Fraïssé-Hintikka Theorem in institutions

Daniel Găină*[1] and Tomasz Kowalski†[2]

[1]Institute of Mathematics for Industry, Kyushu University, Japan
[2]Department of Mathematics and Statistics, La Trobe University, Australia

Abstract

We generalize the characterization of elementary equivalence by Ehrenfeucht-Fraïssé games to arbitrary institutions whose sentences are finitary. These include many-sorted first-order logic, higher-order logic with types, as well as a number of other logics arising in connection to specification languages. The gain for the classical case is that the characterization is proved directly for all signatures, including infinite ones.

1 Introduction

A classical model theorist or logician is dimly aware that in applications it is convenient to have sorts (say, the boolean sort, lists, and the natural numbers), possibly some higher-order ingredients (say, a collection of subsets of $\mathbb{N}$), and perhaps partial functions (square root, for example), but he dismisses these things as trivial nuisances, which, with some effort and perhaps a little cleverness, can be simulated in the classical one sorted first-order setting. An applied logician replies that although simulations are fine in principle, they are not always reasonable in practice, and 'practice' includes the ordinary work of a mathematician. Reverse mathematics is a case in point. Roughly, one considers there extensions of Peano Arithmetic obtained by adding some collections of subsets of $\mathbb{N}$. The resulting structures, although called second-order arithmetics, are in fact two-sorted first-order. Example 8 below gives more details about these two-sorted structures.

The two authors represent these two parties (no prizes for guessing whom which), so this article is also an exercise in reconciliation. One result of this is that the reader familiar with institution theory will find parts of what we say trivial, and the reader familiar with classical model theory will also find parts of what we say trivial. We hope these will not be the same parts.

Institutions. Institution theory arose in response to practical considerations; specifically, as a reaction to a proliferation of algebraic specification languages which did essentially the same in different ways, so the same theorems had to be proved for

*E-mail address: daniel@imi.kyushu-u.ac.jp
†E-mail address: T.Kowalski@latrobe.edu.au

each, over and over. Tired of this, Goguen and Burstall in [12] introduced institutions, an abstract framework for reasoning about properties of logical systems from a meta-perspective. Isolating the essence of a logical system in the abstract *satisfaction relation*, whose essence in turn is that *truth is invariant under change of notation*, and leaving the details open, institutions achieve an appropriate level of generality for the development of abstract model theory, independent of the specific nature of the underlying logic.

Intuitively, an institution can be likened to a high-level programming language: it comes with a syntax, a semantics, and a satisfaction relation linking the two. As we will see later in detail, first-order logic can be presented as an institution: its syntax is what you expect, its semantics is the class of all models in the usual sense, and the satisfaction relation is Tarski's satisfiability. Many other logical systems can be so presented: partial algebras, higher-order logics, intuitionistic logic, modal logics, many-valued logics, in fact any logical framework with a satisfaction relation can be presented as an institution. Institutions may be quite unlike one another, but they share some theorems: these that can be proved independently of any assumptions specific to a particular institution. Proving classical theorems in an *institution independent* way has been a theme in institution theory from its early days, starting from meta-level theorems about algebraic specification languages. A large body of institution independent proofs of classical theorems exists: interpolation and related properties of Beth definability and Robinson joint consistency were studied in [6, 27, 21, 13, 16], ultraproducts and saturated models in [5, 11], model theoretic forcing in [19, 14, 17], elementary chains in [20], quasivarieties and free models in [30, 32, 7, 18], and proof theory in [8, 4, 19, 15]. For a monograph on institution theory, we refer the reader to [9].

However, a classical result in model theory stating that elementary equivalence can be characterized in terms of finite Ehrenfeucht-Fraïssé games has so far not been given an institution independent treatment. The present article fills this gap.

Although the above characterization is commonly called Fraïssé-Hintikka Theorem, we will follow [26] in using the name for something more technical (Theorem 34), of which the game characterization is a corollary (Corollary 35). Since finite games are quite intuitive and easy to describe, Fraïssé-Hintikka Theorem gives a better handle on elementary equivalence than Keisler-Shelah Theorem characterizing elementary equivalence via ultrapowers. A typical example given to illustrate this point is the following. Let $\mathbb{N} \oplus \mathbb{Z}$ be the linear sum of $\mathbb{N}$ and $\mathbb{Z}$ as ordered sets: a copy of $\mathbb{Z}$ on top of a copy of $\mathbb{N}$. It is a nontrivial exercise to find an ultrapower of $\mathbb{N} \oplus \mathbb{Z}$ isomorphic to an ultrapower of $\mathbb{N}$, whereas the fact that the 'existential' player has a winning strategy in all finite games is just staring one in the face: for a fixed k, choose the response high enough in $\mathbb{N}$ (2^{k+1} should suffice) so that you could survive k further steps. Numerous celebrated results were proved using Ehrenfeucht-Fraïssé games: see, e.g., [33] for a quick introduction and survey of three examples important in theoretical computer science.

One of the things required for an institution independent proof of Fraïssé-Hintikka Theorem is a description of Ehrenfeucht-Fraïssé games at an abstract level. We give a suitable description, revealing in the process that a naive generalization of the games for elementary equivalence from single first-order logic to many-sorted first-order logic and higher-order logic setting runs into problems related to infinite signatures.

Notational conventions. The notation we use is standard in institution theory, but since institution theory is nonstandard in logic, we give a few warnings below. Generally, institution theory assumes categorical background, so if any piece of notation seems unclear, an interpretation from category theory (if there is one) will be right. Thus, $|C|$ stands for the set of objects of C; the collection of all signatures and signature morphisms, in particular, forms a category. Composition of morphisms and functors is denoted using the symbol ';' and is considered in diagrammatic order. The word 'arity' in institution theory is used to denote a finite string over the alphabet whose letters are the sorts. So, against etymology, arities formally are not numbers. However, the set of finite strings over a set S is denoted by the standard S^*. We will still say e.g., 'a ternary relation' informally with the usual meaning.

2 Preliminaries

We said that institutions focus on the satisfaction relation, leaving other details open. We did not say details of what. We will do it now. Crucial to the concept of institution is a notion of *signature*, which is more or less what it is in model theory, but whereas in model theory signatures are grudgingly acknowledged and ignored as much as they can be, in institution theory they come to the forefront.

Definition 1 (Institution). *An institution* $\mathtt{I} = (\mathtt{Sig}^{\mathtt{I}}, \mathtt{Sen}^{\mathtt{I}}, \mathtt{Mod}^{\mathtt{I}}, \models^{\mathtt{I}})$ *consists of:*

1. *A category* $\mathtt{Sig}^{\mathtt{I}}$*, whose objects are called* signatures*.*
2. *A functor* $\mathtt{Sen}^{\mathtt{I}} : \mathtt{Sig}^{\mathtt{I}} \to \mathtt{Set}$*, providing for each signature* Σ *a set whose elements are called* (Σ-)sentences*.*
3. *A functor* $\mathtt{Mod}^{\mathtt{I}} : \mathtt{Sig}^{\mathtt{I}} \to \mathtt{Cat}^{op}$*, providing for each signature* Σ *a category whose objects are called* (Σ-)models *and whose arrows are called* (Σ-)homomorphisms*.*
4. *A family of relations* $\models^{\mathtt{I}} = \{\models^{\mathtt{I}}_{\Sigma}\}_{\Sigma \in |\mathtt{Sig}^{\mathtt{I}}|}$*, where* $\models^{\mathtt{I}}_{\Sigma} \subseteq |\mathtt{Mod}(\Sigma)| \times \mathtt{Sen}^{\mathtt{I}}(\Sigma)$ *is called* (Σ-)*satisfaction for all signatures* $\Sigma \in |\mathtt{Sig}|$*, such that the following* satisfaction condition *holds:*
$$\mathfrak{A}' \models_{\Sigma'} \mathtt{Sen}(\varphi)(e) \textit{ iff } \mathtt{Mod}(\varphi)(\mathfrak{A}') \models_{\Sigma} e$$
for all $\varphi\colon \Sigma \to \Sigma' \in \mathtt{Sig}^{\mathtt{I}}$*,* $\mathfrak{A}' \in |\mathtt{Mod}(\Sigma')|$ *and* $e \in \mathtt{Sen}(\Sigma)$*.*

Note that the notion of valuation is conspicuously absent from institutions. Indeed, in institution theory all variables are (treated as) constants, and thus there are no open formulas. Quantification has to be defined accordingly, and it takes some work. See Section 3.2.

In concrete examples, the category of signatures $\mathtt{Sig}$ provides the vocabularies over which the sentences are built and the morphisms in $\mathtt{Sig}^{\mathtt{I}}$, which we will simply call *signature morphisms*, represent a change of notation. Signature morphisms act covariantly on sentences, and contravariantly on models. More concretely, given a signature morphism $\varphi\colon \Sigma \to \Sigma'$ the sentences over the signature Σ are mapped to the sentences over the signature Σ' by the function $\mathtt{Sen}(\varphi)\colon \mathtt{Sen}(\Sigma) \to \mathtt{Sen}(\Sigma')$. The Σ'-models are 'reduced' to the signature Σ by the functor $\mathtt{Mod}(\varphi) : \mathtt{Mod}(\Sigma') \to \mathtt{Mod}(\Sigma)$. We denote the *reduct* functor $\mathtt{Mod}(\varphi)$ by $\_\!\upharpoonright_{\varphi}$ and the function $\mathtt{Sen}(\varphi)$ by φ. If $\mathfrak{A} = \mathfrak{A}'\!\upharpoonright_{\varphi}$ we say that $\mathfrak{A}$ is the φ-*reduct* of $\mathfrak{A}'$, and $\mathfrak{A}'$ is a φ-*expansion* of $\mathfrak{A}$.

