $@_{k_1}\, e$.

[ *For* $\langle\lambda\rangle e$ ] The following are equivalent:

| | | |
|---|---|---|
| 1 | $M^B \models^{\mathsf{SI}} @_k\, \langle\lambda\rangle e$ | |
| 2 | $M^B \models^{w_1} e$ for some $w_1 \in \lvert W^B\rvert$ such that $(W^B_k, w_1) \in W^B_\lambda$ | by the definition of $\models$ |
| 3 | $M^B \models^{\mathsf{SI}} @_k\, \underline{\lambda}(k_1)$ and $M^B \models^{\mathsf{SI}} @_{k_1}\, e$ for some $k_1 \in \mathsf{Nom}^\Delta$ such that $W^B_{k_1} = w_1$ | by Lemma 4.1, since $M^B$ is reachable |
| 4 | $G \Vdash^k \underline{\lambda}(k_1)$ and $G \Vdash^{k_1} e$ for some $k_1 \in \mathsf{Nom}^\Delta$ | by the induction hypothesis |
| 5 | $G \Vdash^k \langle\lambda\rangle e$ | since $G$ is generic |

[ *For* $\downarrow z \cdot e$ ] This case is straightforward since $@_k \downarrow z \cdot e$ is semantically equivalent to $@_k\, \varphi_{z \leftarrow k}(e)$.

[ *For* $\exists\chi \cdot e$ ] Assume that $\chi \colon \Delta \to \Delta'$ and let $w = W^B_k$. The following are equivalent:

| | | |
|---|---|---|
| 1 | $M^B \models^{\mathsf{SI}} @_k\, \exists\chi \cdot e$ | |
| 2 | $N \models^w e$ for some $\chi$-expansion $N$ of $M^B$ | by the definition of $\models$ |
| 3 | $M^B \models^w \theta(e)$ for some substitution $\theta \colon \chi \to 1_\Delta \in \mathsf{St}^{\mathsf{SI}}(\Delta)$ | since $M^B$ is reachable |
| 4 | $G \Vdash^k \theta(e)$ for some substitution $\theta \colon \chi \to 1_\Delta \in \mathsf{St}^{\mathsf{SI}}(\Delta)$ | by the induction hypothesis |
| 5 | $G \Vdash^k \exists\chi \cdot e$ | by the definition of $\Vdash$ |

□

The general results proved in this section for forcing are not instantiated to concrete examples of stratified institutions as they act like an abstract interface for developing other proof-theoretic and model-theoretic results. In this paper, we will use forcing to prove an abstract completeness theorem, which is applicable to many concrete hybrid logics.

## 5 PROOF THEORY FOR STRATIFIED INSTITUTIONS

This section contains the main results of the paper and it revolves around the following concepts:

(1) *entailment system*, which sets the foundation for formal reasoning in institutions; we define a general entailment system to reason formally about the properties of systems described by some hybrid logic;
(2) *soundness*, which says that formal reasoning leads to correct proofs; this property is a consequence of the soundness of each proof rule that defines the underlying entailment system;
(3) *compactness*, a reformulation of compactness defined for institutions; this property holds for our general entailment system, too;
(4) *completeness*, which says that each semantic consequence has a formal proof; this property is significantly more difficult to prove than soundness and compactness, and it requires an

additional infrastructure for its proof; to this end, we define a forcing property based on a notion of syntactic consistency given by the entailment system defined previously.

### 5.1 Entailment systems

Institutions equipped with proof theoretic infrastructure provide a complete description of the intuitive notion of logical system which include sentences, models, satisfaction relation and formal deduction. To reason formally about the semantic consequences of sentences, a notion of *entailment system* is proposed in [51]. The present approach to deduction is slightly more general as the entailment relation is defined between sets of sentences rather than between sets of sentences and single sentences. For a comprehensive approach to proof theory in the framework of institutions, one may look into [21], which provides a more refined framework by discriminating between different proofs and addressing their internal structure.

*Definition 5.1 (Entailment system).* An *entailment system* $\mathcal{E} = (\mathsf{Sig}, \mathsf{Sen}, \vdash)$ consists of a category of signatures Sig, a sentence functor $\mathsf{Sen}\colon \mathsf{Sig} \to \mathbb{S}\mathrm{et}$, and a family $\vdash = \{\vdash_\Sigma\}_{\Sigma \in |\mathsf{Sig}|}$ of entailment relations between sets of sentences with the following properties:

$$\textit{(Monotonicity)}\frac{E \subseteq E_1}{E_1 \vdash_\Sigma E} \qquad \textit{(Transitivity)}\frac{E_1 \vdash_\Sigma E_2 \quad E_2 \vdash_\Sigma E_3}{E_1 \vdash_\Sigma E_3}$$

$$\textit{(Union)}\ \frac{E_1 \vdash_\Sigma E_2 \quad E_1 \vdash_\Sigma E_3}{E_1 \vdash_\Sigma E_2 \cup E_3} \qquad \textit{(Translation)}\frac{E \vdash_\Sigma e}{\varphi(E) \vdash_{\Sigma'} \varphi(e)}\ [\ \varphi\colon \Sigma \to \Sigma'\ ]$$

We may omit the subscript $\Sigma$ from $\vdash_\Sigma$ when it is clear from the context. If $E_1 \vdash E_2$ and $E_2 \vdash E_1$ then we write $E_1 \dashv\vdash E_2$. We say that a set of sentences $E$ is *consistent* if there exists a sentence that is not a consequence of $E$. If a set of sentences is consistent then the set of its syntactic consequences are consistent too.

Lemma 5.2. *If $\Gamma$ is consistent and $\Gamma \vdash E$ then $\Gamma \cup E$ is consistent too.*

Proof. Assume that $\Gamma$ is consistent, i.e. $\Gamma \nvdash_\Sigma \rho$ for some sentence $\rho$. Suppose towards a contradiction that $\Gamma \cup E \vdash \rho$. We have $\Gamma \vdash \Gamma$ and $\Gamma \vdash E$, which implies $\Gamma \vdash \Gamma \cup E$. By the transitivity of the entailment relation, $\Gamma \vdash \rho$, which is a contradiction to $\Gamma \nvdash_\Sigma \rho$. □

The notion of compactness is straightforwardly extended to entailment systems.

*Definition 5.3 (Compactness).* The entailment system $\mathcal{E} = (\mathsf{Sig}, \mathsf{Sen}, \vdash)$ is *compact* whenever for every $\Gamma \subseteq \mathsf{Sen}(\Sigma)$ and each finite $E_f \subseteq \mathsf{Sen}(\Sigma)$ if $\Gamma \vdash_\Sigma E_f$ then there exists $\Gamma_f \subset \Gamma$ finite such that $\Gamma_f \vdash_\Sigma E_f$.

For each entailment system $\mathcal{E} = (\mathsf{Sig}, \mathsf{Sen}, \vdash)$ one can easily construct the *compact entailment subsystem* $\mathcal{E}^c = (\mathsf{Sig}, \mathsf{Sen}, \vdash^c)$ by defining the entailment relation $\vdash^c$ as follows: $\Gamma \vdash^c_\Sigma E$ iff for each finite set $E_f \subseteq E$ there exists a finite set $\Gamma_f \subseteq \Gamma$ such that $\Gamma_f \vdash_\Sigma E_f$.

Lemma 5.4 ([21]). *$\mathcal{E}^c = (\mathsf{Sig}, \mathsf{Sen}, \vdash^c)$ is a compact entailment system.*

*The semantic entailment system* of an institution $\mathtt{I}$ consists of $(\mathsf{Sig}^{\mathtt{I}}, \mathsf{Sen}^{\mathtt{I}}, \models^{\mathtt{I}})$. Soundness and completeness are defined in connection with the semantic entailment relation $\models^{\mathtt{I}}$.

*Definition 5.5 (Soundness & completeness).* An entailment system $\mathcal{E}^{\mathtt{I}} = (\mathsf{Sig}^{\mathtt{I}}, \mathsf{Sen}^{\mathtt{I}}, \vdash^{\mathtt{I}})$ is *sound* (resp. *complete*) for an institution $\mathtt{I}$ if $\Gamma \vdash^{\mathtt{I}}_\Sigma \gamma$ implies $\Gamma \models^{\mathtt{I}}_\Sigma \gamma$ (resp. $\Gamma \models^{\mathtt{I}}_\Sigma \gamma$ implies $\Gamma \vdash^{\mathtt{I}}_\Sigma \gamma$) for all signatures $\Sigma$, each set of $\Sigma$-sentences $\Gamma$ and any $\Sigma$-sentence $\gamma$.

$$(Cons)\ \frac{\varphi(\Gamma) \vdash \varphi(e)}{\Gamma \vdash e}\ [\ \varphi \text{ is conservative }]$$

$$(Subst)\ \frac{\Gamma \vdash @_k\, \theta(e')}{\Gamma \vdash @_k\, \exists\chi \cdot e'}\ [\ \theta\colon 1_\chi \to 1_\Delta\ ]$$

$$(Neg_I)\ \frac{\Gamma \cup \{@_k\, e\} \vdash \bot}{\Gamma \vdash @_k\, \neg e}$$

$$(Neg_E)\ \frac{\Gamma \vdash @_k\, \neg e}{\Gamma \cup \{@_k\, e\} \vdash \bot} \qquad (Neg_D)\ \frac{\Gamma \vdash @_k\, \neg\neg e}{\Gamma \vdash @_k\, e}$$

$$(False_I)\ \frac{\Gamma \vdash @_k\, e \quad \Gamma \vdash @_k\, \neg e}{\Gamma \vdash \bot}$$

$$(False_E)\ \frac{\Gamma \vdash \bot}{\Gamma \vdash E}$$

$$(Disj_I)\ \frac{\Gamma \vdash @_k\, e}{\Gamma \vdash @_k\, \vee E}\ [\ e \in E\ ]$$

$$(Disj_E)\ \frac{\Gamma \vdash @_k\, \vee E \quad \Gamma \cup \{@_k\, e\} \vdash \gamma \text{ for all } e \in E}{\Gamma \vdash \gamma}$$

$$(Pos_I)\ \frac{\chi_z(\Gamma) \cup \{@_k\, \underline{\lambda}(z), @_z\, \chi_z(e)\} \vdash \chi_z(\gamma)}{\Gamma \cup \{@_k\, \langle\lambda\rangle e\} \vdash \gamma}$$

$$(Pos_E)\ \frac{\Gamma \cup \{@_k\, \langle\lambda\rangle e\} \vdash \gamma}{\chi_z(\Gamma) \cup \{@_k\, \underline{\lambda}(z), @_z\, \chi_z(e)\} \vdash \chi_z(\gamma)}$$

$$(Quant_I)\ \frac{\chi(\Gamma) \cup \{@_{k'}\, e'\} \vdash \chi(\gamma)}{\Gamma \cup \{@_k\, \exists\chi \cdot e'\} \vdash \gamma}\ [\ k' = F(\chi)(k)\ ]$$

$$(Quant_E)\ \frac{\Gamma \cup \{@_k\, \exists\chi \cdot e'\} \vdash \gamma}{\chi(\Gamma) \cup \{@_{k'}\, e'\} \vdash \chi(\gamma)}\ [\ k' = F(\chi)(k)\ ]$$

$$(Store_I)\ \frac{\Gamma \vdash @_k\, \varphi_{z \leftarrow k}(e'')}{\Gamma \vdash @_k \downarrow z \cdot e''}$$

$$(Store_E)\ \frac{\Gamma \vdash @_k \downarrow z \cdot e''}{\Gamma \vdash @_k\, \varphi_{z \leftarrow k}(e'')}$$

$$(Ret_I)\ \frac{\Gamma \vdash e}{\Gamma \vdash @_k\, e}$$

$$(Ret_E)\ \frac{\chi_z(\Gamma) \vdash @_z \chi_z(e)}{\Gamma \vdash e}$$

Table 1. Proof rules for stratified institutions

Since a stratified institution is in particular an institution, the definition of entailment system is used also in a framework given by a stratified institution, where it provides the necessary ingredients for a complete description of the idea of stratified logic. The framework in which the results of this section are proved is defined below.

Framework 5.1 ( Stratified logic with structured syntax). *Throughout this section, we work within a stratified institution* $\mathtt{SI}$ *with a structured syntax according to Framework 4.1, for which we assume an entailment system* $\mathcal{E}^{\mathtt{SI}_b} = (\mathsf{Sig}^{\mathtt{SI}}, \mathsf{F}^{\mathtt{SI}}, \mathsf{Sen}_b, \vdash^{\mathtt{SI}_b})$ *for* $\mathtt{SI}_b$. □

A concrete example of $\mathtt{SI}$ is HPL. The entailment system $\mathcal{E}^{\mathsf{HPL}_b}$ is the least entailment system of $\mathsf{HPL}_b$ closed under $(Ret_I)$ and $(Ret_E)$ defined in Table 1, and the proof rules defined in Table 2. Another example of $\mathtt{SI}$ is HFOLR. The entailment system $\mathcal{E}^{\mathsf{HFOLR}_b}$ is is the least entailment system of $\mathsf{HFOLR}_b$ closed under $(Ret_I)$ and $(Ret_E)$ defined in Table 1, and the proof rules defined in Table 4.

One may wonder why we assume an entailment system for $\mathtt{SI}_b$ and not for $\mathtt{SI}_0$. The choice is motivated by practical necessities. In concrete examples of modal logics, it is difficult to prove completeness in the absence of retrieve. Also, the abstract completeness result for $\mathtt{SI}$ is proved under the assumption that $\mathcal{E}^{\mathtt{SI}_b}$ is complete.

*Definition 5.6.* The entailment system $\mathcal{E}^{\mathtt{SI}} = (\mathsf{Sig}^{\mathtt{SI}}, \mathsf{Sen}^{\mathtt{SI}}, \vdash^{\mathtt{SI}})$ of $\mathtt{SI}$ is the least entailment system over $\mathcal{E}^{\mathtt{SI}_b}$ closed under the proof rules defined in Table 1.

In all examples of institutions presented in this work, a signature morphism is conservative iff it is injective. This property relies on the fact that the carrier sets of the models consist of non-empty sets. Therefore, the side condition of the rule (*Cons*) can be easily checked in concrete examples and it doesn't alter the effectiveness of the entailment system.

Since $\bot$ entails any set of sentences, the notion of consistency can be defined in terms of $\bot$.

Remark 5.1. *A set of sentences $\Gamma$ is consistent iff $\Gamma \nvdash \bot$.*

Some useful properties of $\mathcal{E}^{\mathsf{SI}}$ are enumerated below.