When there is no danger of confusion, we omit the superscript $\mathtt{I}$ from the notations of the institution components; for example $\mathtt{Sig}^{\mathtt{I}}$ may be simply denoted by $\mathtt{Sig}$. The notation surrounding the satisfaction relation is standard. Namely, for all signatures Σ, sets of Σ-sentences Γ and E, we have

1. For all Σ-models $\mathfrak{A}$, $(\mathfrak{A} \models E)$ iff $(\mathfrak{A} \models e$ for all $e \in E)$;
2. $\Gamma \models E$ iff for all Σ-models $\mathfrak{A}$ we have $\mathfrak{A} \models \Gamma$ implies $\mathfrak{A} \models E$;
3. $\Gamma \mathrel{|}\!\!\models E$ iff $\Gamma \models E$ and $E \models \Gamma$.

2.1 Examples

We give a few examples of institutions frequently occurring in algebraic specification literature.

Example 2 (First-order logic (FOL) [12])**.**
Signatures. *Signatures are of the form (S, F, P), where S is a set of sorts, $F = \{F_{\mathtt{ar}\to s}\}_{(\mathtt{ar},s)\in S^*\times S}$ is a ($S^*\times S$ -indexed) set of operation symbols, and $P = \{P_{\mathtt{ar}}\}_{\mathtt{ar}\in S^*}$ is a (S^*-indexed) set of relation symbols. If $\mathtt{ar} = \varepsilon$ then an element of $F_{\mathtt{ar}\to s}$ is called a* constant symbol. *Generally, $\mathtt{ar}$ ranges over arities, which are understood here as strings of sorts; in other words an arity gives the number of arguments together with their sorts. We overload the notation and let F and P also denote $\biguplus_{(\mathtt{ar},s)\in S^*\times S} F_{\mathtt{ar}\to s}$ and $\biguplus_{\mathtt{ar}\in S^*} P_{\mathtt{ar}}$, respectively. Therefore, we may write $\sigma \in F_{\mathtt{ar}\to s}$ or $(\sigma\colon \mathtt{ar} \to s) \in F$; both have the same meaning, which is: σ is an operation symbol of type $\mathtt{ar} \to s$.*

A number of usual tricks, such as adding constants, but also, importantly, quantification, are viewed as expansions of the signature, so moving between signatures is common. To make such transitions smooth, the notion of a signature morphism *is introduced. Formally, a signature morphism $\varphi\colon \Sigma \to \Sigma'$, where $\Sigma = (S, F, P)$ and $\Sigma' = (S', F', P')$, is a triple $\varphi = (\varphi^{st}, \varphi^{op}, \varphi^{rl})$ of maps:*

- $\varphi^{st}\colon S \to S'$,
- $\varphi^{op} = \{\varphi^{op}_{\mathtt{ar}\to s}\colon F_{\mathtt{ar}\to s} \to F'_{\varphi^{st}(\mathtt{ar})\to\varphi^{st}(s)} \mid \mathtt{ar} \in S^*, s \in S\}$,
- $\varphi^{rl} = \{\varphi^{rl}_{\mathtt{ar}}\colon P_{\mathtt{ar}} \to P'_{\varphi^{st}(\mathtt{ar})} \mid \mathtt{ar} \in S^*\}$.

When there is no danger of confusion, we may let φ denote either of φ^{st}, $\varphi^{op}_{\mathtt{ar}\to s}$, $\varphi^{rl}_{\mathtt{ar}}$.
Models. *Given a signature $\Sigma = (S, F, P)$, a Σ-model is a triple*

$$\mathfrak{A} = (\{\mathfrak{A}_s\}_{s\in S}, \{\sigma^{\mathfrak{A}}\}_{(\mathtt{ar},s)\in S^*\times S, \sigma\in F_{\mathtt{ar}\to s}}, \{\pi^{\mathfrak{A}}\}_{\mathtt{ar}\in S^*, \pi\in P_{\mathtt{ar}}})$$

interpreting each sort s as a set $\mathfrak{A}_s$, each operation symbol $\sigma \in F_{\mathtt{ar}\to s}$ as a function $\sigma^{\mathfrak{A}}\colon \mathfrak{A}_{\mathtt{ar}} \to \mathfrak{A}_s$ (where $\mathfrak{A}_{\mathtt{ar}}$ stands for $\mathfrak{A}_{s_1} \times \ldots \times \mathfrak{A}_{s_n}$ if $\mathtt{ar} = s_1 \ldots s_n$), and each relation symbol $\pi \in P_{\mathtt{ar}}$ as a relation $\pi^{\mathfrak{A}} \subseteq \mathfrak{A}_{\mathtt{ar}}$. Morphisms between models are the usual Σ-homomorphisms, i.e., S-sorted functions that preserve the structure. For any signature morphism $\varphi\colon \Sigma \to \Sigma'$, where $\Sigma = (S, F, P)$ and $\Sigma' = (S', F', P')$, the model functor $\mathtt{Mod}(\varphi)\colon \mathtt{Mod}(\Sigma') \to \mathtt{Mod}(\Sigma)$ is defined as follows:

1. *The reduct* $\mathfrak{A}'\upharpoonright_\varphi$ *of a* Σ'*-model* $\mathfrak{A}'$ *is a defined by* $(\mathfrak{A}'\upharpoonright_\varphi)_s = \mathfrak{A}'_{\varphi(s)}$ *for each sort* $s \in S$*, and* $x^{\mathfrak{A}'\upharpoonright_\varphi} = \varphi(x)^{\mathfrak{A}'}$*, for each operation symbol* $x \in F$ *or relation symbol* $x \in P$*. Since the models are many-sorted it is useful to have a terminological distinction between* carrier sets $\mathfrak{A}_s$ *of individual sorts, and the* universe $\{\mathfrak{A}_s\}_{s\in S}$ *of the entire model. With this distinction at hand, note that the reduct functor does not modify carrier sets of the sorts, but it can modify the universes of models. For the universe of* $\mathfrak{A}'\upharpoonright_\varphi$ *is* $\{\mathfrak{A}'_{\varphi(s)}\}_{s\in S}$*, which means that the sorts outside the image of* S *are discarded. Otherwise, the notion of reduct is standard.*

2. *The reduct* $h'\upharpoonright_\varphi$ *of a homomorphism* h' *is defined by* $(h'\upharpoonright_\varphi)_s = h'_{\varphi(s)}$ *for all sorts* $s \in S$*.*

One important example of Σ*-model is the absolutely free first-order* Σ*-structure of terms* T_Σ *that interprets each relation as the empty set.*

Sentences. *The set of* Σ*-sentences is given by the following grammar:*

$$e ::= \quad t =_s t' \mid \pi(t_1, \ldots, t_n) \mid \neg e \mid \wedge E \mid \exists X \cdot e'$$

where (i) $t =_s t'$ *is an equation with* $t, t' \in T_{\Sigma,s}$ *and* $s \in S$*,* (ii) $\pi(t_1, \ldots, t_n)$ *is a relational atom with* $\pi \in P_{s_1\ldots s_n}$*,* $t_i \in T_{\Sigma,s_i}$ *and* $s_i \in S$*,* (iii) E *is a finite set of* Σ*-sentences,* (iv) X *is a finite set of variables for* Σ*,* (v) $\exists X \cdot \_$ *is just an abbreviation for* $\exists \chi \cdot \_$ *such that* $\chi \colon \Sigma \hookrightarrow \Sigma[X]$ *is an inclusion,* $\Sigma[X] = (S, F[X], P)$*, and* $F[X]$ *is the family of operation symbols obtained by adding the variables in* X *as constants to* F*,* (vi) e' *is a* $\Sigma[X]$*-sentence. For any signature morphism* $\varphi \colon \Sigma \to \Sigma'$ *the function* $\mathtt{Sen}^{\mathsf{FOL}}(\varphi) : \mathtt{Sen}^{\mathsf{FOL}}(\Sigma) \to \mathtt{Sen}^{\mathsf{FOL}}(\Sigma')$ *translates sentences symbolwise. In Section 3.2 we will give more details about quantification as a signature morphism, and sets of variables for a given signature.*

Satisfaction relation. *Satisfaction is the usual first-order satisfaction and it is defined using the natural interpretations of ground terms* t *as elements* $t^{\mathfrak{A}}$ *in models* $\mathfrak{A}$*. For example,* $\mathfrak{A} \models t_1 =_s t_2$ *iff* $t_1^{\mathfrak{A}} = t_2^{\mathfrak{A}}$*.*

Example 3 (Finitary first-order logic (FOL_f))**.** *This institution is obtained from* FOL *by restricting the semantics to models with finite carrier sets. More concretely,* $\mathsf{FOL}_f = (\mathtt{Sig}^{\mathsf{FOL}}, \mathtt{Sen}^{\mathsf{FOL}}, \mathtt{Mod}^{\mathsf{FOL}_f}, \models^{\mathsf{FOL}})$*, where* $\mathtt{Mod}^{\mathsf{FOL}_f}$ *is a subfunctor of* $\mathtt{Mod}^{\mathsf{FOL}}$ *(i.e.* $\mathtt{Mod}^{\mathsf{FOL}_f}(\Sigma) \subseteq \mathtt{Mod}^{\mathsf{FOL}}(\Sigma)$ *for all* $\Sigma \in |\mathtt{Sig}^{\mathsf{FOL}}|$*) such that for all first-order signatures* $\Sigma = (S, F, P)$ *we have:* $\mathfrak{A} \in |\mathtt{Mod}^{\mathsf{FOL}_f}(\Sigma)|$ *iff* $\mathfrak{A} \in |\mathtt{Mod}^{\mathsf{FOL}}(\Sigma)|$ *and* $\mathfrak{A}_s$ *is finite for all sorts* $s \in S$*.*

Example 4 (Partial algebra (PA))**.** *Here we consider the institution* PA *as employed by the specification language CASL [1].*

A partial algebraic signature is a tuple (S, TF, PF) *such that* $(S, TF \cup PF)$ *is an algebraic signature. Then* TF *is the set of total operations and* PF *is the set of partial operations. A morphism of* PA *signatures* $\varphi \colon (S, TF, PF) \to (S, TF', PF')$ *is just a morphism of algebraic signatures* $(S, TF \cup PF) \to (S, TF' \cup PF')$ *such that* $\varphi(TF) \subseteq TF'$ *and* $\varphi(PF) \subseteq PF'$*.*

A partial algebra $\mathfrak{A}$ *for a* PA *signature* (S, TF, PF) *is just like an ordinary algebra but interpreting the operations of* PF *as partial rather than total functions, which means that* $\sigma^{\mathfrak{A}}$ *might be undefined for some arguments. A partial algebra homomorphism* $h \colon \mathfrak{A} \to \mathfrak{B}$ *is a family of (total) functions* $\{h_s \colon \mathfrak{A}_s \to \mathfrak{B}_s\}_{s\in S}$ *indexed by the* *set of sorts S of the signature such that $h_s(\sigma^{\mathfrak{A}}(a)) = \sigma^{\mathfrak{B}}(h_s(a))$ for each operation symbol $\sigma\colon \mathtt{ar} \to s$ and each string of arguments $a \in \mathfrak{A}_{\mathtt{ar}}$ for which $\sigma^{\mathfrak{A}}(a)$ is defined.*

The sentences have three kinds of atoms: definedness $\mathtt{def}(\_)$, strong *equality* $\overset{s}{=}$ *and existence equality* $\overset{e}{=}$*. The* definedness $\mathtt{def}(t)$ *of a term t holds in a partial algebra $\mathfrak{A}$ when the interpretation $t^{\mathfrak{A}}$ of t is defined. The strong equality $t_1 \overset{s}{=} t_2$ holds when both terms are undefined or both of them are defined and are equal. The existence equality $t_1 \overset{e}{=} t_2$ holds when both terms are defined and are equal. The sentences are formed from these atoms by means of Boolean connectives and quantification over total (first-order) variables. Note that each definedness atom $\mathtt{def}(t)$ is semantically equivalent with $t \overset{e}{=} t$ and any strong equality $t_1 \overset{s}{=} t_2$ is semantically equivalent with $(\mathtt{def}(t_1) \vee \mathtt{def}(t_2)) \Rightarrow t_1 \overset{e}{=} t_2$.*

Example 5 (Higher-order logic (HOL))**.** *This is a simplified version of higher-order logic which does not consider λ-abstraction. One could say* HOL *is λ-calculus without λ, but nobody does. Probably because* HOL *is not calculus.*

For any set S of sorts, let $\mathtt{Type}(S)$ be the set of S-types defined as the least set such that $S \subseteq \mathtt{Type}(S)$ and $s_1 \to s_2 \in \mathtt{Type}(S)$ when $s_1, s_2 \in \mathtt{Type}(S)$. A HOL*-signature is a tuple (S, F), where S is a set of sorts and F is a family of sets of constants $F = \{F_s\}_{s\in\mathtt{Type}(S)}$. A signature morphism $\varphi\colon (S, F) \to (S', F')$ consists of a function $\varphi^{st}\colon S \to S'$ and a family of functions between operation symbols $\{\varphi^{op}_s\colon F_s \to F'_{\varphi^{type}(s)}\}_{s\in\mathtt{Type}(S)}$ where $\varphi^{type}\colon \mathtt{Type}(S) \to \mathtt{Type}(S')$ is the natural extension of φ^{st} to $\mathtt{Type}(S)$. For every signature (S, F), a (S, F)-model $\mathfrak{A}$ interprets each*

(a) sort $s \in S$ as a set $\mathfrak{A}_s$, and

(b) function symbol $\sigma \in F_s$ as an element of $\mathfrak{A}_s$, where for all types $s_1, s_2 \in \mathtt{Type}(S)$, $\mathfrak{A}_{s_1\to s_2} = [\mathfrak{A}_{s_1} \to \mathfrak{A}_{s_2}] = \{f \text{ function } \mid f\colon \mathfrak{A}_{s_1} \to \mathfrak{A}_{s_2}\}$.

An (S, F)-model morphism $h\colon \mathfrak{A} \to \mathfrak{B}$ interprets each type $s \in \mathtt{Type}(S)$ as a function $h_s\colon \mathfrak{A}_s \to \mathfrak{B}_s$ such that $h(\sigma^{\mathfrak{A}}) = \sigma^{\mathfrak{B}}$, for all function symbols $\sigma \in F$, and the following diagram commutes

$$\begin{array}{ccc} \mathfrak{A}_{s_1} & \xrightarrow{\ f\ } & \mathfrak{A}_{s_2} \\ {\scriptstyle h_{s_1}}\downarrow & & \downarrow{\scriptstyle h_{s_2}} \\ \mathfrak{B}_{s_1} & \xrightarrow[h_{s_1\to s_2}(f)]{} & \mathfrak{B}_{s_2} \end{array}$$

for all types $s_1, s_2 \in \mathtt{Type}(S)$ and functions $f \in \mathfrak{A}_{s_1\to s_2}$.

An (S, F)-equation is an equation of the form $t_1 = t_2$, where t_1 and t_2 are terms of the same type. Sentences are constructed from equations by iteration of Boolean connectives and higher-order quantification over sets of variables of any type (which is defined similarly to the first-order quantification).

Example 6 (Higher-order logic with Henkin semantics (HNK))**.** *This institution extends* HOL *by relaxing the condition $\mathfrak{A}_{s_1\to s_2} = [\mathfrak{A}_{s_1} \to \mathfrak{A}_{s_2}]$ to $\mathfrak{A}_{s_1\to s_2} \subseteq [\mathfrak{A}_{s_1} \to \mathfrak{A}_{s_2}]$.* HNK *has been introduced and studied in [3] and [24]. Again, we consider a simplified version close to the "higher-order algebra" of [28] without λ-abstraction.*

Example 7 (Institution of presentations)**.** *In any institution $\mathtt{I} = (\mathtt{Sig}, \mathtt{Sen}, \mathtt{Mod}, \models)$, a presentation is a pair (Σ, E) consisting of a signature $\Sigma \in |\mathtt{Sig}|$ and a set of*

Σ-sentences E. A presentation morphism $\varphi : (\Sigma, E) \to (\Sigma', E')$ is a signature morphism $\varphi\colon \Sigma \to \Sigma'$ such that $E' \models_{\Sigma'} \varphi(E)$. Note that presentation morphisms are closed under composition. The institution of presentations over $\mathtt{I}$, *denoted by* $\mathtt{I}^{pres} = (\mathtt{Sig}^{pres}, \mathtt{Sen}^{pres}, \mathtt{Mod}^{pres}, \models^{pres})$ *is defined as follows:*

- $\mathtt{Sig}^{pres}$ *is the category of presentations of* $\mathtt{I}$,
- $\mathtt{Sen}^{pres}(\Sigma, E) = \mathtt{Sen}(\Sigma)$,
- $\mathtt{Mod}^{pres}(\Sigma, E)$ *is the full subcategory of* $\mathtt{Mod}(\Sigma)$ *of models satisfying E, and*
- $\mathfrak{A} \models^{pres}_{(\Sigma,E)} e$ *iff* $\mathfrak{A} \models_\Sigma e$, *for each (Σ, E)-model $\mathfrak{A}$ and Σ-sentence e.*

Institutions of presentations include presentations of FOL (HNK, PA) with a particular set of axioms, but also, importantly, *theories* over FOL (HNK, PA).

Example 8 (Subsystems of second-order arithmetic). *A concrete example of presentation is* second-order arithmetic Z_2. *It is defined over a first-order signature $\Sigma = (S, F, P)$, with sorts* $\mathtt{Nat}$ *and* $\mathtt{Set}$, *function symbols* $0 :\to \mathtt{Nat}$, $s : \mathtt{Nat} \to \mathtt{Nat}$ $+ : \mathtt{Nat}\ \mathtt{Nat} \to \mathtt{Nat}$ *and* $\times : \mathtt{Nat}\ \mathtt{Nat} \to \mathtt{Nat}$, *and membership relation* $\in : \mathtt{Nat}\ \mathtt{Set}$. *The* number variables *are usually denoted by lower case letters $x, y, \ldots,$ while* set variables *are usually denoted by upper case letters $X, Y, \ldots$. The axioms Γ of* Z_2 *are the usual Peano axioms, together with*

- *the* induction axiom $\forall X \cdot \big(0 \in X \wedge \forall x \cdot (x \in X \Rightarrow s\ x \in X) \Rightarrow \forall x \cdot x \in X\big)$, *and*
- *universal closures of the* comprehension scheme $\exists X \cdot \forall x \cdot (x \in X \Leftrightarrow \rho[x])$, *for any sentence ρ over the signature $\Sigma[x]$, such that X does not occur in ρ.*[1]

Subsystems of Z_2 *are obtained by restricting the formula $\rho[x]$ in certain ways. These systems are central to* reverse mathematics. *A comprehensive handbook of reverse mathematics is [29]. Clearly, any subsystem of* Z_2 *is a presentation over* FOL.

3 Institution independent concepts

In this section, we introduce some concepts necessary to prove our abstract results.

3.1 Internal logic

The following institutional notions dealing with the semantics of Boolean connectives and quantifiers were defined in [31].