Lemma 5.7. *The entailment system $\mathcal{E}^{\mathsf{SI}}$ has the following properties:*
*(1) $\Gamma \vdash @_k \neg\langle\lambda\rangle e$ iff $\chi_z(\Gamma) \cup \{@_z\, \chi_z(e)\} \vdash @_k \neg\underline{\lambda}(z)$;*
*(2) $\Gamma \cup \{@_{k_1}\, e\} \vdash @_k \neg\underline{\lambda}(k_1)$ if $\Gamma \vdash @_k \neg\langle\lambda\rangle e$;*
*(3) $\Gamma \vdash @_k \neg\exists\chi \cdot e'$ iff $\chi(\Gamma) \vdash @_{k'} \neg e'$, where $k' = F^{\mathsf{SI}}(\chi)(k)$;*
*(4) $\Gamma \vdash @_k \forall\chi \cdot e'$ iff $\chi(\Gamma) \vdash @_{k'}\, e'$, where $k' = \mathsf{F}^{\mathsf{SI}}(\chi)(k)$;*
*(5) $@_k\, e_1 \vdash @_k\, e_2$ iff $@_k \neg e_2 \vdash @_k \neg e_1$;*
*(6) $@_k \forall\chi \cdot e' \vdash @_k\, \theta(e')$, where $\theta\colon \chi \to 1_\Delta$ such that $\theta(\neg e) = \neg\theta(e)$ for all $\Delta$-sentences $e$.*

Proof.
(1) $\Gamma \vdash @_k \neg\langle\lambda\rangle e$ iff $\Gamma \cup \{@_k\, \langle\lambda\rangle e\} \vdash \bot$ iff $\chi_z(\Gamma) \cup \{@_k\, \underline{\lambda}(z), @_z\, \chi_z(e)\} \vdash \bot$ iff $\chi_z(\Gamma) \cup \{@_z\, \chi_z(e)\} \vdash @_k \neg\underline{\lambda}(z)$.
(2) Assume that $\Gamma \vdash @_k \neg\langle\lambda\rangle e$. By (1), $\chi_z(\Gamma) \cup \{@_z\, \chi_z(e)\} \vdash @_k \neg\underline{\lambda}(z)$. By *(Translation)* via $\varphi_{z \leftarrow k_1}$, we obtain $\Gamma \cup \{@_{k_1}\, e\} \vdash @_k \neg\underline{\lambda}(k_1)$.
(3) $\Gamma \vdash @_k \neg\exists\chi \cdot e'$ iff $\Gamma \cup \{@_k\, \exists\chi \cdot e'\} \vdash \bot$ iff $\chi(\Gamma) \cup \{@_{k'}\, e'\} \vdash \bot$ iff $\chi(\Gamma) \vdash @_{k'} \neg e'$.
(4) $\Gamma \vdash @_k \forall\chi \cdot e'$ iff $\Gamma \vdash @_k \neg\exists\chi \cdot \neg e'$ iff $\chi(\Gamma) \vdash @_{k'} \neg\neg e'$ iff $\chi(\Gamma) \vdash @_{k'}\, e'$.
(5) Notice that $@_k\, e \vdash @_k \neg\neg e$ iff $\{@_k\, e, @_k \neg e\} \vdash \bot$, which holds by (*False*$_I$). It follows that $@_k \neg e_2 \vdash @_k \neg e_1$ iff $\{@_k \neg e_2, @_k\, e_1\} \vdash \bot$ iff $@_k\, e_1 \vdash @_k \neg\neg e_2$ iff $@_k\, e_1 \vdash @_k\, e_2$.
(6) $@_k \forall\chi \cdot e' \vdash @_k\, \theta(e')$ iff $@_k \neg\exists\chi \cdot \neg e' \vdash @_k\, \theta(e')$ iff $@_k \neg\exists\chi \cdot \neg e' \vdash @_k \neg\neg\theta(e')$ iff $@_k \neg\theta(e') \vdash @_k\, \exists\chi \cdot \neg e'$ iff $@_k\, \theta(\neg e') \vdash @_k\, \exists\chi \cdot \neg e'$, which holds by (*Subst*). □

## 5.2 Soundness

Formal reasoning based on the proof rules defined in Table 1 is sound.

Theorem 5.8 (Soundness). *$\mathcal{E}^{\mathsf{SI}}$ is sound if $\mathcal{E}^{\mathsf{SI}_b}$ is sound.*

Proof. It suffices to show that the proof rules defined in Table 1 are sound. We focus on the cases corresponding to quantification as the remaining cases are similar.

[ *For* (*Quant*$_I$) ] Assume that $\chi(\Gamma) \cup \{@_{k'}\, e'\} \models \chi(\gamma)$. We show that $\Gamma \cup \{@_k\, \exists\chi \cdot e'\} \models \gamma$.

| | | |
|---|---|---|
| 1 | assume $M \models^{\mathsf{SI}} \Gamma \cup \{@_k\, \exists\chi \cdot e'\}$ | |
| 2 | there exists a $\chi$-expansion $M'$ of $M$ such that $M' \models^{w} e'$, where $w$ is the denotation of $k$ in $M$ | since $M \models^{\mathsf{SI}} @_k\, \exists\chi \cdot e'$ |
| 3 | $M' \models^{\mathsf{SI}} \chi(\Gamma)$ | by the satisfaction condition |
| 4 | $M' \models^{\mathsf{SI}} @_{k'}\, e'$ | by 2, since $w$ is the denotation of $k'$ in $M'$ |
| 5 | $M' \models^{\mathsf{SI}} \chi(\gamma)$ | by 3 and 4, since $\chi(\Gamma) \cup \{@_{k'}\, e'\} \models \chi(\gamma)$ |
| 6 | $M \models^{\mathsf{SI}} \gamma$ | by the satisfaction condition |

[ *For* (*Quant*$_E$) ] Assume that $\Gamma \cup \{@_k\, \exists\chi \cdot e'\} \models \gamma$. We show that $\chi(\Gamma) \cup \{@_{k'}\, e'\} \models \chi(\gamma)$.

| | | |
|---|---|---|
| 1 | assume $M' \models^{\mathsf{SI}} \chi(\Gamma) \cup \{@_{k'}\, e'\}$ | |
| 2 | $M' \restriction_\chi \models^{\mathsf{SI}} \Gamma$ | by the satisfaction condition |
| 3 | $M' \models^{w} e'$, where $w$ is the denotation of $k'$ in $M'$ | since $M' \models^{\mathsf{SI}} @_{k'}\, e'$ |
| 4 | $M' \restriction_\chi \models^{w} \exists\chi \cdot e'$ | by the definition of $\models$ |
| 5 | $M' \restriction_\chi \models^{\mathsf{SI}} @_k\, \exists\chi \cdot e'$ | since $w$ is the denotation of $k$ in $M' \restriction_\chi$ |
| 6 | $M' \restriction_\chi \models^{\mathsf{SI}} \gamma$ | by 2 and 5, since $\Gamma \cup \{@_k\, \exists\chi \cdot e'\} \models \gamma$ |

7 $M' \models^{\mathtt{SI}} \chi(\gamma)$ by the satisfaction condition □

### 5.3 Compactness

The entailment system of SI is compact as the proof rules defined in Table 1 have a finite number of premises.

Theorem 5.9 (Compactness). *$\mathcal{E}^{\mathtt{SI}}$ is compact if $\mathcal{E}^{\mathtt{SI}_b}$ is compact.*

Proof. Let $\mathcal{E}^c = (\mathsf{Sig}^{\mathtt{SI}}, \mathsf{Sen}^{\mathtt{SI}}, \vdash^c)$ be the compact entailment subsystem of $\mathcal{E}^{\mathtt{SI}}$. By compactness of $\mathcal{E}^{\mathtt{SI}_b}$, we have $\vdash^{\mathtt{SI}_b}_{\Delta} \subseteq \vdash^c_{\Delta}$ for all $\Delta \in |\mathsf{Sig}^{\mathtt{SI}}|$. If we show that $\mathcal{E}^c$ is closed under the proof rules defined in Table 1 then since $\mathcal{E}^{\mathtt{SI}}$ is the least entailment system over $\mathcal{E}^{\mathtt{SI}_b}$ closed under the proof rules defined in Table 1, we get $\mathcal{E}^c = \mathcal{E}^{\mathtt{SI}}$. We focus on ($Disj_E$) and ($Quant_I$) as the remaining cases are similar.

[ *For* ($Disj_E$) ] Assume that $\Gamma \vdash^c @_k \vee E$ and $\Gamma \cup \{@_k\, e\} \vdash^c \gamma$ for all $e \in E$. There exist (a) $\Gamma_f \subseteq \Gamma$ finite such that $\Gamma_f \vdash @_k \vee E$, and (b) $\Gamma_e \subseteq \Gamma$ finite such that $\Gamma_e \cup \{@_k\, e\} \vdash \gamma$ for all $e \in E$. Since $E$ is finite, the set $\Gamma' = \Gamma_f \cup (\bigcup_{e \in E} \Gamma_e)$ is finite too. By *(Monotonicity)* and *(Transitivity)*, $\Gamma' \vdash @_k \vee E$ and $\Gamma' \cup \{@_k\, e\} \vdash \gamma$ for all $e \in E$. It follows that $\Gamma' \vdash \gamma$. By the definition of $\vdash^c$, $\Gamma \vdash^c \gamma$.

[ *For* ($Quant_I$) ] Assume that $\chi(\Gamma) \cup \{@_{k'}\, e'\} \vdash^c \chi(\gamma)$. There exists $\Gamma_f \subseteq \Gamma$ finite such that $\chi(\Gamma_f) \cup \{@_{k'}\, e'\} \vdash \chi(\gamma)$. It follows that $\Gamma_f \cup \{@_k\, \exists\chi \cdot e'\} \vdash \gamma$. By the definition of $\vdash^c$, $\Gamma \cup \{@_k\, \exists\chi \cdot e'\} \vdash^c \gamma$.

□

The compactness result is used for the proof of completeness in the next subsection.

### 5.4 Completeness

Completeness is much more difficult to establish than soundness or compactness and it requires a quite sophisticated infrastructure for its proof. We give a proof of completeness using the forcing technique defined in Section 4. Based on the entailment relation generated by the proof rules in Table 1, one can define a forcing property for the vertex of an extension.

*Definition 5.10 (Canonical forcing property).* Let $\langle \mathcal{V}, \vartheta \rangle$ be an extension of $\Delta$ as in Definition 3.17. The *canonical forcing property* $\mathbb{P} = \langle \mathsf{P}, \leq, f \rangle$ over $\langle \mathcal{V}, \vartheta \rangle$ is defined as follows:

(1) $\mathsf{P} = \{p \mid p \subseteq \vartheta_{C'}(\mathsf{Sen}(\Delta_{C'}))$ for some $C' \in \mathcal{P}_\alpha(C)$ and $p$ is consistent$\}$,
(2) $f(p) = p \cap \mathsf{Sen}_b(\Delta_C)$ for all $p \in \mathsf{P}$, and
(3) $\leq$ is the inclusion relation $\subseteq$.

Definition 5.10 is at the core of the proof of completeness. In concrete examples, the condition $p \subseteq \vartheta_{C'}(\mathsf{Sen}(\Delta_{C'}))$ says that the cardinality of the set of all constants from $C$ that occur in $p$ is strictly less than $\alpha$. The proposition below shows that the canonical forcing property is well-defined provided that the entailment system of $\mathtt{SI}_b$ is complete.

Lemma 5.11. *If $\mathcal{E}^{\mathtt{SI}_b}$ is complete then $\mathbb{P}$ described in Definition 5.10 is indeed a forcing property.*

Proof. All conditions described in Definition 4.2, except the last one, obviously hold for $\mathbb{P}$. Assume a condition $p \in \mathsf{P}$ and a sentence $@_k\, e \in \mathsf{Sen}_b(\Delta)$ such that $f(p) \models^{\mathtt{SI}} @_k\, e$. We show that $p \cup \{@_k\, e\} \in \mathsf{P}$:

By Definition 5.10, $p \subseteq \vartheta_{C'}(\mathsf{Sen}(\Delta_{C'}))$ for some $C' \in \mathcal{P}_\alpha(C)$. By Definition 3.17, $@_k\, e \in \vartheta_{C''}(\mathsf{Sen}(\Delta_{C''}))$ for some $C'' \in \mathcal{P}_\alpha(C)$. Since $\mathcal{P}_\alpha(C)$ is closed under unions, $p \cup \{@_k\, e\} \subseteq \vartheta_{C'''}(\mathsf{Sen}(\Delta_{C'''}))$, where $C''' = C' \cup C'' \in \mathcal{P}_\alpha(C)$. Since $f(p) \models^{\mathtt{SI}} @_k\, e$ and $\mathcal{E}^{\mathtt{SI}_b}$ is complete, $f(p) \vdash @_k\, e$. By *(Monotonicity)* and *(Transitivity)*, $p \vdash @_k\, e$. By Lemma 5.2, $p \cup \{@_k\, e\}$ is consistent. It follows that $p \cup \{@_k\, e\} \in \mathsf{P}$.

Hence, $@_k\, e \in f(p \cup \{@_k\, e\})$. □

Since completeness of $\mathcal{E}^{\mathtt{SI}_b}$ is sufficient for proving that $\mathbb{P}$ is a forcing property, this assumption will be a part of the hypotheses of all results that follow. The definition of extension describes in a category-based setting the extension of the initial language $\Delta$ with an infinite number of Henkin witness constants $C$. The following result presents a general technique for assigning witness constants to the quantified variables that appear in the conditions of the canonical forcing property.

Proposition 5.12. *Assume that $\mathcal{E}^{\mathtt{SI}_b}$ is complete. Let $\mathbb{P} = \langle \mathsf{P}, \leq, f\rangle$ be a canonical forcing property over an extension $\langle \mathcal{V}, \vartheta\rangle$ as described in Definition 5.10. Then $\mathbb{P}$ has the following properties:*

*(1) If $p \in \mathsf{P}$ and $@_k \vee E \in p$ then $p \cup \{@_k\, e\} \in \mathsf{P}$ for some $e \in E$.*

*(2) If $p \in \mathsf{P}$ and $@_k \langle\lambda\rangle e \in p$ such that $\langle\lambda\rangle e \notin \mathsf{Sen}_0(\Delta_C)$ then $p \cup \{@_k\, \underline{\lambda}(k_1), @_{k_1}\, e\} \in \mathsf{P}$ for some nominal $k_1$.*

*(3) If $p \in \mathsf{P}$ and $@_k\, \exists\chi \cdot e' \in p$ such that $\exists\chi \cdot e' \notin \mathsf{Sen}_0(\Delta_C)$, where $\chi\colon \Delta_C \to \Delta'$, then $p \cup \{@_k\, \varphi(e')\} \in \mathsf{P}$ for some signature morphism $\varphi\colon \Delta' \to \Delta_C$ such that $\chi;\varphi = 1_{\Delta_C}$.*

Proof.