Definition 9 (Internal logic). *Given a signature Σ in an institution, a Σ-sentence γ is a semantic*

1. negation *of a Σ-sentence e if for each Σ-model $\mathfrak{A}$,*

   $\mathfrak{A} \models \gamma$ *iff* $\mathfrak{A} \not\models e$;

[1] If X could occur in ρ, then the formula $x \notin X$ would produce an inconsistent comprehension axiom $\exists X \cdot \forall x \cdot (x \in X \Leftrightarrow x \notin X)$.

2. conjunction *of a (finite) set of* Σ*-sentences* E *if for each* Σ*-model* $\mathfrak{A}$,

   $\mathfrak{A} \models \gamma$ *iff* $\mathfrak{A} \models e$ *for all* $e \in E$;

3. universal χ-quantification *of a* Σ'*-sentence* e', *where* $\chi\colon \Sigma \to \Sigma'$, *if for each* Σ*-model* $\mathfrak{A}$,

   $\mathfrak{A} \models \gamma$ *iff* $\mathfrak{A}' \models_{\Sigma'} e'$ *for all* χ*-expansions* $\mathfrak{A}'$ *of* $\mathfrak{A}$.

Distinguished negation is usually denoted by $\neg\_$, *distinguished conjunction by* $\wedge\_$, *and distinguished universal* χ*-quantification by* $\forall\chi \cdot \_$. □

One should notice a peculiar treatment of quantifiers here, because it is one of distinctive characteristics of institution theory. Intuitively, the sentence $\forall x \cdot e[x]$ should hold in $\mathfrak{A}$ if and only if the open formula $e[x]$ is satisfied by $\mathfrak{A}$ on all valuations v into $\mathfrak{A}$. This is equivalent to saying that for all expansions $(\mathfrak{A}, a)$ of $\mathfrak{A}$, we have that $(\mathfrak{A}, a)$ satisfies $e[x/a]$; and it is the way quantification is rendered in institutions. Namely, let $\chi\colon \Sigma \to \Sigma[x]$ be a signature morphism that adds the variable x as a new constant to Σ. The sentence $\forall x \cdot e'$ is an abbreviation of $\forall\chi \cdot e'$ and the third clause in Definition 9 ensures that we consider all χ-expansions of $\mathfrak{A}$. Thus, the classical notion of valuation is incorporated into expansions. Variables do not belong to the language: they can be imagined as coming from some external pool, but technically variables are just special constants. Succinctly, one can say that variables in institutions are arbitrary constants, and so the usual messy caveats about avoiding accidental binding of free variables are not needed. We devote Section 3.2 below to precise institutional treatment of quantifiers.

Sentence building operators such as Boolean connectives and quantifiers are part of the metalanguage and they are used to construct sentences which belong to the internal language of individual institutions using the universal semantics presented above. One can also define $\vee\_$, $\exists\chi \cdot \_$ using the classical definitions. For example, $\exists\chi \cdot e' \coloneqq \neg\forall\chi \cdot \neg e'$ and $\top \coloneqq \wedge\emptyset$.

In this article, we consider conjunctions only over finite sets of sentences. The concept of quantification we use is very general, and it can cover second order quantification in classical model theory. The universal quantifier $\forall X \cdot \_$ is regarded as an abbreviation for $\forall\chi \cdot \_$, where $\chi\colon \Sigma \hookrightarrow \Sigma[X]$ is an inclusion of signatures and $\Sigma[X]$ denotes the extension of Σ with the variables from X as constants.

An institution $\mathtt{I}$ is said to be semantically closed under negation (conjunction, universal quantification, etc.) if every sentence in $\mathtt{I}$ has a semantic negation (conjunction, universal quantification, etc.) according to Definition 9.

Dealing with standard logical operators, we adopt the following convention about their binding strength: $\neg$ binds stronger than $\wedge$, which binds stronger than $\vee$, which binds stronger than $\Rightarrow$, which binds stronger than quantifiers; quantifiers $\exists$ and $\forall$ have the same binding strength.

3.2 Quantification space

Quantification comes with some subtle issues related to the translation of quantified sentences along signature morphisms. We alluded to that in Section 3.1; now we will make it precise. Let us begin by a motivating example. Consider a first-order signature Σ and a Σ-sentence $\forall x \cdot \rho$. In institutional model theory, $\forall x \cdot \rho$ is an abbreviation

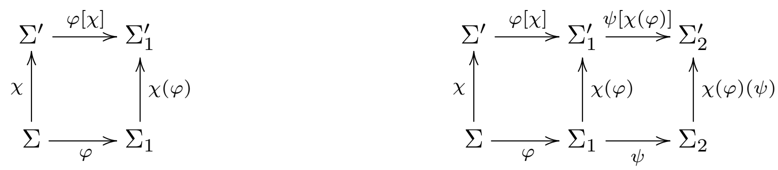

Figure 1: Quantification pushouts

for $\forall\chi\cdot\rho$, where $\chi\colon \Sigma \hookrightarrow \Sigma[x]$ is an inclusion of signatures, $\Sigma[x]$ is the signature obtained from Σ by adding the variable x as a constant, and ρ is a sentence over $\Sigma[x]$. Clearly, $\forall\chi\cdot\rho$ is not a sentence over $\Sigma[x]$ because the domain of χ is not $\Sigma[x]$ but Σ. If we want a counterpart of $\forall x\cdot\rho$ in $\Sigma[x]$ we need to translate it along χ, and to do that, we need to rename x. The need to rename x can happen for essentially any signature morphism φ, not only for χ. Diagramatically, we have $\Sigma[x] \overset{\chi}{\hookleftarrow} \Sigma \overset{\varphi}{\to} \Sigma_1$, and a variable x_1 for Σ_1. This implies that there are two further signature morphisms:

1. $\chi(\varphi)\colon \Sigma_1 \hookrightarrow \Sigma_1[x_1]$, which adds the variable x_1 to Σ_1, and
2. $\varphi[\chi]\colon \Sigma[x] \to \Sigma_1[x_1]$, which extends φ by maping x to x_1.

making $\{\Sigma[x] \overset{\chi}{\hookleftarrow} \Sigma \overset{\varphi}{\to} \Sigma_1, \Sigma[x] \overset{\varphi[\chi]}{\to} \Sigma_1[x_1] \overset{\chi(\varphi)}{\hookleftarrow} \Sigma_1\}$ a pushout of signatures, which with $\Sigma[x] = \Sigma'$ and $\Sigma_1[x_1] = \Sigma'_1$ is precisely the left-hand side diagram in Figure 1.

The renaming procedure can be iterated. In terms of diagrams: composing such pushouts horizontally produces such pushouts again. Now for the definition.

Definition 10 (Quantification space [10])**.** *Given a category* $\mathtt{Sig}$*, a subclass of arrows* $\mathtt{Q} \subseteq \mathtt{Sig}$ *is called a* quantification space *if,*

1. *for any signature morphisms* $\Sigma \overset{\chi}{\to} \Sigma' \in \mathtt{Q}$ *and* $\Sigma \overset{\varphi}{\to} \Sigma_1 \in \mathtt{Sig}$ *there is a designated pushout depicted on the left in Figure 1 such that* $\chi(\varphi) \in \mathtt{Q}$*, and*
2. *the horizontal composition of such designated pushouts is again a designated pushout:*
   (a) $\chi(1_\Sigma) = \chi$, $1_\Sigma[\chi] = 1_{\Sigma'}$*, and*
   (b) *for all pushouts such as the ones depicted on the right in Figure 1 we have* $\varphi[\chi];\psi[\chi(\varphi)] = (\varphi;\psi)[\chi]$ *and* $\chi(\varphi)(\psi) = \chi(\varphi;\psi)$*.*

In concrete examples of institutions, the quantification space is fixed and the translation of a quantified sentence $\exists\chi\cdot\rho \in \mathtt{Sen}(\Sigma)$ along φ is $\exists\chi(\varphi)\cdot\varphi[\chi](\rho)$. The second condition in Definition 10 is required by the functoriality of the translations.

3.2.1 Examples

We give some examples of quantification spaces for the institutions defined above. To this end, we fix a countably infinite set $\{\mathtt{x}_i \mid i \in \mathbb{N}\}$ of variable names.

Example 11 ($\mathtt{Q}^{\mathsf{FOL}}$)**.** *A* FOL *variable for a signature* $\Sigma = (S, F, P)$ *is a triple* $(\mathtt{x}_i, s, \Sigma)$*, where (a)* $i \in \mathbb{N}$ *and* $\mathtt{x}_i$ *is the name of the variable, and (b)* $s \in S$ *is the sort of* *the variable. In* FOL*, the quantification space* $\mathsf{Q}^{\mathsf{FOL}}$ *consists of signature extensions with a finite number of variables* $\chi\colon \Sigma \hookrightarrow \Sigma[X]$*, where* Σ *is a first-order signature and* $X = \{X_s\}_{s\in S}$ *is a finite set of variables for* Σ*. Given a signature morphism* $\varphi\colon \Sigma \to \Sigma_1$ *in* FOL*, where* $\Sigma_1 = (S_1, F_1, P_1)$*, then*

- $\chi(\varphi)\colon \Sigma_1 \hookrightarrow \Sigma_1[X^\varphi]$ *is an inclusion, where* $X^\varphi = \{(\mathtt{x}_i, \varphi(s), \Sigma_1) \mid (\mathtt{x}_i, s, \Sigma) \in X\}$*,*
- $\varphi[\chi]$ *is the extension of* φ *that maps each* $(\mathtt{x}_i, s, \Sigma)$ *to* $(\mathtt{x}_i, \varphi(s), \Sigma_1)$*.*

Example 12 (Q^{PA})**.** *A* PA *variable for a signature* $\Sigma = (S, TF, PF)$ *is a triple* $(\mathtt{x}_i, s, \Sigma)$*, where (a)* $i \in \mathbb{N}$ *and* $\mathtt{x}_i$ *is the name of the variable, and (b)* $s \in S$ *is the sort of the variable. The quantification space* Q^{PA} *consists of signature extensions with a finite number of variables* $\chi\colon \Sigma \hookrightarrow \Sigma[X]$*, where* Σ *is a* PA *signature,* $X = \{X_s\}_{s\in S}$ *is a finite set of variables for* Σ*, and* $\Sigma[X] = (S, TF \cup X, PF)$*. The translation of variables along signature morphisms is defined as in the case of* FOL*.*

Example 13 ($\mathsf{Q}^{\mathsf{HNK}}$)**.** *A* HNK *variable for a signature* $\Sigma = (S, F)$ *is a triple* $(\mathtt{x}_i, s, \Sigma)$*, where (a)* $i \in \mathbb{N}$ *and* $\mathtt{x}_i$ *is the name of the variable, and (b)* $s \in \mathtt{Type}(S)$ *is the type of the variable. The quantification space* $\mathsf{Q}^{\mathsf{HNK}}$ *consists of signature extensions with a finite number of variables* $\chi\colon \Sigma \hookrightarrow \Sigma[X]$*, where* Σ *is a* HNK *signature,* $X = \{X_s\}_{s\in \mathtt{Type}(S)}$ *is a finite set of variables for* Σ *such that* $\Sigma[X] = (S, F \cup X)$*. The translation of variables along signature morphisms is defined as in the case of* FOL*.*

4 Fraïssé-Hintikka Theorem

In this section we prove an institution independent version of Fraïssé-Hintikka Theorem .

Definition 14 (Unnested terms and unnested first-order atomic sentences)**.** *For any first-order signature* $\Sigma = (S, F, P)$*, the set of unnested terms* T^u_Σ *is defined as follows:*

1. $c \in T^u_{\Sigma,s}$ *for all constants* $(c :\to s) \in F$*, and*
2. $\sigma(c_1, \ldots, c_n) \in T^u_{\Sigma,s}$ *for all operation symbols* $(\sigma\colon s_1 \ldots s_n \to s) \in F$ *and all constants* $(c_1 :\to s_1), \ldots, (c_n :\to s_n) \in F$*.*

An unnested atomic Σ*-sentence is any atomic* Σ*-sentence of the form:*

1. $t = c$*, where* $s \in S$*,* $t \in T^u_{\Sigma,s}$ *and* $(c \to s) \in F$*, or*
2. $\pi(c_1, \ldots, c_n)$*, where* $(\pi\colon s_1 \ldots s_n) \in P$ *and* $(c_i :\to s_i) \in F$*.*

Note that the translation of any unnested atomic sentence along a first-order signature morphism is also atomic and unnested. This means that the institutions described in the following example are well-defined.

Example 15 (Unnested first-order logic (FOL_u))**.** *The following institutions are obtained from first-order logic by restricting the syntax:*

1. FOL_0*, the restriction of* FOL *to unnested atomic sentences, and*
2. FOL_u*, the restriction of* FOL *to sentences obtained from the sentences in* FOL_0 *by applying Boolean connectives and quantification over* $\mathsf{Q}^{\mathsf{FOL}}$*.*

Similarly, one can define PA_0, PA_u and HNK_0, HNK_u.

Definition 16 (Equally strong institutions, $\mathtt{I} \sim \mathtt{I}'$)**.** *Let* $\mathtt{I} = (\mathtt{Sig}, \mathtt{Sen}, \mathtt{Mod}, \models)$ *and* $\mathtt{I}' = (\mathtt{Sig}, \mathtt{Sen}', \mathtt{Mod}, \models')$ *be two institutions that share* $\mathtt{Mod}\colon \mathtt{Sig} \to \mathtt{Cat}^{op}$.

1. $\mathtt{I}'$ *is at least as strong as* $\mathtt{I}$*, in symbols* $\mathtt{I} \lesssim \mathtt{I}'$*, iff for every signature* $\Sigma \in |\mathtt{Sig}|$ *and every sentence* $\rho \in \mathtt{Sen}(\Sigma)$ *there exists a sentence* $\rho' \in \mathtt{Sen}'(\Sigma)$ *such that* $|\mathtt{Mod}(\Sigma, \rho)| = |\mathtt{Mod}(\Sigma, \rho')|$.
2. $\mathtt{I}$ *and* $\mathtt{I}'$ *are equally strong, in symbols* $\mathtt{I} \sim \mathtt{I}'$*, iff* $\mathtt{I} \lesssim \mathtt{I}'$ *and* $\mathtt{I}' \lesssim \mathtt{I}$.

Lemma 17. *We have* $\mathsf{FOL}_u \sim \mathsf{FOL}$*,* $\mathsf{PA}_u \sim \mathsf{PA}$ *and* $\mathsf{HNK}_u \sim \mathsf{HNK}$*.*

Proof. We focus on first-order case as the other two cases are analogous.

It suffices to show that any first-order atomic sentence is semantically equivalent to a sentence obtained from the sentences in FOL_0 by applying Boolean connectives and quantification over finite sets of variables. There are two cases to consider.

1. For equations we proceed by induction on the structure of the terms. The base case $c_1 = c_2$, where $(c_1 :\to s), (c_2 :\to s) \in F$, is ensured by the fact that $c_1 = c_2$ is unnested.

   For the induction step, we prove the statement for $\sigma(t_1, \ldots, t_n) = t$ assuming that it holds for all equations constructed from terms that have a depth less or equal than the maximum depth of the terms $t_1, \ldots, t_n, t$:

1	$\sigma(t_1,\ldots,t_n) = t \mathrel{\models\!\!\!\mid} \forall x_1,\ldots,x_n,x \cdot \bigwedge_{i=1}^{n}(t_i = x_i) \wedge (t = x) \Rightarrow \sigma(x_1,\ldots,x_n) = x$	
2	there exist $\rho_i, \rho \in \mathtt{Sen}^{\mathsf{FOL}_u}(\Sigma[x_1,\ldots,x_n,x])$ such that $t_i = x_i \mathrel{\models\!\!\!\mid} \rho_i$ and $t = x \mathrel{\models\!\!\!\mid} \rho$	by induction hypothesis
3	$\sigma(t_1,\ldots,t_n) = t \mathrel{\models\!\!\!\mid} \forall x_1,\ldots,x_n,x \cdot \bigwedge_{i=1}^{n} \rho_i \wedge \rho \Rightarrow \sigma(x_1,\ldots,x_n) = x$	

2. For relations $\pi(t_1, \ldots, t_n)$:

1	$\pi(t_1,\ldots,t_n) \mathrel{\models\!\!\!\mid} \forall x_1 \ldots x_n \cdot \bigwedge_{i=1}^{n}(t_i = x_i) \Rightarrow \pi(x_1,\ldots,x_n)$	
2	there exist $\rho_i \in \mathtt{Sen}^{\mathsf{FOL}_u}(\Sigma[x_1,\ldots,x_n])$ such that $\rho_i \mathrel{\models\!\!\!\mid} t = x_i$	by the proof above
3	$\pi(t_1,\ldots,t_n) \mathrel{\models\!\!\!\mid} \forall x_1 \ldots x_n \cdot \bigwedge_{i=1}^{n} \rho_i \Rightarrow \pi(x_1,\ldots,x_n)$	

□

Definition 18 (Institution with structured syntax)**.** *An institution of the form* $\mathtt{I} = (\mathtt{Sig}, \mathtt{Sen}, \mathtt{Mod}, \models)$ *has a* structured syntax *if it is equipped with (a) a subfunctor* $\mathtt{Sen}_b\colon \mathtt{Sig} \to \mathtt{Set}$ *of* $\mathtt{Sen}\colon \mathtt{Sig} \to \mathtt{Set}$ *(i.e.* $\mathtt{Sen}_b(\Sigma) \subseteq \mathtt{Sen}(\Sigma)$ *for all signatures* $\Sigma \in |\mathtt{Sig}|$*), and (b) a quantification space* $\mathtt{Q} \subseteq \mathtt{Sig}$ *such that*

1. $\mathtt{I}$ *is semantically closed under Boolean connectives and quantification over the signature morphisms in* $\mathtt{Q}$*, and*

2. $\mathtt{I} \sim \mathtt{I}_u$, *where* $\mathtt{I}_u$ *is obtained from* $\mathtt{I}$ *by restricting the syntax to sentences obtained from the sentences of* $\mathtt{I}_b = (\mathtt{Sig}, \mathtt{Sen}_b, \mathtt{Mod}, \models)$ *by applying Boolean connectives and quantification over* $\mathtt{Q}$.

An example of institution with structured syntax, or, as we will also say, *syntactically structured institution*, is FOL.

Example 19. FOL *is a syntactically structured institution, where*

1. $\mathtt{I}_b$ *is* FOL_0, *the restriction of* FOL *to unnested atomic sentences, and*
2. $\mathtt{Q}$ *is* $\mathtt{Q}^{\mathsf{FOL}}$, *the quantification space defined in Example 11.*

By Lemma 17, PA and HNK are also syntactically structured.

Lemma 20. *Let* $\mathtt{I} = (\mathtt{Sig}, \mathtt{Sen}, \mathtt{Mod}, \models)$ *be a syntactically structured institution as described in Definition 18. Then* $\mathtt{I}^{pres} = (\mathtt{Sig}^{pres}, \mathtt{Sen}^{pres}, \mathtt{Mod}^{pres}, \models^{pres})$ *is a syntactically structured institution with the following parameters:*

1. *the sentence functor* $\mathtt{Sen}_b^{pres} \colon \mathtt{Sig}^{pres} \to \mathtt{Set}$ *is defined by* $\mathtt{Sen}_b^{pres}(\Sigma, E) = \mathtt{Sen}_b(\Sigma)$ *for all presentations* $(\Sigma, E) \in |\mathtt{Sig}^{pres}|$, *and*
2. *the quantification space* $\mathtt{Q}^{pres}$ *consists of all presentation morphisms* $\chi \colon (\Sigma, E) \to (\Sigma', \chi(E))$ *with* $\chi \colon \Sigma \to \Sigma' \in \mathtt{Q}$.