(1) Let $p \in \mathsf{P}$ and $@_k \vee E \in p$. By *(Monotonicity)*, $p \vdash @_k \vee E$. By Definition 5.10, $p \nvdash \bot$.
Suppose that $p \cup \{@_k\, e\} \vdash \bot$ for all $e \in E$ then since $p \vdash @_k \vee E$, by $(Disj_E)$, $p \vdash \bot$; by Definition 5.10, $p \notin \mathsf{P}$, which is a contradiction to $p \in \mathsf{P}$.
It follows that $p \cup \{@_k\, e\} \nvdash \bot$ for some $e \in E$. By Definition 5.10, $p \cup \{@_k\, e\} \in \mathsf{P}$ for some $e \in E$.

(2) Let $p \in \mathsf{P}$ and $@_k \langle\lambda\rangle e \in p$. By Definition 5.10, $p = \vartheta_{C'}(p')$ for some $C' \in \mathcal{P}_\alpha(C)$ and $p' \subseteq \mathsf{Sen}(\Delta_{C'})$. Since $\langle\lambda\rangle e \notin \mathsf{Sen}_0(\Delta_C)$, by Framework 4.1 (F4 and F3), there exists a sentence $@_{k'} \langle\lambda'\rangle e' \in p'$ such that $\mathsf{F}(\vartheta_{C'})(k') = k$, $\mathsf{F}(\vartheta_{C'})(\lambda') = \lambda$ and $\vartheta_{C'}(e') = e$. Let $z'$ be a nominal variable for $\Delta_{C'}$. By Definition 3.2, there exists a designated pushout as depicted in the left side of Figure 2.

$$\begin{tikzcd}
 & \Delta_{C'}[z'] \arrow[r, dashed, "v'"] & \Delta_C[z] \\
\Delta_{C'} \arrow[ur, dashed, "\chi_{z'}"] \arrow[rr, "\vartheta_{C'}"'] \arrow[urr, phantom, "pushout"] & & \Delta_C \arrow[u, "\chi_z"']
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \Delta_{C'}[z'] \arrow[r, dashed, "v'"] \arrow[d, dashed, "\varphi_{z'}"] & \Delta_C[z] \arrow[d, dashed, "\varphi_z"] \\
\Delta_{C'} \arrow[ur, dashed, "\chi_{z'}"] \arrow[r, "\mathcal{V}(C' \subseteq C'')"'] & \Delta_{C''} \arrow[r, "\vartheta_{C''}"'] & \Delta_C \arrow[r, "1_\Delta"'] \arrow[u, "\chi_z"'] & \Delta_C
\end{tikzcd}$$

Fig. 2. Pushout image

By Definition 3.17, there exist $(C' \subseteq C'') \in (\mathcal{P}_\alpha(C), \subseteq)$ and $\varphi_{z'}\colon \Delta_{C'}[z'] \to \Delta_{C''}$ conservative such that $\chi_{z'};\varphi_{z'} = \mathcal{V}(C' \subseteq C'')$. By the pushout depicted in Figure 2, since $\chi_{z'};(\varphi_{z'};\vartheta_{C''}) = \vartheta_{C'};1_\Delta$, there exists $\varphi_z\colon \Delta_C[z] \to \Delta_C$ such that $\chi_z;\varphi_z = 1_{\Delta_C}$ and $v';\varphi_z = \varphi_{z'};\vartheta_{C''}$.
Suppose that $\chi_{z'}(p') \cup \{@_{k'}\, \underline{\lambda}'(z'), @_{z'}\, \chi_{z'}(e')\} \vdash \bot$; then by $(Pos_I)$, $p' \cup \{@_{k'} \langle\lambda'\rangle e'\} \vdash \bot$; since $@_{k'} \langle\lambda'\rangle e' \in p'$, we have $p' \vdash \bot$; by *(Translation)*, $p \vdash \bot$, and by Definition 5.10, $p \notin \mathsf{P}$; hence, we obtained a contradiction to $p \in \mathsf{P}$.
It follows that $\chi_{z'}(p') \cup \{@_{k'}\, \underline{\lambda}'(z'), @_{z'}\, \chi_{z'}(e')\} \nvdash \bot$. By applying $(Cons)$ to $(\varphi_{z'};\vartheta_{C''})$, we obtain $\vartheta_{C''}(\,\varphi_{z'}\,(\chi_{z'}(p') \cup \{@_{k'}\, \underline{\lambda}'(z'), @_{z'}\, \chi_{z'}(e')\}\,)\,) \nvdash \bot$. By the commutativity of the diagrams depicted in the right side of Figure 2, $p \cup \{@_k\, \underline{\lambda}(k_1), @_{k_1}\, e\} \nvdash \bot$, where $k_1 = \mathsf{F}(\varphi_{z'};\vartheta_{C''})(z')$. By Definition 5.10, $p \cup \{@_k\, \underline{\lambda}(k_1), @_{k_1}\, e\} \in \mathsf{P}$.

(3) Let $p \in \mathsf{P}$ and $@_k\, \exists\chi \cdot e' \in p$. By Definition 5.10, $p = \vartheta_{C'}(p')$ for some $C' \in \mathcal{P}_\alpha(C)$ and $p' \subseteq \mathsf{Sen}(\Delta_{C'})$. Since $\exists\chi \cdot e' \notin \mathsf{Sen}_0(\Delta_C)$, by Framework 4.1 (F4 and F3), there exist
   (a) a sentence $@_{k'}\, \exists\chi' \cdot e'' \in p'$, where $\chi'\colon \Delta_{C'} \to \Delta''$, and
   (b) a designated pushout depicted in the left side of Figure 3,

such that $\mathsf{F}(\vartheta_{C'})(k') = k$, $\upsilon'(e'') = e'$ and $\vartheta_{C'}(\exists\chi' \cdot e'') = \exists\chi \cdot e'$.

$$\begin{tikzcd}
 & \Delta'' \arrow[r, dashed, "\upsilon'"] & \Delta' \\
\Delta_{C'} \arrow[ur, dashed, "\chi'"] \arrow[rr, "\vartheta_{C'}"'] & \textit{pushout} & \Delta_C \arrow[u, "\chi"']
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \Delta'' \arrow[r, dashed, "\upsilon'"] \arrow[d, dashed, "\varphi'"] & \Delta' \arrow[d, dashed, "\varphi"] \\
\Delta_{C'} \arrow[ur, dashed, "\chi'"] \arrow[r, "\mathcal{V}(C' \subseteq C'')"'] & \Delta_{C''} \arrow[r, "\vartheta_{C''}"'] & \Delta_C \arrow[u, "\chi"'] \arrow[r, "1_\Delta"'] & \Delta_C
\end{tikzcd}$$

Fig. 3. Pushout image

By Definition 3.17, there exist $(C' \subseteq C'') \in (\mathcal{P}_\alpha(C), \subseteq)$ and $\varphi' \colon \Delta'' \to \Delta_{C''}$ conservative such that $\chi'; \varphi' = \mathcal{V}(C' \subseteq C'')$. By the pushout depicted in Figure 3, since $\chi'; (\varphi'; \vartheta_{C''}) = \vartheta_{C'}; 1_\Delta$, there exists $\varphi \colon \Delta' \to \Delta_C$ such that $\chi; \varphi = 1_{\Delta_C}$ and $\upsilon'; \varphi = \varphi'; \vartheta_{C''}$.

Suppose that $\chi'(p') \cup \{@_{k''} e''\} \vdash \bot$, where $k'' = \mathsf{F}(\chi')(k')$; then by (*Quant*$_I$), we have $p' \cup \{@_{k'} \exists\chi' \cdot e''\} \vdash \bot$; since $@_{k'} \exists\chi' \cdot e'' \in p'$, we get $p' \vdash \bot$; by *(Translation)*, $p \vdash \bot$, and by Definition 5.10, $p \notin \mathsf{P}$; hence, we obtained a contradiction to $p \in \mathsf{P}$.

It follows that $\chi'(p') \cup \{@_{k''} e''\} \nvdash \bot$. By applying (*Cons*) to $(\varphi'; \vartheta_{C''})$, we get $\vartheta_{C''}(\varphi'(\chi'(p') \cup \{@_{k''} e''\})) \nvdash \bot$. By the commutativity of the diagrams depicted in the right side of Figure 3, $p \cup \{@_k \varphi(e')\} \nvdash \bot$. By Definition 5.10, $p \cup \{@_k \varphi(e')\} \in \mathsf{P}$. □

Proposition 5.12 sets the basis for the following important result concerning canonical forcing properties, which says that all sentences of a given condition are forced eventually by the some condition greater or equal than the initial one.

Theorem 5.13. *Assume that $\mathcal{E}^{\mathsf{SI}_b}$ is complete. Let $\mathbb{P} = \langle \mathsf{P}, \leq, f \rangle$ be a canonical forcing property over an extension $\langle \mathcal{V}, \vartheta \rangle$ as described in Definition 5.10. Then for all $\Delta_C$-sentences $e$, nominals $k$ and conditions $p \in \mathsf{P}$ we have:*

$$q \Vdash^k e \text{ for some } q \geq p \text{ iff } p \cup \{@_k\, e\} \in \mathsf{P}.$$

Proof. We proceed by induction on the structure of $e$.

[ *For* $e \in \mathsf{Sen}_0(\Delta_C)$ ] Assume that there is $q \geq p$ such that $q \Vdash^k e$. We show that $p \cup \{@_k\, e\} \in \mathsf{P}$:

| | | |
|---|---|---|
| 1 | $@_k\, e \in q$ | by Definition 4.3 |
| 2 | $p \cup \{@_k\, e\} \subseteq q$ | since $q \geq p$ |
| 3 | $p \cup \{@_k\, e\}$ is consistent | since $q$ is consistent and $p \cup \{@_k\, e\} \subseteq q$ |
| 4 | $p \cup \{@_k\, e\} \in \mathsf{P}$ | by Definition 5.10 |

Assume that $p \cup \{@_k\, e\} \in \mathsf{P}$. Let $q = p \cup \{@_k\, e\}$. By Definition 4.3, $q \Vdash^k e$.

[ *For* $\neg e$ ] By the induction hypothesis, for each $q \in \mathsf{P}$ we have

(S1) $r \Vdash^k e$ for some $r \geq q$ iff $q \cup \{@_k\, e\} \in \mathsf{P}$, which is equivalent to

(S2) $r \nVdash^k e$ for all $r \geq q$ iff $q \cup \{@_k\, e\} \notin \mathsf{P}$, which is equivalent to

(S3) $q \Vdash^k \neg e$ iff $q \cup \{@_k\, e\} \notin \mathsf{P}$.

Assume that $q \Vdash^k \neg e$ for some $q \geq p$. We show that $p \cup \{@_k \neg e\} \in \mathsf{P}$:

| | | |
|---|---|---|
| 1 | $q \cup \{@_k\, e\} \notin \mathsf{P}$ | by statement S3 |
| 2 | $q \cup \{@_k\, e\} \vdash \bot$ | by Definition 5.10 |
| 3 | $q \vdash @_k \neg e$ | by (*Neg*$_I$) |
| 4 | $q \cup \{@_k \neg e\}$ is consistent | by Lemma 5.2 |
| 5 | $p \cup \{@_k \neg e\}$ is consistent | since $p \cup \{@_k \neg e\} \subseteq q \cup \{@_k \neg e\}$ |
| 6 | $p \cup \{@_k \neg e\} \in \mathsf{P}$ | by Definition 5.10 |

Assume that $p \cup \{@_k \neg e\} \in \mathsf{P}$. We show that $q \Vdash^k \neg e$ for some $q \geq p$:

| | | |
|---|---|---|
| 1 | let $q = p \cup \{@_k \neg e\}$ | |
| 2 | $q \cup \{@_k e\} \notin \mathsf{P}$ | since $@_k \neg e \in q$ |
| 3 | $q \Vdash^k \neg e$ | by statement S3 |

[ *For* $\vee E$ ] Assume that there exists $q \geq p$ such that $q \Vdash^k \vee E$. We show that $p \cup \{@_k \vee E\} \in \mathsf{P}$:

| | | |
|---|---|---|
| 1 | $q \Vdash^k e$ for some $e \in E$ | by Definition 4.3 |
| 2 | $p \cup \{@_k e\}$ is consistent | by the induction hypothesis |
| 3 | $@_k e \vdash @_k \vee E$ | by (*Disj*$_I$) |
| 4 | $p \cup \{@_k e\} \vdash @_k \vee E$ | by *(Monotonicity)* and *(Transitivity)* |
| 5 | $p \cup \{@_k e, @_k \vee E\}$ is consistent | by Lemma 5.2 |
| 6 | $p \cup \{@_k \vee E\}$ is consistent | $p \cup \{@_k \vee E\} \subseteq p \cup \{@_k e, @_k \vee E\}$ |

Assume that $p \cup \{@_k \vee E\} \in \mathsf{P}$. We show that $q \Vdash^k \vee E$ for some $q \geq p$:

| | | |
|---|---|---|
| 1 | $(p \cup \{@_k \vee E\}) \cup \{@_k e\} \in \mathsf{P}$ for some $e \in E$ | by Proposition 5.12 (1) |
| 2 | $q \Vdash^k e$ for some $q \geq p \cup \{@_k \vee E\}$ | by the induction hypothesis |
| 3 | $q \Vdash^k \vee E$ for some $q \geq p$ | by Definition 4.3 |

[ *For* $@_{k_1} e$ ] Straightforward, since $@_k @_{k_1} e \dashv @_{k_1} e$.