Proof. Let $\mathtt{I}_u = (\mathtt{Sig}, \mathtt{Sen}_u, \mathtt{Mod}, \models)$ be as described in Definition 18, and $\mathtt{I}_u^{pres}$ the institution obtained from $\mathtt{I}^{pres}$ by restricting the syntax to the sentences obtained from the sentences of $\mathtt{I}_b^{pres} = (\mathtt{Sig}^{pres}, \mathtt{Sen}_b^{pres}, \mathtt{Mod}^{pres}, \models^{pres})$ by applying Boolean connectives and quantification over $\mathtt{Q}^{pres}$. $\mathtt{I}^{pres}$ is semantically closed under Boolean connectives, as $\mathtt{I}$ is semantically closed under Boolean connectives. We show that $\mathtt{I}^{pres} \sim \mathtt{I}_u^{pres}$:

1	let $(\Sigma, E) \in |\mathtt{Sig}^{pres}|$ and $\rho \in \mathtt{Sen}(\Sigma)$	
2	$\rho \models_\Sigma \gamma$ and $\gamma \models_\Sigma \rho$ for some $\gamma \in \mathtt{Sen}_u(\Sigma)$	since $\mathtt{I} \sim \mathtt{I}_u$
3	$\rho \models^{pres}_{(\Sigma,E)} \gamma$ and $\gamma \models^{pres}_{(\Sigma,E)} \rho$ for some $\gamma \in \mathtt{Sen}_u(\Sigma)$	by the definition of $\models^{pres}$
4	$\mathtt{I}^{pres} \sim \mathtt{I}_u^{pres}$	since $(\Sigma, E) \in |\mathtt{Sig}^{pres}|$ and $\rho \in \mathtt{Sen}(\Sigma)$ were arbitrarily chosen

Hence, $\mathtt{I}^{pres}$ is syntactically structured. □

A direct consequence of Lemma 17 and Lemma 20 is the following corollary.

Corollary 21. $\mathsf{FOL}^{pres} \sim \mathsf{FOL}_u^{pres}$, $\mathsf{PA}^{pres} \sim \mathsf{PA}_u^{pres}$ *and* $\mathsf{HNK}^{pres} \sim \mathsf{HNK}_u^{pres}$.

Another important class of institutions that fall under the framework developed in the present contribution are the institutions obtained by restricting the semantics of some standard logical system.

Lemma 22. *Let* $\mathtt{I} = (\mathtt{Sig}, \mathtt{Sen}, \mathtt{Mod}, \models)$ *and* $\mathtt{I}' = (\mathtt{Sig}, \mathtt{Sen}, \mathtt{Mod}', \models)$ *be two institutions such that* $\mathtt{Mod}'$ *is a subfunctor of* $\mathtt{Mod}$. *If* $\mathtt{I}$ *is syntactically structured then* $\mathtt{I}'$ *is syntactically structured as well.*

Proof. Assume that $\mathtt{I}$ is syntactically structured with the following parameters: (a) the sentence functor is $\mathtt{Sen}_b$, and (b) the quantification space is $\mathtt{Q}$. We show that $\mathtt{I}$ is syntactically structured with the same parameters. Since $\mathtt{I}$ is semantically closed under

Boolean connectives and quantification over $\mathtt{Q}$, $\mathtt{I}'$ is semantically closed under Boolean connectives and quantification over $\mathtt{Q}$. Let $\mathtt{I}_u = (\mathtt{Sig}, \mathtt{Sen}_u, \mathtt{Mod}, \models)$ be as described in Definition 18, and $\mathtt{I}'_u = (\mathtt{Sig}, \mathtt{Sen}_u, \mathtt{Mod}', \models)$. Then:

1	let $\Sigma \in \lvert\mathtt{Sig}\rvert$ and $\rho \in \mathtt{Sen}(\Sigma)$	
2	$\lvert\mathtt{Mod}^{pres}(\Sigma,\rho)\rvert = \lvert\mathtt{Mod}^{pres}(\Sigma,\gamma)\rvert$ for some $\gamma \in \mathtt{Sen}_u(\Sigma)$	since $\mathtt{I} \sim \mathtt{I}_u$
3	$\lvert\mathtt{Mod}'^{pres}(\Sigma,\rho)\rvert = \lvert\mathtt{Mod}'(\Sigma)\rvert \cap \lvert\mathtt{Mod}^{pres}(\Sigma,\rho)\rvert$	by the definition of $\mathtt{Mod}'^{pres}$
4	$\lvert\mathtt{Mod}'^{pres}(\Sigma,\gamma)\rvert = \lvert\mathtt{Mod}'(\Sigma)\rvert \cap \lvert\mathtt{Mod}^{pres}(\Sigma,\gamma)\rvert$	by the definition of $\mathtt{Mod}'^{pres}$
5	$\lvert\mathtt{Mod}'^{pres}(\Sigma,\rho)\rvert = \lvert\mathtt{Mod}'^{pres}(\Sigma,\gamma)\rvert$	from 2, 3 and 4
6	$\mathtt{I}' \sim \mathtt{I}'_u$	since Σ and ρ were arbitrarily chosen

Hence, $\mathtt{I}'$ is syntactically structured. □

A direct consequence of Lemma 17 and Lemma 22 is the following corollary.

Corollary 23. *FOL_f and HOL are syntactically structured.*

Proof. By noting that FOL_f are HOL are obtained by restricting the semantics of FOL and HNK, respectively. □

Ehrenfeucht-Fraïssé games in institutions. We now define an abstract equivalence between models of a syntactically structured institution, based on Ehrenfeucht-Fraïssé games.

Definition 24 (Trees)**.** *Given a syntactically structured institution $\mathtt{I}$ as described in Definition 18, a $\mathtt{Q}$-tree $\mathtt{tr}$ is inductively defined as follows:*

1. *any signature Σ is a $\mathtt{Q}$-tree with the root Σ and height 0;*
2. *if λ is a nonzero cardinal, $\{\chi_i \colon \Sigma \to \Sigma_i\}_{i<\lambda}$ is a family of signature morphisms from $\mathtt{Q}$ and $\{\mathtt{tr}_i\}_{i<\lambda}$ is a family of $\mathtt{Q}$-trees of height at most k such that*
   (a) the root of $\mathtt{tr}_i$ is Σ_i for all $i < \lambda$, and
   (b) the height of $\mathtt{tr}_j$ is k for some $j < \lambda$,

   then $\Sigma\{\overset{\chi_i}{\multimap} \mathtt{tr}_i\}_{i<\lambda}$ is a $\mathtt{Q}$-tree with root Σ and height $k+1$.

The definition above formalizes trees of finite height and possibly infinitely branched with nodes signatures and edges signature morphisms used for quantification.

Definition 25 (Game equivalence)**.** *Let $\mathtt{I}$ be a syntactically structured institution as described in Definition 18. The* game equivalence *between two models $\mathfrak{A}$ and $\mathfrak{B}$ over the same signature Σ is defined via $\mathtt{Q}$-trees $\mathtt{tr}$ with the root Σ:*

[$\mathtt{height}(\mathtt{tr}) = 0$] *$\mathfrak{A} \approx_{\mathtt{tr}} \mathfrak{B}$ iff the following property holds:*

$\mathfrak{A} \models \rho$ *iff* $\mathfrak{B} \models \rho$*, for all* $\rho \in \mathtt{Sen}_b(\Sigma)$.

[$\mathtt{height}(\mathtt{tr}) > 0$] *Let $\mathtt{tr} = \Sigma\{\overset{\chi_i}{\multimap} \mathtt{tr}_i\}_{i<\lambda}$ and assume that the relation $\mathfrak{A}_i \approx_{\mathtt{tr}_i} \mathfrak{B}_i$ has defined for all $i < \lambda$ and any Σ_i-models $\mathfrak{A}_i$ and $\mathfrak{B}_i$, where $\Sigma_i = \mathtt{root}(\mathtt{tr}_i)$. Then $\mathfrak{A} \approx_{\mathtt{tr}} \mathfrak{B}$ iff the following properties hold:*

1. *for any $i < \lambda$ and all χ_i-expansions $\mathfrak{A}_i$ of $\mathfrak{A}$ there exists a χ_i-expansion $\mathfrak{B}_i$ of $\mathfrak{B}$ such that $\mathfrak{A}_i \approx_{\mathtt{tr}_i} \mathfrak{B}_i$, and*
2. *for any $i < \lambda$ and all χ_i-expansions $\mathfrak{B}_i$ of $\mathfrak{B}$ there exists a χ_i-expansion $\mathfrak{A}_i$ of $\mathfrak{A}$ such that $\mathfrak{A}_i \approx_{\mathtt{tr}_i} \mathfrak{B}_i$.*

The definition of game equivalence is stated, of course, with Ehrenfeucht-Fraïssé games in mind. In the abstract institutional setting, the game $E_{\mathtt{tr}}(\mathfrak{A}, \mathfrak{B})$ is parameterized by two models $\mathfrak{A}$ and $\mathfrak{B}$ and a $\mathtt{Q}$-tree $\mathtt{tr}$. We think of the models $\mathfrak{A}$ and $\mathfrak{B}$ as game pieces, and of the tree as a gameboard. The play of the game proceeds in stages, corresponding to the levels of $\mathtt{tr}$; at each stage the play climbs one level up the tree. The image here is that the pieces (models) move on a board consisting of signature morphisms: each move along a morphism corresponds to picking an element. This image is in stark contrast to the classical image where models themselves are the board, and moves are choices of elements. Although either image is just a visualization, we claim that our image agrees better with the inner workings of institutions. After all, "picking an element" must be rendered as a signature morphism (nothing else is there to play that role); a signature morphism is an arrow, and each move consists in a piece moving along an arrow.