[ *For* $\langle\lambda\rangle e$ ] Assume that there exists $q \geq p$ such that $q \Vdash^k \langle\lambda\rangle e$. We show that $p \cup \{@_k \langle\lambda\rangle e\} \in \mathsf{P}$:

| | | |
|---|---|---|
| 1 | $q \Vdash^k \underline{\lambda}(k_1)$ and $q \Vdash^{k_1} e$ for some nominal $k_1$ | from $q \Vdash^k \langle\lambda\rangle e$, by Definition 4.3 |
| 2 | $q \cup \{@_{k_1} e\} \in \mathsf{P}$ | from $q \leq q$ and $q \Vdash^{k_1} e$, by the induction hypothesis |
| 3 | $@_k \underline{\lambda}(k_1) \in q$ | from $q \Vdash^k \underline{\lambda}(k_1)$, by Definition 4.3 |
| 4 | $q \cup \{@_k \langle\lambda\rangle e\} \nvdash \bot$ | if $q \cup \{@_k \langle\lambda\rangle e\} \vdash \bot$ then by (*Neg*$_I$), $q \vdash @_k \neg\langle\lambda\rangle e$ and by Lemma 5.7 (2), $q \cup \{@_{k_1} e\} \vdash @_k \neg\underline{\lambda}(k_1)$, which is a contradiction to $@_k \underline{\lambda}(k_1) \in q$ |
| 5 | $p \cup \{@_k \langle\lambda\rangle e\} \nvdash \bot$ | since $p \subseteq q$ |
| 6 | $p \cup \{@_k \langle\lambda\rangle e\} \in \mathsf{P}$ | by Definition 5.10 |

Assume that $p \cup \{@_k \langle\lambda\rangle e\} \in \mathsf{P}$. We show that $q \Vdash^k \langle\lambda\rangle e$ for some $q \geq p$:

| | | |
|---|---|---|
| 1 | $(p \cup \{@_k \langle\lambda\rangle e\}) \cup \{@_k \underline{\lambda}(k_1), @_{k_1} e\} \in \mathsf{P}$ for some nominal $k_1$ | by Proposition 5.12 (2) |
| 2 | $q \Vdash^{k_1} e$ for some $q \geq p \cup \{@_k \langle\lambda\rangle e, @_k \underline{\lambda}(k_1)\}$ | by the induction hypothesis |
| 3 | $q \Vdash^k \underline{\lambda}(k_1)$ | since $@_k \underline{\lambda}(k_1) \in f(q)$ |
| 4 | $q \Vdash^k \langle\lambda\rangle e$ | by 3 and 2 |

[ *For* $\downarrow z \cdot e$ ] Straightforward, since $@_k \downarrow z \cdot e \dashv @_k \varphi_{z \leftarrow k}(e)$.

[ *For* $\exists\chi \cdot e$ ] Assume that $q \Vdash^k \exists\chi \cdot e$ for some $q \geq p$. We show that $p \cup \{@_k \exists\chi \cdot e\} \in \mathsf{P}$:

| | | |
|---|---|---|
| 1 | $q \Vdash^k \theta(e)$ for some substitution $\theta\colon \chi \to 1_{\Delta_C}$ | by Definition 4.3 |
| 2 | $p \cup \{@_k \theta(e)\}$ is consistent | by the induction hypothesis |
| 3 | $p \cup \{@_k \theta(e)\} \vdash @_k \exists\chi \cdot e$ | by (*Subst*) |
| 4 | $p \cup \{@_k \theta(e), @_k \exists\chi \cdot e\}$ is consistent | by Lemma 5.2 |
| 5 | $p \cup \{@_k \exists\chi \cdot e\}$ is consistent | since $p \cup \{@_k \exists\chi \cdot e\} \subseteq p \cup \{@_k \theta(e), @_k \exists\chi \cdot e\}$ |
| 6 | $p \cup \{@_k \exists\chi \cdot e\} \in \mathsf{P}$ | by Definition 5.10 |

We assume that $p \cup \{@_k \exists\chi \cdot e\} \in \mathsf{P}$. We show that $q \Vdash^k \exists\chi \cdot e$ for some $q \geq p$:

| | | |
|---|---|---|
| 1 | $(p \cup \{@_k\, \exists\chi \cdot e\}) \cup \{@_k\, \varphi(e)\} \in \mathsf{P}$ and $\chi; \varphi = 1_{\Delta_C}$ for some $\varphi\colon \Delta' \to \Delta_C$ | by Proposition 5.12 (3) |
| 2 | $q \Vdash^k \varphi(e)$ for some $q \geq p \cup \{@_k\, \exists\chi \cdot e\}$ | by the induction hypothesis |
| 3 | $q \Vdash^k \exists\chi \cdot e$ for some $q \geq p$ | by Definition 4.3 |

□

The following result is a corollary of Theorem 5.13. It shows that each generic set of a given canonical forcing property has a reachable model that satisfies all its conditions. This is essential for proving the completeness theorem.

Corollary 5.14. *Assume that $\mathcal{E}^{\mathsf{SI}_b}$ is complete. Let $\mathbb{P} = \langle \mathsf{P}, \leq, f \rangle$ be a canonical forcing property over an extension $\langle \mathcal{V}, \vartheta \rangle$. Then for each generic set $G$ we have:*

*(1) $G \Vdash^k e$ for all conditions $p \in G$, sentences $e \in p$ and nominals $k$.*

*(2) Under the hypotheses of Theorem 4.8, there exists a generic model for $G$ which is reachable and satisfies each condition $p \in G$.*

Proof.

(1) We show that $G \Vdash^k e$ for all conditions $p \in G$, sentences $e \in p$ and nominals $k$. Suppose towards a contradiction that $G \nVdash^k e$ for some $p \in G$, $e \in p$ and nominal $k$. Then:

| | | |
|---|---|---|
| 1 | $q \Vdash^k \neg e$ for some $q \in G$ | since $G \nVdash^k e$ and $G$ is generic |
| 2 | $r \geq p$ and $r \geq q$ for some $r \in G$ | since $G$ is generic |
| 3 | $e \in r$ | since $e \in p$ and $r \geq p$ |
| 4 | $r \vdash @_k\, e$ | by $(Ret_I)$ |
| 5 | $r \cup \{@_k\, e\}$ is consistent | by Lemma 5.2 |
| 6 | $s \Vdash^k e$ for some $s \geq r$ | by Theorem 5.13, since $r \cup \{@_k\, e\} \in \mathsf{P}$ |
| 7 | $s \Vdash^k \neg e$ | by Lemma 4.4 (2), since $s \geq q$ and $q \Vdash^k \neg e$ |
| 8 | contradiction | by Lemma 4.4 (4), since $s \Vdash^k e$ and $s \Vdash^k \neg e$ |

It follows that $G \Vdash^k e$ for all $p \in G$, $e \in p$ and nominals $k$.

(2) By Theorem 4.8, there exists a generic model $M$ for $G$ which is reachable. Let $e \in p$ and $w \in \mathsf{K}_\Delta(M)$. We show that $M \models^w e$: by Lemma 4.1, $w = K_\Delta(M)_k$ for some nominal $k$; by the first part of the proof, $G \Vdash^k e$; since $M$ is a model for $G$, $M \models^{\mathsf{SI}} @_k\, e$; hence, $M \models^w e$. □

Lemma 4.6 ensures the existence of generic sets for forcing properties defined over signatures of countable power. In what follows, we prove that generic sets exist for canonical forcing properties defined over signatures of any power.

Proposition 5.15. *Assume that $\mathcal{E}^{\mathsf{SI}_b}$ is complete and compact. Let $\mathbb{P} = \langle \mathsf{P}, \leq, f \rangle$ be a canonical forcing property over an extension $\langle \mathcal{V}, \vartheta \rangle$. Every condition $p \in \mathsf{P}$ belongs to a generic set.*

Proof. We define the following chain of conditions $p_0 \leq p_1 \leq \ldots$ in $\mathsf{P}$ by induction on ordinals. Since $\mathsf{SI}$ is closed under retrieve, the cardinality of the set of the $\Delta_C$-sentences with retrieve as top operator is $\alpha$. Let $\{@_{k_i}\, e_i \mid i < \alpha\}$ be an enumeration of the $\Delta_C$-sentences with retrieve as top operator.

[ $i = 0$ ] $p_0 = p$. Let $C_0 \in \mathcal{P}_\alpha(C)$ such that $p_0 \subseteq \vartheta_{C_0}(\mathsf{Sen}(\Delta_{C_0}))$.

[ *$i < \alpha$ successor ordinal* ]
If $p_{i-1} \Vdash^{k_{i-1}} \neg e_{i-1}$ then let $p_i = p_{i-1}$ else choose $p_i \geq p_{i-1}$ such that $p_i \Vdash^{k_{i-1}} e_{i-1}$.
Note that there exists $(C_{i-1} \subseteq C_i) \in (\mathcal{P}_\alpha(C), \subseteq)$ such that $p_i \subseteq \vartheta_{C_i}(\mathsf{Sen}(\Delta_{C_i}))$.

[ *$\beta < \alpha$ limit ordinal* ] $p_\beta = \bigcup_{i<\beta} p_i$.
Let $C_\beta = \bigcup_{i<\beta} C_i$ and note that $C_\beta \in \mathcal{P}_\alpha(C)$ and $p_\beta \subseteq \vartheta_{C_\beta}(\mathsf{Sen}(\Delta_{C_\beta}))$.

Suppose towards a contradiction that $p_\beta$ is not consistent then by compactness, there exists $p' \subseteq p_\beta$ finite such that $p'$ is not consistent; since $p'$ is finite, $p' \subseteq p_i$ for some $i < \beta$; it follows that $p_i$ is not consistent, which is a contradiction. Hence, $p_\beta$ is consistent.

The set $G = \{q \in \mathsf{P} \mid q \leq p_i \text{ for some } i < \alpha\}$ is generic and contains $p$. □

Corollary 5.14 and Proposition 5.15 enable the proof of completeness.

Theorem 5.16 (Completeness). *Assume that*

*(1) $\mathcal{E}^{\mathtt{SI}_b}$ is complete and compact, and*
*(2) each signature $\Delta$ has an extension $\langle \mathcal{V}, \vartheta \rangle$ as in Definition 3.17 such that each subset of $\mathsf{Sen}_b(\Delta_C)$ is locally basic and it has a basic model that is reachable.*

*Then the entailment system $\mathcal{E}^{\mathtt{SI}} = (\mathsf{Sig}^{\mathtt{SI}}, \mathsf{Sen}^{\mathtt{SI}}, \vdash^{\mathtt{SI}})$ over $\mathcal{E}^{\mathtt{SI}_b}$ generated by the proof rules in Table 1 is complete.*

Proof. Firstly, we prove local completeness: assuming that $\Gamma \models_\Delta @_k\, e$, we show $\Gamma \vdash_\Delta @_k\, e$. Suppose towards a contradiction that $\Gamma \nvdash @_k\, e$. Then:

| | | |
|---|---|---|
| 1 | $\Gamma \cup \{@_k \neg e\} \nvdash \bot$ | if $\Gamma \cup \{@_k \neg e\} \vdash \bot$ then by ($Neg_I$), $\Gamma \vdash @_k \neg\neg e$, and by ($Neg_D$), $\Gamma \vdash @_k\, e$, which is a contradiction to $\Gamma \nvdash @_k\, e$ |
| 2 | let $\langle \mathcal{V}, \vartheta \rangle$ be an extension of $\Delta$ and let $\mathbb{P} = \langle \mathsf{P}, \leq, f \rangle$ be the canonical forcing property over $\langle \mathcal{V}, \vartheta \rangle$ as described in Definition 5.10 | |
| 3 | $\vartheta_\emptyset(\Gamma \cup \{@_k \neg e\}) \nvdash \bot$ | by (*Cons*), since $\vartheta_\emptyset$ is conservative |
| 4 | $\vartheta_\emptyset(\Gamma \cup \{@_k \neg e\}) \in \mathsf{P}$ | by Definition 5.10 |
| 5 | $\vartheta_\emptyset(\Gamma \cup \{@_k \neg e\}) \in G$ for some generic set $G$ | by Proposition 5.15 |
| 6 | there exists a generic model $M$ for $G$ | by Theorem 4.8 |
| 7 | $M \models^{\mathtt{SI}} \vartheta_\emptyset(\Gamma \cup \{@_k \neg e\})$ | by Corollary 5.14 (2) |
| 8 | $M\restriction_{\vartheta_\emptyset} \models^{\mathtt{SI}} \Gamma \cup \{@_k \neg e\}$ | by the satisfaction condition |
| 9 | contradiction to $\Gamma \models @_k\, e$ | since $M\restriction_{\vartheta_\emptyset} \models^{\mathtt{SI}} \Gamma$ and $M\restriction_{\vartheta_\emptyset} \not\models^{\mathtt{SI}} @_k\, e$ |

It follows that $\Gamma \vdash @_k\, e$.

Secondly, we show that completeness is a consequence of local completeness:

| | | |
|---|---|---|
| 1 | assume that $\Gamma \models e$ and let $z$ be a nominal variable | |
| 2 | $\chi_z(\Gamma) \models \chi_z(e)$ | by the satisfaction condition |
| 3 | $\chi_z(\Gamma) \models @_z \chi_z(e)$ | since $\chi_z(e) \models @_z \chi_z(e)$ |
| 4 | $\chi_z(\Gamma) \vdash @_z \chi_z(\rho)$ | by local completeness |
| 5 | $\Gamma \vdash \rho$ | by ($Ret_E$) |

□

## 6 CONCRETE ENTAILMENT SYSTEMS

We apply the general results developed in the previous section to concrete examples of stratified institutions. To this end we need to provide proof rules for the sentences obtained by applying at most one time retrieve to atoms.

### 6.1 Entailment system of HPL

We set the parameters of the completeness theorem for HPL as follows:

- the stratified institution $\mathtt{SI}$ is HPL;
- the substitution functor $\mathsf{SSt}^{\mathsf{HPL}} \colon \mathsf{D}^{\mathsf{HPL}} \to \mathbb{C}\mathrm{at}^{op}$ is defined in Example 3.15;
- the quantification space $\mathsf{Q}^{\mathsf{HPL}}$ is defined in Example 3.4;

$$(R^n)\ \frac{}{\Gamma \vdash @_k\, k} \qquad (P^n)\ \frac{\Gamma \vdash @_k\, \underline{\lambda}(k_1) \quad \Gamma \vdash @_{k_1}\, k_2}{\Gamma \vdash @_k\, \underline{\lambda}(k_2)} \qquad (W^n)\ \frac{\Gamma \vdash @_k\, \rho \quad \Gamma \vdash @_k\, k_1}{\Gamma \vdash @_{k_1}\, \rho}$$

Table 2. Proof rules for $\mathsf{HPL}_b$

- the entailment system $\mathcal{E}^{\mathsf{HPL}_b}$ is the least entailment system of $\mathsf{HPL}_b$ closed under $(Ret_I)$ and $(Ret_E)$ defined in Table 1, and the proof rules defined in Table 2.

The general results developed in the previous sections rely on good proof theoretic properties of the base logic layer. We show that HPL has a foundation that supports the instantiation of the abstract infrastructure defined in Section 5.

Proposition 6.1. *$\mathcal{E}^{\mathsf{HPL}_b}$ is sound and compact.*

Proof. For soundness, we need to show that all proof rules enumerated in Table 2 are sound, which is straightforward.

For compactness, let $\mathcal{E}^c = (\mathsf{Sig}^{\mathsf{HPL}}, \mathsf{F}^{\mathsf{HPL}}, \mathsf{Sen}^{\mathsf{HPL}_b}, \vdash^c)$ be the compact entailment subsystem of $\mathsf{HPL}_b$. It suffices to prove that $\mathcal{E}^c$ is closed under the proof rules defined in Table 2, which is again straightforward. See Theorem 5.9 for a hint. □

The proof of the following theorem is conceptually the same as the proof of [36, Theorem 6.2].

Theorem 6.2. *$\mathcal{E}^{\mathsf{HPL}_b}$ is complete.*

Theorem 6.2 enables the application of Theorem 5.16 to HPL.

Theorem 6.3. *$\mathcal{E}^{\mathsf{HPL}}$ is sound, compact and complete.*

Proof. Since soundness and compactness are straightforward, we focus on completeness. We need to show that the conditions of Theorem 5.16 hold in HPL:

(1) By Proposition 6.2, $\mathcal{E}^{\mathsf{HPL}_b}$ is complete.
(2) Let $\Delta$ be any signature. Assume that $\mathsf{card}(\mathsf{Sen}^{\mathsf{HPL}}(\Delta)) = \alpha$ and consider a set $C$ of new nominals of cardinality $\alpha$. We define the diagram $\mathcal{V} : \mathcal{P}_\alpha(C) \to \mathsf{Sig}^{\mathsf{HPL}}$ as follows:
   (a) $\mathcal{V}(C') = \Delta[C']$ is the extension of $\Delta$ with nominals from $C'$, for all $C' \in \mathcal{P}_\alpha(C)$, and
   (b) $\mathcal{V}(C' \subseteq C'')\colon \Delta[C'] \hookrightarrow \Delta[C'']$ is an inclusion for all $(C' \subseteq C'') \in (\mathcal{P}_\alpha(C), \subseteq)$.

   We define the colimit $\vartheta : \mathcal{V} \Rightarrow \Delta[C]$ by $\vartheta_{C'}\colon \Delta[C'] \hookrightarrow \Delta[C]$ for all $C' \in \mathcal{P}_\alpha(C)$. One can straightforwardly prove that $\langle \mathcal{V}, \vartheta \rangle$ is an extension of $\Delta$ by applying the same arguments used in the proof of Proposition 3.18.

   Since $\Delta[C]$ is non-void, by Proposition 3.30, each set of $\Delta[C]$-sentences is epi-basic and it has a basic model which is reachable. □

Using the same parameters as for HPL, one can apply Theorem 5.16 to HPLQ to prove its completeness.

### 6.2 Entailment system of HREL

We set the parameters of the completeness theorem for HREL as follows:

- the stratified institution SI is HREL;
- $\mathsf{D}^{\mathsf{HREL}}$ consists of signature extensions with nominals and first-order constants.
- $\mathsf{SSt}^{\mathsf{HREL}}\colon \mathsf{D}^{\mathsf{HREL}} \to \mathbb{C}\mathsf{at}^{op}$ maps each signature $(\mathsf{Nom}, \Lambda, (F, P))$ to the category of stratified substitutions represented by pairs of functions $\langle \theta_a : C_1 \to \mathsf{Nom}[C_2], \theta_b : D_1 \to F[D_2] \rangle$, where $C_1$ and $C_2$ are sets of nominals different from the nominals in Nom, and $D_1$ and $D_2$ are sets

| | | |
|---|---|---|
| $(R^n)\ \dfrac{}{\Gamma \vdash @_k\, k}$ | $(P^n)\ \dfrac{\Gamma \vdash @_k\, \underline{\lambda}(k_1) \quad \Gamma \vdash @_{k_1}\, k_2}{\Gamma \vdash @_k\, \underline{\lambda}(k_2)}$ | $(W^n)\ \dfrac{\Gamma \vdash @_k\, \rho \quad \Gamma \vdash @_k\, k_1}{\Gamma \vdash @_{k_1}\, \rho}$ |
| $(R^I)\ \dfrac{\Gamma \vdash d_1 = d_2}{\Gamma \vdash @_k\, (d_1 = d_2)}$ | $(R^E)\ \dfrac{\Gamma \vdash @_k\, (d_1 = d_2)}{\Gamma \vdash d_1 = d_2}$ | $(R^h)\ \dfrac{}{\Gamma \vdash d_1 = d_1}$ |
| $(S^h)\ \dfrac{\Gamma \vdash d_1 = d_2}{\Gamma \vdash d_2 = d_1}$ | $(T^h)\ \dfrac{\Gamma \vdash d_1 = d_2 \quad \Gamma \vdash d_2 = d_3}{\Gamma \vdash d_1 = d_3}$ | $(P^h)\ \dfrac{\Gamma \vdash @_k\, \pi(d_1) \quad \Gamma \vdash d_1 = d_2}{\Gamma \vdash @_k\, \pi(d_2)}$ |

Table 3. Proof rules for $\mathsf{HREL}_b$

of first-order constants different from the constants in $F$; this definition is conceptually the same as the definition of $\mathsf{SSt}^{\mathsf{HFOLR}} \colon \mathsf{D}^{\mathsf{HFOLR}} \to \mathbb{C}\mathrm{at}^{op}$ in Example 3.16;

- the quantification space $\mathsf{Q}^{\mathsf{HREL}}$ consists of signature extensions with a finite number of both nominal and first-order variables;
- the entailment system $\mathcal{E}^{\mathsf{HREL}_b}$ is the least entailment system of $\mathsf{HREL}_b$ closed under the proof rules $(Ret_I)$ and $(Ret_E)$ defined in Table 1, and the proof rules defined in Table 3.

Notice that HREL allows only a weak form of quantification over nominals through possibility over binary modalities, similarly to HPL.

Proposition 6.4. *$\mathcal{E}^{\mathsf{HREL}_b}$ is sound and compact.*

Proof. Straightforward. □

Theorem 6.5. *$\mathcal{E}^{\mathsf{HREL}_b}$ is complete.*

Proof. Consider a HREL signature $\Delta$. Let $c$ be a new nominal and $d$ a new constant. We denote by $\chi$ the inclusion $\Delta \hookrightarrow \Delta[c, d]$. Notice that $\Delta[c, d]$ is non-void. If we prove that

$$B \models \rho \text{ implies } B \vdash \rho \text{ for all } B \subseteq \mathsf{Sen}^{\mathsf{HREL}_b}(\Delta[c, d]) \text{ and all } \rho \in \mathsf{Sen}^{\mathsf{HREL}_b}(\Delta[c, d])$$

then by $(Cons)$ applied to $\chi$,

$$B \models \rho \text{ implies } B \vdash \rho \text{ for all } B \subseteq \mathsf{Sen}^{\mathsf{HREL}_b}(\Delta) \text{ and all } \rho \in \mathsf{Sen}^{\mathsf{HREL}_b}(\Delta).$$

We focus on proving that the restriction $\mathsf{HREL}'_b$ of $\mathsf{HREL}_b$ to non-void signatures is complete.

Let $B$ be a set of $\mathsf{HREL}_b$ sentences over a non-void signature $\Delta = (\mathsf{Nom}, \Lambda, \Sigma)$, where $\Sigma = (F, P)$. We define the relation $\equiv^{\mathsf{n}} = \{(k_1, k_2) \mid B \vdash @_{k_1}\, k_2\}$. We show that $\equiv^{\mathsf{n}}$ is an equivalence:

[ $\equiv^{\mathsf{n}}$ *is reflexive* ] by $(R^n)$;

[ $\equiv^{\mathsf{n}}$ *is symmetric* ] Assume that $k_1 \equiv^{\mathsf{n}} k_2$. By the definition of $\equiv^{\mathsf{n}}$, $B \vdash @_{k_1}\, k_2$. By $(R^n)$, $B \vdash @_{k_1}\, k_1$. By $(W^n)$, $B \vdash @_{k_2}\, k_1$. Hence, $k_2 \equiv^{\mathsf{n}} k_1$.

[ $\equiv^{\mathsf{n}}$ *is transitive* ] Assume that $k_1 \equiv^{\mathsf{n}} k_2$ and $k_2 \equiv^{\mathsf{n}} k_3$. By the definition of $\equiv^{\mathsf{n}}$, $B \vdash @_{k_1}\, k_2$ and $B \vdash @_{k_2}\, k_3$. By the symmetry of $\equiv^{\mathsf{n}}$, $B \vdash @_{k_2}\, k_1$. Since $B \vdash @_{k_2}\, k_3$ and $B \vdash @_{k_2}\, k_1$, by $(W^n)$, we get $B \vdash @_{k_1}\, k_3$. Hence, $k_1 \equiv^{\mathsf{n}} k_3$.

We denote by $[k]$ the equivalence class of $k$, for all $k \in \mathsf{Nom}$ and by $[\mathsf{Nom}]$ the set $\mathsf{Nom}/_{\equiv^{\mathsf{n}}}$. We define the relation $\equiv^{\mathsf{r}} = \{(d_1, d_2) \mid B \vdash d_1 = d_2\}$ on $F$. By $(R^h)$, $(S^h)$ and $(T^h)$, the relation $\equiv^{\mathsf{r}}$ is an equivalence on $F$. We denote by $[d]$ the equivalence class of $d$, for all $d \in F$, and by $[F]$ the set $\mathsf{F}/_{\equiv^{\mathsf{r}}}$. We define the model $(W^B, M^B)$ as follows:

- $|W^B| = [\mathsf{Nom}]$ and $W^B_\lambda = \{([k], [k_1]) \mid \mathsf{B} \vdash @_k\, \underline{\lambda}(k_1)\}$ for all $\lambda \in \Lambda_2$;
- $|M^B_{[k]}| = [F]$ and $M^B_{[k],\pi} = \{[d] \mid B \vdash @_k\, \pi(d)\}$, for all $[k] \in [\mathsf{Nom}]$ and all $\pi \in P$.

$M^B_{[k],\pi}$ is well-defined:

Assume that $B \vdash @_k\, \pi(d)$, $[k] = [k_1]$ and $[d] = [d_1]$. By the definition of $\equiv^{\mathsf{n}}$, $B \vdash @_k\, k_1$. By $(W^n)$, $B \vdash @_{k_1}\, \pi(d)$. By the definition of $\equiv^{\mathsf{r}}$, $B \vdash d = d_1$. By $(P^h)$, $B \vdash @_{k_1}\, \pi(d_1)$.

By construction, $(W^B, M^B) \models @_k\, \rho$ iff $B \vdash @_k\, \rho$, for all sentences $@_k\, \rho \in \mathsf{Sen}^{\mathsf{HPL}_b}(\Delta)$. Since $(W^B, M^B)$ is reachable, $(W^B, M^B) \models B$. We show that $\mathsf{HREL}'_b$ is locally complete:

Assume that $B \models @_k\, \rho$ for some $@_k\, \rho \in \mathsf{Sen}^{\mathsf{HPL}_b}(\Delta)$. Since $(W^B, M^B) \models B$, we have $(W^B, M^B) \models @_k\, \rho$, which is equivalent to $B \vdash @_k\, \rho$.

By $(Ret_E)$, locally completeness implies completeness. See the second part of the proof of Theorem 5.16. Hence, $\mathsf{HREL}'_b$ is complete. □

Theorem 6.5 enables the application of Theorem 5.16 to HREL.

Theorem 6.6. *$\mathcal{E}^{\mathsf{HREL}}$ is sound, compact and complete.*

Proof. Since soundness and compactness are straightforward, we focus on completeness. We need to show that the conditions of Theorem 5.16 hold in HREL:

(1) By Proposition 6.5, $\mathcal{E}^{\mathsf{HREL}_b}$ is complete.
(2) Let $\Delta = (\mathsf{Nom}, \Lambda, \Sigma)$ be any signature, where $\Sigma = (F, P)$. Assume that $\mathsf{card}(\mathsf{Sen}^{\mathsf{HPL}}(\Delta)) = \alpha$. Let $C = \{C_{\mathsf{n}}, C_{\mathsf{r}}\}$ be a family of sets of cardinality $\alpha$, where $C_{\mathsf{n}}$ is a set of new nominals and $C_{\mathsf{r}}$ is a set of new first-order constants. We define the diagram $\mathcal{V} : \mathcal{P}_\alpha(C) \to \mathsf{Sig}^{\mathsf{HREL}}$ as follows: (a) $\mathcal{V}(C') = \Delta[C']$ is the extension of $\Delta$ with elements from $C'$, for all $C' \in \mathcal{P}_\alpha(C)$, and (b) $\mathcal{V}(C' \subseteq C'')\colon \Delta[C'] \hookrightarrow \Delta[C'']$ is an inclusion, for all $(C' \subseteq C'') \in (\mathcal{P}_\alpha(C), \subseteq)$. We define the colimit $\vartheta : \mathcal{V} \Rightarrow \Delta[C]$ by $\vartheta_{C'}\colon \Delta[C'] \hookrightarrow \Delta[C]$ for all $C' \in \mathcal{P}_\alpha(C)$. One can straightforwardly prove that $\langle \mathcal{V}, \vartheta \rangle$ is an extension of $\Delta$ applying the same arguments used in the proof of Proposition 3.18.
Let $\Delta'$ be the HFOLR signature $(\mathsf{Nom}, \Lambda, (F, \emptyset) \subseteq (F, P))$. By Proposition 3.33, any set of atomic $\Delta'[C]$-sentences (in HFOLR) is locally epi-basic and it has a basic model which is reachable. It follows that any set of atomic $\Delta[C]$-sentences (in HREL) is locally epi-basic and it has a basic model which is reachable. □

### 6.3 Entailment system of HFOLR

We set the parameters of the completeness theorem for HFOLR as follows:

- the institution $\mathtt{SI}$ is HFOLR;
- the substitution functor $\mathsf{SSt}^{\mathsf{HFOLR}}\colon \mathsf{D}^{\mathsf{HFOLR}} \to \mathbb{C}\mathsf{at}^{op}$ is defined in Example 3.16;
- the quantification space $\mathsf{Q}^{\mathsf{HFOLR}}$ is defined in Example 3.5;
- the entailment system $\mathcal{E}^{\mathsf{HFOLR}_b}$ is the least entailment system of $\mathsf{HFOLR}_b$ closed under $(Ret_I)$ and $(Ret_E)$ defined in Table 1, and the proof rules defined in Table 4, where the signature morphism $\varphi_{k \leftarrow k_1} : \Delta \to \Delta$ maps $k_1$ to $k$ and it is the identity on the rest of the symbols.