The play starts at the root, labelled by Σ. At the initial (0th) stage of the play, the universal quantifier player ($\forall$belard, as tradition has it) picks the structure $\mathfrak{A}$ or $\mathfrak{B}$ and if $\mathfrak{A}$ and $\mathfrak{B}$ do not satisfy precisely the same sentences from $\mathtt{Sen}_b(\Sigma)$ the existential quantifier player ($\exists$loise, of course) loses immediately. Otherwise the play continues, and at the end of stage j, say, it has reached a node labelled by a signature Σ' at level j with some expansions $\mathfrak{A}'$ and $\mathfrak{B}'$ of $\mathfrak{A}$ and $\mathfrak{B}$, such that $\mathfrak{A}'$ and $\mathfrak{B}'$ satisfy precisely the same sentences from $\mathtt{Sen}_b(\Sigma')$. Now, if Σ' is a leaf node, $\forall$belard loses. Otherwise, he picks a signature morphism χ_i labelling an upward edge of Σ' and a χ_i-expansion $\mathfrak{A}'_i$ of $\mathfrak{A}'$ (or a χ_i-expansion $\mathfrak{B}'_i$ of $\mathfrak{B}'$; we will stick to $\mathfrak{A}'$ for brevity). $\exists$loise has to respond with a χ_i-expansion $\mathfrak{B}'_i$ of $\mathfrak{B}'$ in such a way that the expansions $\mathfrak{A}'_i$ and $\mathfrak{B}'_i$ satisfy precisely the same sentences from $\mathtt{Sen}_b(\Sigma')$. If she cannot do it, she loses; if she can, the play climbs up the edge χ_i and continues at the node one level higher.

As usual, we say that $\exists$loise has a winning strategy in $E_{\mathtt{tr}}(\mathfrak{A}, \mathfrak{B})$, if she can win regardless of the moves $\forall$belard makes. We should remark here that our abstract definition of Ehrenfeucht-Fraïssé games differs from what the reader may expect in that the $\mathtt{Q}$-tree for the game is given beforehand, generally limiting $\forall$belard's choices. We do not do it to give female players an edge, but because we wish to apply the games to logics with infinite signatures, for which certain restrictions of the possible moves are necessary in order to define game sentences (to come shortly).

Definition 26 (Perfect trees)**.** *A perfect λ-ary tree, where λ is a cardinal, is a tree of finite height such that (a) every node has λ descendants, and (b) each leaf node is at the same height.*

Since every tree of finite height is a subtree of a perfect tree, it suffices to consider only perfect $\mathtt{Q}$-trees. The lemma below makes it explicit.

Lemma 27. *$\exists$loise has a winning strategy for all games $E_{\mathtt{tr}}(\mathfrak{A}, \mathfrak{B})$ with arbitrary $\mathtt{Q}^{\mathtt{I}}$-trees $\mathtt{tr}$ if and only if she has a winning strategy for all games $E_{\mathtt{tr}}(\mathfrak{A}, \mathfrak{B})$ with perfect $\mathtt{Q}$-trees $\mathtt{tr}$.*

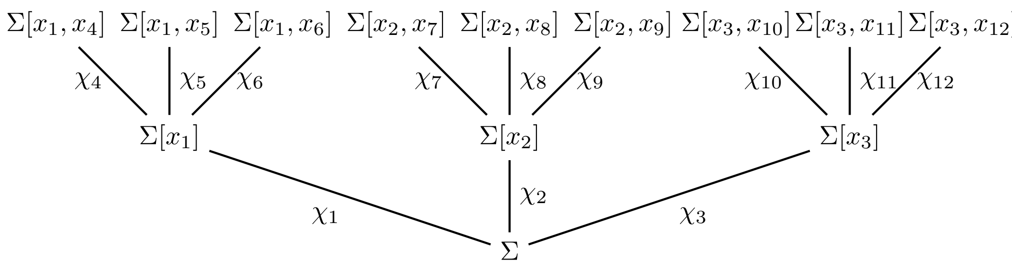

Figure 2: Perfect ternary tree of height 2

Ehrenfeucht-Fraïssé games in first-order logic. We will now instantiate our abstract description of Ehrenfeucht-Fraïssé games in first-order logic. It will turn out that in the single-sorted case we get precisely the usual Ehrenfeucht-Fraïssé games, whereas in many-sorted case we obtain a notion slightly different from the usual one.

Definition 28 (First-order gameboard trees)**.** *Let Σ be a* FOL *signature with a (possibly infinite) set S of sorts. A* gameboard $\mathtt{Q}^{\mathsf{FOL}}$-tree $\mathtt{tr}$ *with the root Σ is a perfect λ-ary tree such that (a) $\lambda \leq card(S)$, and (b) for each internal node Σ' the upward edges of Σ' are signature morphisms of the form $\chi_i \colon \Sigma' \hookrightarrow \Sigma'[x_i]$, for all $i < \lambda$, such that for any $i \neq j$, x_i and x_j are variables of different sorts.*

Let $\Sigma = (S, F, P)$ be a FOL signature such that $S = \{s_1, s_2, s_3\}$. A perfect ternary tree of height 2 is depicted in Figure 2. The labels $\chi_1, \ldots, \chi_{12}$ are signature extensions with one variable; the variables x_1, x_2, x_3 have sorts s_1, s_2, s_3, respectively; similarly, the variables x_4, x_5, x_6 have sorts s_1, s_2, s_3, respectively.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be two first-order structures over the same signature $\Sigma = (S, F, P)$. The *unnested Ehrenfeucht-Fraïssé game* $E_k(\mathfrak{A}, \mathfrak{B})$ of length $k \in \mathbb{N}$, is played in k steps, as follows. At the jth step of a play, $\forall$belard takes one of the structures $\mathfrak{A}$ or $\mathfrak{B}$, say $\mathfrak{A}$, and chooses an element a of any sort s of this structure. Then $\exists$loise chooses an element of the same sort from $\mathfrak{B}$. At the jth step of the play, both players have chosen sequences of elements $a_1 \ldots a_j$ and $b_1 \ldots b_j$ such that a_i and b_i have the same sorts s_i, for all $i \in \{1, \ldots, j\}$. In the argot of institutions, this amounts to choosing j variables $x_1, \ldots, x_j$ of sorts $s_1, \ldots, s_j$, respectively, and the expansions $(\mathfrak{A}, a_1, \ldots, a_j)$ and $(\mathfrak{B}, b_1, \ldots, b_j)$ of $\mathfrak{A}$ and $\mathfrak{B}$ to the signature $\Sigma[x_1, \ldots, x_j]$. Proceeding this way, at the end of the play, sequences $\overline{a} = a_1 \ldots a_k$ and $\overline{b} = b_1 \ldots b_k$ have been chosen. The play with these choices is a win for $\exists$loise, if $(\mathfrak{A}, \overline{a}) \approx_{\Sigma[x_1, x_2 \ldots, x_k]} (\mathfrak{B}, \overline{b})$, that is $(\mathfrak{A}, \overline{a})$ and $(\mathfrak{B}, \overline{b})$ satisfy the same unnested atomic sentences.

What we have just described is an instance of the definition of Ehrenfeucht-Fraïssé games in institutions, which corresponds to the classical definition of Ehrenfeucht-Fraïssé games in first-order logic. In other words, $E_k(\mathfrak{A}, \mathfrak{B})$ denotes the game $E_{\mathtt{tr}}(\mathfrak{A}, \mathfrak{B})$, where $\mathtt{tr}$ is a gameboard tree of height k with branching λ, where $\lambda = card(S)$. The reader can consult [26] for (much) more on games for elementary equivalence in single-sorted first-order logic. In many-sorted case (if you treat sorts seriously, as it is done in most computer science applications), our definition of the game differs from the usual one. To see where the difference lies, here is a simple example.

Example 29. *Let $k \geq 2$ be a natural number. Let a first-order signature $\Sigma = (S, F, P)$ be: (a) $S = \{s_1, \ldots, s_k, s'_k\}$, (b) $F = \emptyset$, and (c) $P = \{(\pi\colon s_1 \ldots s_k), (\varpi\colon s_1 \ldots s_{k-1}s'_k)\}$. Next, we define two Σ-models $\mathfrak{A}$ and $\mathfrak{B}$ putting:*

(a) $\mathfrak{A}_{s_1} = \{a_1, c_1\}$, $\mathfrak{A}_{s_2} = \{a_2\}$, $\ldots$, $\mathfrak{A}_{s_k} = \{a_k\}$, $\mathfrak{A}_{s'_k} = \{a'_k\}$ *and* $\pi^{\mathfrak{A}} = \{(a_1, \ldots, a_k)\}, \varpi^{\mathfrak{A}} = \{(a_1, \ldots, a_{k-1}, a'_k)\}$.

(b) $\mathfrak{B}_{s_1} = \{b_1, d_1\}$, $\mathfrak{B}_{s_2} = \{b_2\}$ $\ldots$, $\mathfrak{B}_{s_k} = \{b_k\}$, $\mathfrak{B}_{s'_k} = \{b'_k\}$ *and* $\pi^{\mathfrak{B}} = \{(d_1, b_2, \ldots, b_k)\}, \varpi^{\mathfrak{B}} = \{(b_1, \ldots, b_{k-1}, b'_k)\}$.