Proposition 6.7 (Soundness & compactness of $\mathsf{HFOLR}_b$). *$\mathcal{E}^{\mathsf{HFOLR}_b}$ is sound and compact.*

Proof. Straightforward. □

The completeness proof of $\mathsf{HFOLR}_b$ is similar to the completeness proof for the basic layer of hybrid-dynamic first-order logic with rigid symbols presented in [39]. Since hybrid-dynamic first-order logic with rigid symbols from [39] relies only on rigid terms, some non-trivial adjustments are necessary.

Lemma 6.8 (Least Kripke structure of a set of nominal sentences). *Let $\Gamma^{\mathsf{n}}$ be a set of $\Delta$-sentences obtained from nominals by applying at most one time retrieve, where $\Delta = (\mathsf{Nom}, \Lambda, \Sigma^{\mathsf{r}} \subseteq \Sigma)$*

$$(R^n)\ \frac{}{\Gamma \vdash @_k\, k} \qquad (P^n)\ \frac{\Gamma \vdash @_k\, \underline{\lambda}(k_1,\ldots,k_n) \quad \Gamma \vdash @_{k_i}\, k'_i \text{ for all } i \in \{1,\ldots,n\}}{\Gamma \vdash @_k\, \underline{\lambda}(k'_1,\ldots,k'_n)}$$

$$(W^n)\ \frac{\Gamma \vdash @_k\, \rho \quad \Gamma \vdash @_k\, k_1}{\Gamma \vdash @_{k_1}\, \rho} \qquad (W^h)\ \frac{\Gamma \vdash \rho \quad \Gamma \vdash @_k\, k_1}{\Gamma \vdash \varphi_{k \leftarrow k_1}(\rho)} \qquad (W^r)\ \frac{\Gamma \vdash @_k\, k_1}{\Gamma \vdash \varphi_{k \leftarrow k_1}(t) =_s t}\ [\, s \in S^{\mathsf{r}}\,]$$

$$(R^h)\ \frac{}{\Gamma \vdash t = t} \qquad (S^h)\ \frac{\Gamma \vdash t_1 = t_2}{\Gamma \vdash t_2 = t_1} \qquad (T^h)\ \frac{\Gamma \vdash t_1 = t_2 \quad \Gamma \vdash t_2 = t_3}{\Gamma \vdash t_1 = t_3}$$

$$(F^h)\ \frac{\Gamma \vdash t_1 = t_2}{\Gamma \vdash \sigma(t_1) = \sigma(t_2)}\ [\, \sigma \in \overline{F}\,] \qquad (P^h)\ \frac{\Gamma \vdash t_1 = t_2 \quad \Gamma \vdash \pi(t_1)}{\Gamma \vdash \pi(t_2)}\ [\, \pi \in \overline{P}]$$

$$(R^E)\ \frac{\Gamma \vdash @_k\, \rho}{\Gamma \vdash \mathsf{at}_k\, \rho} \qquad (R^I)\ \frac{\Gamma \vdash \mathsf{at}_k\, \rho}{\Gamma \vdash @_k\, \rho}$$

Table 4. Proof rules for $\mathsf{HFOLR}_b$

*is a non-void signature. Then there exists a reachable initial model* $(W^{\mathsf{n}}, M^{\mathsf{n}})$ *such that* $\Gamma^{\mathsf{n}} \vdash @_k\, \rho$ *iff* $(W^{\mathsf{n}}, M^{\mathsf{n}}) \models @_k\, \rho$*, for all nominals* $k$ *and atomic sentences* $\rho$ *over the signature* $\Delta$*.*

Proof. We define a binary relation on nominals $\equiv^{\mathsf{n}} = \{(k_1, k_2) \in \mathsf{Nom} \times \mathsf{Nom} \mid \Gamma^{\mathsf{n}} \vdash @_{k_1}\, k_2\}$. By $(R^n)$ and $(W^n)$ defined in Table 4, the relation $\equiv^{\mathsf{n}}$ is an equivalence on Nom. See the first part of the proof of Theorem 6.5.

Let $[\_]\colon (\mathsf{Nom}, \Lambda, \Sigma^{\mathsf{r}} \subseteq \Sigma) \to (\mathsf{Nom}/_{\equiv^{\mathsf{n}}}, \Lambda, \Sigma^{\mathsf{r}} \subseteq \Sigma)$ be the signature morphism which maps each nominal $k \in \mathsf{Nom}$ to its equivalence class $k/_{\equiv^{\mathsf{n}}}$ and it is the identity on the remaining symbols. We define $(W^{\mathsf{n}}, M^{\mathsf{n}}) = (W^{[\Delta]}, M^{[\Delta]})\restriction_{[\_]}$, where $[\Delta] = (\mathsf{Nom}/_{\equiv^{\mathsf{n}}}, \Lambda, \Sigma^{\mathsf{r}} \subseteq \Sigma)$ and $(W^{[\Delta]}, M^{[\Delta]})$ is the initial model of $[\Delta]$ (see Lemma 2.18). It is straightforward to show that $(W^{\mathsf{n}}, M^{\mathsf{n}})$ is the initial model of $\Gamma^{\mathsf{n}}$.

Note that $\Gamma^{\mathsf{n}} \vdash @_{k_1}\, k_2$ iff $[k_1] = [k_2]$ iff $(W^{\mathsf{n}}, M^{\mathsf{n}}) \models^{\mathsf{HFOLR}_b} @_{k_1}\, k_2$, for all $@_{k_1}\, k_2 \in \mathsf{Sen}^{\mathsf{HFOLR}_b}(\Delta)$. In the following we focus on proving

$$\begin{array}{l} \Gamma^{\mathsf{n}} \vdash @_k\, (t_1 = t_2) \text{ iff } (W^{\mathsf{n}}, M^{\mathsf{n}}) \models^{\mathsf{HFOLR}} @_k\, (t_1 = t_2), \\ \text{for all nominals } k \in \mathsf{Nom} \text{ and hybrid terms } t_1, t_2 \in T_{\overline{\Sigma}} \end{array} \tag{1}$$

By $(R^I)$ and $(R^E)$, property (1) is equivalent to

$$\Gamma^{\mathsf{n}} \vdash t_1 = t_2 \text{ iff } (W^{\mathsf{n}}, M^{\mathsf{n}}) \models^{\mathsf{HFOLR}} t_1 = t_2, \text{ for all rigid terms } t_1, t_2 \in T_{@\Sigma} \tag{2}$$

By soundness, it suffices to show that

$$\begin{array}{l} \Gamma \vdash t_1 = \varphi_{k_1 \leftarrow k_2}(t_2) \text{ if } [t_1] = [t_2] \text{ and } [k_1] = [k_2], \\ \text{for all sorts } s \in S, \text{ nominals } k_i \in \mathsf{Nom} \text{ and rigid terms } t_i \in T_{@\Sigma, @_{k_i} s}, \text{ where } i \in \{1, 2\} \end{array} \tag{3}$$

We proceed by induction on the structure of $t_1$. We only consider the case when $t_1 = (@_{k_1}\sigma)(t'_1)$, where $k_1 \in \mathsf{Nom}$, $\sigma\colon \mathsf{ar} \to s \in F^{\mathsf{f}}$ and $s \in S^{\mathsf{r}}$, as the remaining cases are similar:

| | | |
|---|---|---|
| 1 | $t_2 = (@_{k_2}\sigma)(t'_2)$ such that $[t'_1] = [t'_2]$ for some $t'_2 \in T_{@\Sigma, @_{k_2}\mathsf{ar}}$ | since $(@_{[k_1]}\sigma)([t'_1]) = [t_1] = [t_2]$ |
| 2 | $\Gamma \vdash @_{k_1}\, k_2$ | since $[k_1] = [k_2]$ |
| 3 | $\Gamma \vdash t'_1 = \varphi_{k_1 \leftarrow k_2}(t'_2)$ | by the induction hypothesis |
| 4 | $\Gamma \vdash (@_{k_1}\sigma)(t'_1) = (@_{k_1}\sigma)(\varphi_{k_1 \leftarrow k_2}(t'_2))$ | by $(F^h)$ |
| 5 | $\Gamma \vdash (@_{k_1}\sigma)(t'_1) = \varphi_{k_1 \leftarrow k_2}((@_{k_2}\sigma)(t'_2))$ | by the definition of $\varphi_{k_1 \leftarrow k_2}$ |

| | | |
|---|---|---|
| 6 | $\Gamma \vdash \varphi_{k_1 \leftarrow k_2}((@_{k_2}\sigma)(t'_2)) = (@_{k_2}\sigma)(t'_2)$ | by $(W^r)$ |
| 7 | $\Gamma \vdash (@_{k_1}\sigma)(t'_1) = (@_{k_2}\sigma)(t'_2)$ | from 5 and 6, by $(T^h)$ |
| 8 | $\Gamma \vdash t_1 = t_2$ | since $(@_{k_i}\sigma)(t'_i) = t_i$ |

□

The following proposition shows that a set $\Gamma$ of hybrid equations generates a congruence on a reachable Kripke structure $(W, M)$ when $\Gamma$ entails all equations satisfied by $(W, M)$. In particular, the result holds when $\Gamma$ includes the set of all equations that are satisfied by $(W, M)$.

Proposition 6.9 (Congruence generated by a set of equations). *Let $\Delta = (\mathsf{Nom}, \Lambda, \Sigma^{\mathsf{r}} \subseteq \Sigma)$ be a non-void signature. Assume (a) a set $\Gamma$ of $\Delta$-sentences of the form $@_k\, \rho$, where $k$ is a nominal and $\rho$ is either a nominal or a hybrid equation, and (b) a reachable $\Delta$-model $(W, M)$, such that $(W, M) \models @_k\, \rho$ implies $\Gamma \vdash @_k\, \rho$, for all nominals $k$ and all nominal sentences or hybrid equations $\rho$ over $\Delta$. For each possible world $w \in |W|$, let $\equiv_w$ be the binary relation on $M_w$ defined by $\tau_1 \equiv_w \tau_2$ iff $\Gamma \vdash t_1 = t_2$ for some $k \in \mathsf{Nom}$, $s \in S$ and $t_1, t_2 \in T_{@\Sigma, @_k s}$ such that $w = W_k$ and $\tau_i = M_{w,t_i}$. Then:*

*P1) $[t_1] \equiv_{[k]} [t_2]$ iff $\Gamma \vdash t_1 = t_2$, for all nominals $k \in \mathsf{Nom}$, sorts $s \in S$ and terms $t_1, t_2 \in T_{@\Sigma, @_k s}$, where $[\_] : (W^\Delta, M^\Delta) \to (W, M)$ is the unique arrow from $(W^\Delta, M^\Delta)$ to $(W, M)$ given by Lemma 2.18;*

*P2) $\equiv = \{\equiv_w\}_{w \in |W|}$ is a $\Delta$-congruence on $(W, M)$, i.e. $\equiv_w$ is a first-order congruence on $M_w$ for all $w \in |W|$, and $\equiv_{w_1,s} = \equiv_{w_2,s}$ for all $w_1, w_2 \in |W|$ and all $s \in S^{\mathsf{r}}$.*

Proof.

P1) In regard to the characterization of $\equiv$, the "if" part follows immediately by the very definition of $\equiv$. Therefore, we focus on the "only if" part. To that end, suppose $[t_1] \equiv_{[k]} [t_2]$. Then:

| | | |
|---|---|---|
| 1 | $\Gamma \vdash t'_1 = t'_2$ for some $k' \in \mathsf{Nom}$, $s \in S$ and $t'_1, t'_2 \in T_{@\Sigma, @_{k'} s}$ such that $[k] = [k']$ and $[t_i] = [t'_i]$ | by the definition of $\equiv$ |
| 2 | $\Gamma \vdash @_k\, k'$ | since $(W, M) \models @_k\, k'$ |
| 3 | $\Gamma \vdash \varphi_{k \leftarrow k'}(t'_1) = \varphi_{k \leftarrow k'}(t'_2)$ | from $\Gamma \vdash t'_1 = t'_2$, by $(W^h)$ |
| 4 | $[t_i] = [t'_i] = [\varphi_{k \leftarrow k'}(t'_i)]$ | since $[k] = [k']$ |
| 5 | $\Gamma \vdash t_i = \varphi_{k \leftarrow k'}(t'_i)$ | since $(W, M) \models t_i = \varphi_{k \leftarrow k'}(t'_i)$ and $t_i = \varphi_{k \leftarrow k'}(t'_i)$ is a rigid sentence [10] |
| 6 | $\Gamma \vdash t_1 = t_2$ | from 5 and 3, by $(T^h)$ |

P2) For each nominal $k$, the reflexivity, symmetry, and transitivity of $\equiv_{[k]}$ are straightforward consequences of the proof rules $(R^h)$, $(S^h)$, and $(T^h)$, of the characterization given at (P1), and of the fact that $(W, M)$ is reachable. For the compatibility of $\equiv$ with the operations in $F$, assume that $(\sigma\colon \mathsf{ar} \to s) \in F$ and $\tau_1, \tau_2 \in M_{[k],\mathsf{ar}}$ such that $\tau_1 \equiv_{[k],\mathsf{ar}} \tau_2$. There is no significant distinction between the case where $\sigma$ is rigid and the case where $\sigma$ is flexible. Therefore, we choose to focus on the latter case, corresponding to $(\sigma\colon \mathsf{ar} \to s) \in F^{\mathsf{f}}$:

| | | |
|---|---|---|
| 1 | $\Gamma \vdash t_1 = t_2$ for some $t_i \in T_{@\Sigma, @_k \mathsf{ar}}$ such that $\tau_i = [t_i]$ | by (P1) |
| 2 | $\Gamma \vdash (@_k\sigma)(t_1) = (@_k\sigma)(t_2)$ | by $(F^h)$ |
| 3 | $[(@_k\sigma)(t_1)] \equiv_{[k],s} [(@_k\sigma)(t_2)]$ | by the definition of $\equiv_{[k]}$ |
| 4 | $M_{[k],\sigma}(\tau_1) \equiv_{[k],s} M_{[k],\sigma}(\tau_2)$ | since $[(@_k\, \sigma)(t_i)] = M_{[k],\sigma}(\tau_i)$ |

It remains to check that $(\equiv_{[k_1],s}) = (\equiv_{[k_2],s})$ for all nominals $k_1, k_2 \in \mathsf{Nom}$ and all rigid sorts $s \in S^{\mathsf{r}}$. This follows easily from (P1) and the fact that $@_k\, s = s$ for all rigid sorts $s \in S^{\mathsf{r}}$. □

The following result says that in $\mathsf{HFOLR}_b$, all sets of sentences have an initial model which encapsulates formal deductions.

[10] This means that $t_i = \varphi_{k \leftarrow k'}(t'_i) \vdash @_{k''}(t_i = \varphi_{k \leftarrow k'}(t'_i))$ and $t_i = \varphi_{k \leftarrow k'}(t'_i) \dashv\vdash @_{k''}(t_i = \varphi_{k \leftarrow k'}(t'_i))$, for all $k'' \in \mathsf{Nom}$.

Theorem 6.10 (Initiality & entailments). *In* $\mathsf{HFOLR}_b$*, every set* $\Gamma$ *of sentences over a non-void signature* $\Delta = (\mathsf{Nom}, \Lambda, \Sigma^{\mathsf{r}} \subseteq \Sigma)$ *has a reachable initial model* $(W^\Gamma, M^\Gamma)$ *such that* $\Gamma \vdash @_k\, \rho$ *iff* $(W^\Gamma, M^\Gamma) \models @_k\, \rho$*, for all nominals* $k$ *and all atomic sentences* $\rho$ *over* $\Delta$*.*

Proof. Let $\Gamma^{\mathsf{n}}$ be the subset of $\Gamma$ of sentences of the form $@_k\, k'$, where $k, k' \in \mathsf{Nom}$. By Lemma 6.8, there exists a initial model $(W^{\mathsf{n}}, M^{\mathsf{n}})$ of $\Gamma^{\mathsf{n}}$ such that $\Gamma^{\mathsf{n}} \vdash @_k\, \rho$ iff $(W^{\mathsf{n}}, M^{\mathsf{n}}) \models @_k\, \rho$, for all nominals $k$ and atomic sentences $\rho$ over $\Delta$. Then $(W^{\mathsf{n}}, M^{\mathsf{n}})$ satisfies the hypotheses of Proposition 6.9. Let $[\_] : (W^\Delta, M^\Delta) \to (W^{\mathsf{n}}, M^{\mathsf{n}})$ be the unique arrow from $(W^\Delta, M^\Delta)$ to $(W^{\mathsf{n}}, M^{\mathsf{n}})$ given by Lemma 2.18. It follows that the relation $\equiv$ defined by $[t_1] \equiv_{[k]} [t_2]$ whenever $\Gamma \vdash t_1 = t_2$, for all sorts $s \in S$, all nominals $k \in \mathsf{Nom}$, and all rigid terms $t_1, t_2 \in T_{@\Sigma, @_k s}$, is a congruence on $(W^{\mathsf{n}}, M^{\mathsf{n}})$. We define $(W^\Gamma, M^\Gamma)$ as follows:

- $W^\Gamma = W^{\mathsf{n}}$, and $M^\Gamma_{[k]} = M^{\mathsf{n}}_{[k]}/_{\equiv}$ for all nominals $k \in \mathsf{Nom}$.
- $(W^\Gamma, M^\Gamma)$ interprets
  (a) each modality $\lambda \in \Lambda$ as $W^\Gamma_\lambda = \{([k], [k_1], \ldots, [k_n]) \mid \Gamma \vdash @_k\, \underline{\lambda}(k_1, \ldots, k_n)\}$, and
  (b) each relation $\pi \in P$ as $M^\Gamma_{[k],\pi} = \{[t]/_{\equiv_{[k]}} \in M^\Gamma_{[k]} \mid \Gamma \vdash (@_k \pi)(t)\}$.

Note that the interpretation of $\pi \in P$ is independent of the choice of the nominal $k$:

> if $\Gamma \vdash (@_k \pi)(t)$ and $[k'] = [k]$ then $\Gamma \vdash @_{k'}\, k$, and by $(W^h)$, $\Gamma \vdash \varphi_{k' \leftarrow k}((@_k \pi)(t))$; by the definition of $\varphi_{k' \leftarrow k}$, we obtain $\Gamma \vdash (@_{k'} \pi)(\varphi_{k' \leftarrow k}(t))$, where $[t] \equiv_{[k]} [\varphi_{k' \leftarrow k}(t)]$.

The fact that $(W^\Gamma, M^\Gamma)$ is a reachable model of $\Gamma$ follows in a straightforward manner by construction. Therefore, we focus on the initiality property. Let $(W, M)$ be a $\Delta$-model that satisfies $\Gamma$. In particular, $(W, M)$ satisfies $\Gamma^{\mathsf{n}}$. By Lemma 6.8, we deduce that there exists a unique homomorphism $h \colon (W^{\mathsf{n}}, M^{\mathsf{n}}) \to (W, M)$. We also know that $(W, M)$ satisfies all hybrid equations in $\Gamma$, which implies that $\equiv_{[k]} \subseteq \ker(h_{[k]})$ for all $k \in \mathsf{Nom}$. This means that there exists a unique homomorphism $h' \colon (W^\Gamma, M^\Gamma) \to (W, M)$ such that $(\_/_{\equiv_{[k]}}); h'_{[k]} = h_{[k]}$ for all $k \in \mathsf{Nom}$. It is straightforward to prove that $h'$ preserves the interpretation of all relation symbols.

Lastly, we show that $\Gamma \vdash @_k\, \rho$ iff $(W^\Gamma, M^\Gamma) \models @_k\, \rho$, for all nominals $k$ and atomic sentences $\rho$. The "only if" part is straightforward since $(W^\Gamma, M^\Gamma)$ is a model of $\Gamma$. For the "if" part, we proceed by case analysis on the structure of $\rho$. The more interesting cases are those of hybrid relations. Suppose, for instance, that $(W^\Gamma, M^\Gamma) \models @_k\, \pi(t)$, where $(\pi \colon \mathsf{ar}) \in P^{\mathsf{f}}$, $k \in \mathsf{Nom}$, and $t \in T_{\overline{\Sigma}, \mathsf{ar}}$.

| | | |
|---|---|---|
| 1 | $(W^\Gamma, M^\Gamma) \models (@_k \pi)(t')$, where $t' = \mathsf{at}_k\, t$ | since $@_k\, \pi(t) \models\!\!\dashv (@_k \pi)(t')$ |
| 2 | $[t']/_{\equiv_{[k]}} \in M^\Gamma_{[k],\pi}$ | by the definition of $\models$ |
| 3 | $\Gamma \vdash (@_k \pi)(t'')$ for some $t'' \in T_{@\Sigma, @_k \mathsf{ar}}$ such that $[t'] \equiv_{[k]} [t'']$ | by the definition of $M^\Gamma_{[k],\pi}$ |
| 4 | $\Gamma \vdash t'' = t'$ | by Proposition 6.9 |
| 5 | $\Gamma \vdash (@_k \pi)(t')$ | by $(P^h)$ |
| 6 | $\Gamma \vdash @_k\, \pi(t)$ | by $(R^I)$ |

□

Theorem 6.11. $\mathcal{E}^{\mathsf{HFOLR}}$ *is sound, compact and complete.*

Proof. Since soundness and compactness are straightforward, we focus on completeness. More concretely we need to show that the conditions of Theorem 5.16 hold for HFOLR.

(1) By Theorem 6.10 and (*Cons*), $\mathsf{HFOLR}_b$ is complete.
(2) By Proposition 3.19, each signature $\Delta$ has an extension $\langle \mathcal{V}, \vartheta \rangle$ such that the vertex $\Delta[C]$ of the colimit $\vartheta \colon \mathcal{V} \Rightarrow \Delta[C]$ is non-void. By Proposition 3.33, each set of $\mathsf{HFOLR}_b$ sentences over $\Delta[C]$ is locally epi-basic and it has a basic model which is reachable.

□

## 7 A CASE STUDY

In this section, we describe briefly (1) *the specification method*, and (2) *the verification technique* we intend to build on top of the foundation developed in the present contribution.

### 7.1 Specification method

We envision a specification methodology where the rigid data types are built outside the hybrid specification. For example, a hybrid specification in HFOLR has a signature $(\mathsf{Nom}, \Lambda, \Sigma^{\mathsf{r}} \subseteq \Sigma)$, where $\Sigma^{\mathsf{r}} = (S^{\mathsf{r}}, F^{\mathsf{r}}, P^{\mathsf{r}})$ and $\Sigma = (S, F, P)$. Practitioners will start by specifying the rigid data types, i.e. a first-order specification with the signature $\Sigma^{\mathsf{r}}$. This is followed by the definition of (a) nominals, (b) accessibility relations between states, and (c) flexible data types, in such a way that no "junk" and no "confusion" are added to the rigid data types, i.e. the $\Sigma^{\mathsf{r}}$-models previously defined are preserved. For the sake of simplicity, in practice, a variable is identified only by its name; by a slight abuse of notation, for each inclusion $\chi\colon \Delta \hookrightarrow \Delta'$ and any $\Delta$-sentence $\gamma$, we let $\gamma$ denote also $\chi(\gamma)$.

In this paper, only basic specifications SP are considered, that is $\mathsf{SP} = (\mathsf{Sig}(\mathsf{SP}), \mathsf{Sen}(\mathsf{SP}))$, where $\mathsf{Sig}(\mathsf{SP})$ is a signature and $\mathsf{Sen}(\mathsf{SP})$ is a set of sentences over $\mathsf{Sig}(\mathsf{SP})$.

*Example 7.1.* We define the following specification of lists in FOL:

spec LIST
sorts Elt List
op $\mathtt{empty}\colon \to \mathsf{List}$
op $\_ \fatsemi \_ : \mathsf{List}\mathsf{Elt} \to \mathsf{List}$
vars L Q : List
var F : Elt
eq-1 $\forall \mathsf{L} \cdot \mathsf{L} = \mathtt{empty} \vee \exists \mathsf{Q}, \mathsf{F} \cdot \mathsf{L} = \mathsf{Q} \fatsemi \mathsf{F}$

By the above sentence, a list is either empty or it is obtained from another list by adding one element. LIST provides the rigid data types for the hybrid specification presented next.

*Example 7.2.* The hybrid specification BUFFER defined below consists of a buffer with two distinct operation modes: (a) "lifo", where it behaves as a stack, and (b) "fifo", where it behaves as a queue. The alternation of configurations is triggered by an event "shift".

spec BUFFER[LIST]
nominals `lifo` `fifo`
modality `shift` : 2
op `del` : List $\to$ List
vars E F : Elt
var L : List
rel-1 $@_{\mathtt{fifo}}\, \underline{\mathtt{shift}}(\mathtt{lifo})$
rel-2 $@_{\mathtt{lifo}}\, \underline{\mathtt{shift}}(\mathtt{fifo})$
eq-2 $\mathtt{del}(\mathtt{empty}) = \mathtt{empty}$
eq-3 $\forall \mathsf{L}, \mathsf{E} \cdot (@_{\mathtt{lifo}}\mathtt{del})(\mathsf{L} \fatsemi \mathsf{E}) = \mathsf{L}$
eq-4 $\forall \mathsf{E} \cdot (@_{\mathtt{fifo}}\mathtt{del})(\mathtt{empty} \fatsemi \mathsf{E}) = \mathtt{empty}$
eq-5 $\forall \mathsf{L}, \mathsf{E}, \mathsf{F} \cdot (@_{\mathtt{fifo}}\mathtt{del})(\mathsf{L} \fatsemi \mathsf{E} \fatsemi \mathsf{F}) = (@_{\mathtt{fifo}}\mathtt{del})(\mathsf{L} \fatsemi \mathsf{E}) \fatsemi \mathsf{F}$

The REL component of the hybrid signature consists of two nominals `fifo` and `lifo` and one binary modality `shift`. The signature of rigid symbols is the signature of LIST. There is one flexible operation symbol, `del`.

The system has two operation modes, `lifo`, when it behaves like a stack, and `fifo`, when it behaves like a queue. The binary modality `shift` makes the transition between `lifo` and `fifo`

modes according to the nominal relations $@_{\mathtt{fifo}}\ \underline{\mathtt{shift}}(\mathtt{lifo})$ and $@_{\mathtt{lifo}}\ \underline{\mathtt{shift}}(\mathtt{fifo})$. The function symbol del denotes the operation "pop". Notice that del play different roles in each operation mode.

The models of BUFFER consists of all Kripke structures $(W, M)$ over the signature of BUFFER which (a) have a rigid LIST structure, that is $M_{\mathtt{lifo}}\restriction_{\mathsf{Sig(LIST)}} = M_{\mathtt{fifo}}\restriction_{\mathsf{Sig(LIST)}}$ is a model of LIST, and (b) satisfy the axioms defined in Example 7.2. This construction is particularly useful for structured specifications which are obtained from basic specifications by applying specification building operators such as union, translation, hiding or freeness [66]. As for Example 7.2, notice that any Kripke structure over Sig(BUFFER) which satisfies Sen(BUFFER) (i.e. all sentences defined in Example 7.1 and Example 7.2) is a model of BUFFER.

### 7.2 Formal verification

This section is dedicated to proving that BUFFER satisfies the following property:

$$\forall \mathsf{L} \cdot (@_{\mathtt{lifo}}\mathtt{del})((@_{\mathtt{fifo}}\mathtt{del})(\mathsf{L})) = (@_{\mathtt{fifo}}\mathtt{del})((@_{\mathtt{lifo}}\mathtt{del})(\mathsf{L}))$$

This means that the order of deleting the front and the top element from a list is irrelevant w.r.t. the final result. In order to implement efficient proof strategies, one often needs to derive new proof rules from the original ones.

$$(\mathit{Ref})\ \frac{}{\Gamma \vdash \forall X \cdot t = t} \qquad (\mathit{Sym})\ \frac{\Gamma \vdash \forall X \cdot t_1 = t_2}{\Gamma \vdash \forall X \cdot t_2 = t_1}$$

$$(\mathit{Trans})\ \frac{\Gamma \vdash \forall X \cdot t_1 = t_2 \quad \Gamma \vdash \forall X \cdot t_2 = t_3}{\Gamma \vdash \forall X \cdot t_1 = t_3} \qquad (\mathit{Rew})\ \frac{\Gamma \vdash \forall X \cdot t_1 = t_2}{\Gamma \vdash \forall Y \cdot t[\theta(t_1)_p] = t[\theta(t_2)]_p}\ [\ \theta\colon X \to T_{@\Sigma}(Y)\ ]$$

Table 5. Derived proof rules for HFOLR

Since the present contribution is not dedicated to the development of a formal method, minimally, we derive four new proof rules presented in Table 5, which allows one to avoid complex formal proofs for obvious properties. Notice that $e[t_1 \leftarrow t_2]$ is the sentence obtained from $e$ by substituting $t_2$ for $t_1$, while $t|_p$ is the subterm of $t$ at position $p$ and $t[\theta(t_i)]_p$ is the term obtained from $t$ by substituting $\theta(t_i)$ for $t|_p$ at position $p$.

For the sake of simplicity, we denote by $\Delta$ the signature Sig(BUFFER), by $\Gamma$ the set of sentences Sen(BUFFER), and by PR(L) the formula $(@_{\mathtt{lifo}}\mathtt{del})((@_{\mathtt{fifo}}\mathtt{del})(\mathsf{L})) = (@_{\mathtt{fifo}}\mathtt{del})((@_{\mathtt{lifo}}\mathtt{del})(\mathsf{L}))$.

Lemma 7.3. *We assume that the variable* L *is of sort* List*, and the variables* E *and* F *are of sort* Elt.

*(1)* $\Gamma \vdash \mathsf{PR}(\mathtt{empty})$*;*
*(2)* $\Gamma \vdash \forall \mathsf{E} \cdot \mathsf{PR}(\mathtt{empty} \fatsemi \mathsf{E})$*;*
*(3)* $\Gamma \vdash \forall \mathsf{L}, \mathsf{E}, \mathsf{F} \cdot \mathsf{PR}(\mathsf{L} \fatsemi \mathsf{E} \fatsemi \mathsf{F})$*;*
*(4)* $\Gamma \vdash \forall \mathsf{L}, \mathsf{E} \cdot \mathsf{PR}(\mathsf{L} \fatsemi \mathsf{E})$*;*
*(5)* $\Gamma \vdash \forall \mathsf{L} \cdot \mathsf{PR}(\mathsf{L})$.

Proof. The first two assertions are straightforward to prove. We start with the third assertion.

| | | |
|---|---|---|
| 1 | $\Gamma \vdash \forall \mathsf{L}, \mathsf{E}, \mathsf{F} \cdot (@_{\mathtt{lifo}}\mathtt{del})((@_{\mathtt{fifo}}\mathtt{del})(\mathsf{L} \fatsemi \mathsf{E} \fatsemi \mathsf{F})) = (@_{\mathtt{lifo}}\mathtt{del})((@_{\mathtt{fifo}}\mathtt{del})(\mathsf{L} \fatsemi \mathsf{E}) \fatsemi \mathsf{F})$ | from eq-5, by (*Rew*) |
| 2 | $\Gamma \vdash \forall \mathsf{L}, \mathsf{E}, \mathsf{F} \cdot (@_{\mathtt{lifo}}\mathtt{del})((@_{\mathtt{fifo}}\mathtt{del})(\mathsf{L} \fatsemi \mathsf{E}) \fatsemi \mathsf{F}) = (@_{\mathtt{fifo}}\mathtt{del})(\mathsf{L} \fatsemi \mathsf{E})$ | from eq-3, by (*Rew*) |
| 3 | $\Gamma \vdash \forall \mathsf{L}, \mathsf{E}, \mathsf{F} \cdot (@_{\mathtt{fifo}}\mathtt{del})((@_{\mathtt{lifo}}\mathtt{del})(\mathsf{L} \fatsemi \mathsf{E} \fatsemi \mathsf{F})) = (@_{\mathtt{fifo}}\mathtt{del})(\mathsf{L} \fatsemi \mathsf{E})$ | from eq-3, by (*Rew*) |
| 4 | $\Gamma \vdash \forall \mathsf{L}, \mathsf{E}, \mathsf{F} \cdot (@_{\mathtt{fifo}}\mathtt{del})(\mathsf{L} \fatsemi \mathsf{E}) = (@_{\mathtt{fifo}}\mathtt{del})((@_{\mathtt{lifo}}\mathtt{del})(\mathsf{L} \fatsemi \mathsf{E} \fatsemi \mathsf{F}))$ | from 3, by (*Sym*) |
| 5 | $\Gamma \vdash \forall \mathsf{L}, \mathsf{E}, \mathsf{F} \cdot (@_{\mathtt{lifo}}\mathtt{del})((@_{\mathtt{fifo}}\mathtt{del})(\mathsf{L} \fatsemi \mathsf{E} \fatsemi \mathsf{F})) = (@_{\mathtt{fifo}}\mathtt{del})((@_{\mathtt{lifo}}\mathtt{del})(\mathsf{L} \fatsemi \mathsf{E} \fatsemi \mathsf{F}))$ | by applying (*Trans*) to 1, 2 and 4 |

We prove the fourth assertion:

| | | |
|---|---|---|
| 1 | $\Gamma \vdash_{\Delta[\mathsf{L},\mathsf{E}]} \mathsf{L} = \mathtt{empty} \vee \exists \mathsf{Q}, \mathsf{F} \cdot \mathsf{L} = \mathsf{Q} \fatsemi \mathsf{F}$ | from eq-1, by Lemma 5.7 (6) and the fact that eq-1 is a rigid sentence |
| 2 | $\Gamma \cup \{\mathsf{L} = \mathtt{empty}\} \vdash_{\Delta[\mathsf{L},\mathsf{E}]} \mathsf{PR}[\mathsf{L} \fatsemi \mathsf{E}]$ | by the following proof steps |
| 2.1 | $\Gamma \vdash_{\Delta[\mathsf{L},\mathsf{E}]} \mathsf{PR}[\mathtt{empty} \fatsemi \mathsf{E}]$ | from the second assertion, by (*Rew*) |
| 2.2 | $\Gamma \cup \{\mathsf{L} = \mathtt{empty}\} \vdash_{\Delta[\mathsf{L},\mathsf{E}]} \mathsf{PR}[\mathtt{empty}; \mathsf{E}]$ | by *(Transitivity)* and *(Monotonicity)* |
| 2.3 | $\Gamma \cup \{\mathsf{L} = \mathtt{empty}\} \vdash_{\Delta[\mathsf{L},\mathsf{E}]} \mathsf{PR}[\mathsf{L} \fatsemi \mathsf{E}]$ | from 2.2, by (*Sym*) and (*Rew*) |
| 3 | $\Gamma \cup \{\exists \mathsf{Q}, \mathsf{F} \cdot \mathsf{L} = \mathsf{Q} \fatsemi \mathsf{F}\} \vdash_{\Delta[\mathsf{L},\mathsf{E}]} \mathsf{PR}[\mathsf{L} \fatsemi \mathsf{E}]$ | by the following proof steps |
| 3.1 | $\Gamma \cup \{\mathsf{L} = \mathsf{Q} \fatsemi \mathsf{F}\} \vdash_{\Delta[\mathsf{L},\mathsf{Q},\mathsf{E},\mathsf{F}]} \mathsf{PR}[\mathsf{Q} \fatsemi \mathsf{F} \fatsemi \mathsf{E}]$ | from the third assertion, by (*Rew*) |
| 3.2 | $\Gamma \cup \{\mathsf{L} = \mathsf{Q} \fatsemi \mathsf{F}\} \vdash_{\Delta[\mathsf{L},\mathsf{Q},\mathsf{E},\mathsf{F}]} \mathsf{PR}[\mathsf{L} \fatsemi \mathsf{E}]$ | from 3.1, by (*Sym*) and (*Rew*) |
| 3.3 | $\Gamma \cup \{\exists \mathsf{Q}, \mathsf{F} \cdot \mathsf{L} = \mathsf{Q} \fatsemi \mathsf{F}\} \vdash_{\Delta[\mathsf{L},\mathsf{E}]} \mathsf{PR}[\mathsf{L} \fatsemi \mathsf{E}]$ | from 3.2, by ($Quant_I$) and the fact that $\exists \mathsf{Q}, \mathsf{F} \cdot \mathsf{L} = \mathsf{Q} \fatsemi \mathsf{F}$ is a rigid sentence |
| 4 | $\Gamma \vdash_{\Delta[\mathsf{L},\mathsf{E}]} \mathsf{PR}(\mathsf{L} \fatsemi \mathsf{E})$ | by applying ($Disj_E$) to 1, 2 and 3 |
| 5 | $\Gamma \vdash_{\Delta} \forall \mathsf{L}, \mathsf{E} \cdot \mathsf{PR}(\mathsf{L} \fatsemi \mathsf{E})$ | from 4, by Lemma 5.7 (4) and the fact that $\forall \mathsf{L}, \mathsf{E} \cdot \mathsf{PR}[\mathsf{L} \fatsemi \mathsf{E}]$ is a rigid sentence |

The proof of the fifth assertion resembles the proof of the fourth assertion. □

It is worth noting that the expressivity of the basic layer of HFOLR allows a simple description of the property to prove $\forall \mathsf{L} \cdot (@_{\mathtt{lifo}}\mathtt{del})((@_{\mathtt{fifo}}\mathtt{del})(\mathsf{L})) = (@_{\mathtt{fifo}}\mathtt{del})((@_{\mathtt{lifo}}\mathtt{del})(\mathsf{L}))$. Using ideas from Section 2.2.4, the same property can be expressed in HFOLS as follows: $\forall \mathsf{L} \cdot \exists x_1, x_2, y_1, y_2 \cdot x_1 = y_1 \wedge @_{\mathtt{lifo}}(x_1 = \mathtt{del}(x_2))) \wedge @_{\mathtt{fifo}}(x_2 = \mathtt{del}(\mathsf{L})) \wedge @_{\mathtt{fifo}}(y_1 = \mathtt{del}(y_2))) \wedge @_{\mathtt{lifo}}(y_2 = \mathtt{del}(\mathsf{L}))$, where $x_i$, $y_i$ are variables of sort List. It is not difficult to see that the expressivity of $\mathsf{HFOLR}_b$ has deep ramifications in formal verification; for example, the proofs become much simpler.

## 8 CONCLUSIONS

*Contributions.* The present contribution introduces the forcing technique in the abstract model-theoretic setting of stratified institutions [38]. A forcing property based on syntactic consistency is defined in the context given by an entailment system designed for an abstract stratified institution. The result is the completeness property for an arbitrary stratified institution satisfying properties which can be straightforwardly checked in concrete examples of hybrid logics. The paper advances the knowledge on hybrid logics. For example, the definition of locally basic set of sentences for stratified institutions generalizes in a non-trivial way the notion of basic set of sentences for first-order institutions. See Example 3.29 for details. Definition 3.28 characterizes the atomic sentences of hybrid logics and plays a key role in the developments of the present results. Also, the essence of a concrete hybrid logic is separated from the features which are common to all hybrid logics. This is based on the idea that the fragment given by the sentences obtained by applying at most one time retrieve to atomic sentences gives the particularity of a given hybrid logic. This is referred to as the basic fragment as its sentences are locally basic. As a consequence of this fact, completeness is proved in two steps: (a) Concrete entailment systems are developed for the basic fragments of each individual hybrid logic; their completeness is proved separately, case by case. (b) An entailment system is defined for an arbitrary stratified institution; its completeness is proved by assuming the completeness of the basic layer. While the first step depends on the logic, the second step is institution-independent. We have obtained complete entailment systems for HPL, HPLQ, HREL and HFOLR. Similar completeness results can be developed for other hybrid logics with rigid

symbols defined on top of preordered algebra [25], order-sorted algebra [32], partial algebra [3, 55], membership algebra [52] or higher-order algebra [53].

*Related work.* The results based on forcing and presented in [34, 42] are applicable to first-order institutions which restricts the syntax to signatures of a countable power. Due to Proposition 5.15, this restriction is no longer needed in the present contribution. Similar ideas used in the proof of Proposition 5.15 can be applied to [34, 42]. A Birkhoff style result for stratified institutions is given in [36], where the sentences are restricted to Horn clauses, which have the potential of making the specifications executable by rewriting or resolution. Another abstract completeness result for hybridized institutions, i.e. institutions obtained by applying the hybridization process proposed in [26], can be found in [57], which refines the results presented in [58]. The authors claim that soundness and completeness can still be obtained in a multiple hybridization scenario, e.g. the hybridization is applied twice to a base logic. This is not possible in the present contribution, as the results are based on the existence of a basic layer for which the sentences are locally basic. However, the work presented in [57] does not cover quantification. Also, it is not instantiated to any version of hybrid first-order logic (with or without quantification) and it is difficult to see how it can be applied in such scenarios. It is worth mentioning that HFOLR cannot be obtained as an instance of the hybridization process, and therefore, do not fall in the framework of [57]. Another general completeness result for temporal logics can be found in [29], which appears to be the source of inspiration for [57]. A completeness proof for rigid first-order hybrid logic based on Henkin's approach can be found in [9].

The hybrid specification from Example 7.2 is conceptually the same as the "plastic" buffer specification presented in [26]. The difference consists of the underlying logic used for defining the specification. The case study conducted in [26] is based on a hybrid logic obtained from partial algebra [3, 55] by applying the aforementioned hybridization process; the result is a variation of HFOLS with partial operation symbols instead of relation symbols. The formal verification reported in [26] relies on a general "encoding" of the underlying hybrid logic in first-order logic. The results of the translations look rather complex when compared with the inputs. Then the corresponding proofs are performed by using SPASS [73] automatic first-order logic prover through the Hets system [56]. The advantage is that the established verification methodology underlying SPASS can be immediately applied to concrete case studies. This idea is explored further in works such as [12, 72]. However, there are shortcomings: (a) the verification method implemented in SPASS is inherent to first-order logic, and therefore, it is not designed to make use of the specific features of a hybrid logic, and (b) the complexity of the translations, which is irrelevant for automatic theorem proving, but it is inconvenient for interactive proofs. In the present contribution, the proofs are performed directly in the underlying hybrid logic, which has several benefits that include the possibility of improving both the verification technique and the specification itself. For short, the formal proof reported in [26] is based on the methodology implemented in SPASS, while the present verification technique is to be developed.

*Future work.* In the future, the ideas sketched in Section 7 will be developed further into a proper specification and verification methodology of reconfigurable systems. Such a formal method will be implemented into a dedicated software tool for assisting formal proofs of properties satisfied by reconfigurable systems. The present contribution is a part of a larger research agenda, where the target is to provide a technology to support the design of reliable software systems based on a hybrid logic. It is fair to say that the present contribution paves the way for such a technology by providing a solid proof-theoretic foundation.

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