Lemma 30. *If $\mathfrak{A}$ and $\mathfrak{B}$ are the models defined in Example 29 then $\mathfrak{A} \approx_c \mathfrak{B}$ for all chains of length at most k, but $\mathfrak{A} \not\approx_{\mathtt{tr}} \mathfrak{B}$ for a perfect gameboard tree of height k.*

Proof. For the sake of simplicity, we assume that $k = 2$, as the generalization to any natural number k is straightforward.

Firstly, we show that $\mathfrak{A} \not\approx_{\mathtt{tr}} \mathfrak{B}$, where $\mathtt{tr}$ is the perfect 3-ary tree of height 2 depicted in Figure 2 such that (a) the leftmost variable has the sort s_1, (b) the variable in the middle has the sort s_2, and (c) the rightmost variable has the sort s'_2. Now, suppose ∀belard chooses a_1 then ∃loise can chose either b_1 or d_1:

1. if ∃loise chooses b_1 then ∀belard can choose a_2. In this case, the only possible choice for ∃loise is b_2. Hence, $\mathfrak{A} \models \pi(a_1, a_2)$ but $\mathfrak{B} \not\models \pi(b_1, b_2)$
2. if ∃loise chooses d_1 then ∀belard can choose a'_2. In this case, the only possible choice for ∃loise is b'_2. Hence, $\mathfrak{A} \models \varpi(a_1, a'_2)$ but $\mathfrak{B} \not\models \varpi(d_1, b'_2)$.

Secondly, we show that $\mathfrak{A} \approx_c \mathfrak{B}$ for any chain c of length 2. The interesting cases are when the variables added are of different sorts. We enumerate the non-trivial cases below:

1. $c \coloneqq \Sigma \hookrightarrow \Sigma[x_1] \hookrightarrow \Sigma[x_1, x_2]$, where x_1 is of sort s_1 and x_2 is of sort s_2. All unnested atoms over $\Sigma[x_1, x_2]$ are $x_1 = x_1$, $x_2 = x_2$ and $\pi(x_1, x_2)$. The identities $x_1 = x_1$ and $x_2 = x_2$ are not relevant as they are satisfied by all models. There are only two expansions of $\mathfrak{A}$, $(\mathfrak{A}, a_1, a_2)$ and $(\mathfrak{A}, c_1, a_2)$, and two expansions of $\mathfrak{B}$, $(\mathfrak{B}, b_1, b_2)$ and $(\mathfrak{B}, d_1, b_2)$. The game is won by ∃loise, as all choices of ∀belard lead to the following expansions:
   (a) $(\mathfrak{A}, a_1, a_2)$ which can be matched by $(\mathfrak{B}, d_1, b_2)$, that is $\mathfrak{A} \models \pi(a_1, a_2)$ and $\mathfrak{B} \models \pi(d_1, b_2)$,
   (b) $(\mathfrak{A}, c_1, a_2)$ which can be matched by $(\mathfrak{B}, b_1, b_2)$, that is $\mathfrak{A} \not\models \pi(c_1, a_2)$ and $\mathfrak{B} \not\models \pi(b_1, b_2)$, and
   (c) $(\mathfrak{B}, d_1, b_2)$ which can be matched by $(\mathfrak{A}, a_1, a_2)$,
   (d) $(\mathfrak{B}, b_1, b_2)$ which can be matched by $(\mathfrak{A}, c_1, a_2)$.
2. $c \coloneqq \Sigma \hookrightarrow \Sigma[x_1] \hookrightarrow \Sigma[x_1, x'_2]$, where x_1 is of sort s_1 and x'_2 is of sort s'_2. As in the case above, all unnested atoms over $\Sigma[x_1, x_2]$ are $x_1 = x_1$, $x'_2 = x'_2$ and $\varpi(x_1, x'_2)$. There are only two expansions of $\mathfrak{A}$, $(\mathfrak{A}, a_1, a'_2)$ and $(\mathfrak{A}, c_1, a'_2)$, and two expansions of $\mathfrak{B}$, $(\mathfrak{B}, b_1, b'_2)$ and $(\mathfrak{B}, d_1, b'_2)$. The game is won by ∃loise, as all ∀belard's choices lead to the following expansions:

(a) $(\mathfrak{A}, a_1, a'_2)$ which can be matched by $(\mathfrak{B}, d_1, b'_2)$, that is $\mathfrak{A} \models \varpi(a_1, a'_2)$ and $\mathfrak{B} \models \pi(d_1, b'_2)$,

(b) $(\mathfrak{A}, c_1, a'_2)$ which can be matched by $(\mathfrak{B}, b_1, b'_2)$, that is $\mathfrak{A} \not\models \pi(c_1, a'_2)$ and $\mathfrak{B} \not\models \pi(b_1, b'_2)$, and

(c) $(\mathfrak{B}, d_1, b'_2)$ which can be matched by $(\mathfrak{A}, a_1, a'_2)$,

(d) $(\mathfrak{B}, b_1, b'_2)$ which can be matched by $(\mathfrak{A}, c_1, a'_2)$. □

The reason for the difference is that some trees that are not perfect (such as chains) limit severely the choices ∀belard can make. Of course, knowing the tree beforehand ∃loise can adjust her strategy accordingly. Two more remarks are in order.

1. In single-sorted first-order logic, any gameboard tree $\mathtt{tr}$ is a chain, and $E_{\mathtt{tr}}(\mathfrak{A}, \mathfrak{B})$ always coincides with the usual game, so our formulation agrees with the classical formulation.
2. For two first-order structures over a signature with a set of sorts with at least two elements, the game which is equivalent to the "classically expected game" is the game over a perfect λ-ary tree, where λ is the cardinal of the set of sorts: at any point during the game, ∀belard can choose any element of any sort from the (many-sorted) universes of two models; in order to make this possible, any inner node of the gameboard tree needs a λ-ramification such that each child adds (to the parent signature) exactly one variable and the sorts of the added variables cover all sorts of the signature.

To sum up, we see the perfect gameboard trees as stepping stones from intuition to formalization. We could formalize the games differently, yet some generalizations are necessary in dealing with infinite signatures. We chose infinite branching, as we believe it maintains balance between intuition and formalization.

PA and HNK gameboard trees are defined analogously to first-order logic case. In HNK, there are infinitely branching gameboard trees for each signature with at least one sort, since the set of types over such signatures is infinite. Instantiating the abstract description of Ehrenfeucht-Fraïssé games to these concrete institutions, using only gameboard trees, is straightforward.

Finite trees and game sentences. With finitely branching Q-trees of height k, one can associate special sentences, usually called *game formulas*, characterizing winning strategies in Ehrenfeucht-Fraïssé games up to length k. For single sorted FOL it was done by Hintikka in [25]. Note that in single-sorted FOL, gameboard trees are actually finite chains.

As we mentioned a number of times already, in order to define game formulas, we need to restrict the signatures so that the set of unnested atomic sentences is finite.

Definition 31 (Finitary signatures)**.** *Let* I *be a syntactically structured institution as described in Definition 18. A signature* Σ *in* I *is* finitary *iff for all* Q*-chains* $c = \{\chi_i \colon \Sigma_i \to \Sigma_{i+1}\}_{i<k}$ *with* $k \in \mathbb{N}$ *and* $\Sigma_0 = \Sigma$*, the set* $\mathtt{Sen}_b(\Sigma_k)$ *is finite.*

In concrete examples, finitary signatures have the set of function and relation symbols finite, however, the set of sorts of finitary signatures may be infinite as the next lemma shows.

Lemma 32 (First-order finitary signatures)**.** *A first-order signature* $\Sigma = (S, F, P)$ *is finitary iff both sets* F *and* P *are finite.*

Proof. The forward implication is straightforward. We focus on the converse implication. It suffices to prove that for any finite set of variables X for Σ, the set of unnested atomic sentences $\mathtt{Sen}^{\mathsf{FOL}_0}(\Sigma)$ is finite. Since $F \cup X$ is finite, the set of terms of depth less than or equal to two is finite. Since P is finite, the set of unnested atomic sentences over $\Sigma[X]$ is finite too. □

In all examples of institutions given in this article, sentences are built from a finite number of symbols. Therefore, any sentence is included in a signature with a finite number of symbols.

Definition 33 (Game sentences). *Let* $\mathtt{I} = (\mathtt{Sig}, \mathtt{Sen}, \mathtt{Mod}, \models)$ *be a syntactically structured institution as described in Definition 18. For all finite* $\mathtt{Q}$*-trees* $\mathtt{tr}$ *such that* $\mathtt{root}(\mathtt{tr})$ *is finitary, the finite set of sentences* $\Theta_{\mathtt{tr}}$ *is defined as follows:*

[$\mathtt{height}(\mathtt{tr}) = 0$] *In this case,* $\mathtt{tr}$ *consists of a single root node, labelled by* Σ*, and*

$$\Theta_{\mathtt{tr}} = \{ \bigwedge_{e \in \mathtt{Sen}_b(\Sigma)} e^{f(e)} \mid f \colon \mathtt{Sen}_b(\Sigma) \to \{0,1\} \},$$

where e^1 *stands for* e *and* e^0 *stands for* $\neg e$ *for all* $e \in \mathtt{Sen}_b(\Sigma)$.

[$\mathtt{height}(\mathtt{tr}) > 0$] *Let* $\mathtt{tr} = \Sigma\{\overset{\chi_1}{\multimap} \mathtt{tr}_1, \ldots, \overset{\chi_n}{\multimap} \mathtt{tr}_n\}$ *be a tree and assume that the finite set of sentences* Θ_{tr_i} *has been defined for all* $i \in \{1, \ldots, n\}$*. Then:*

- $\Theta_{\mathtt{tr}} = \{\gamma_{E_1} \wedge \ldots \wedge \gamma_{E_n} \mid E_1 \subseteq \Theta_{\mathtt{tr}_1}, \ldots, E_n \subseteq \Theta_{\mathtt{tr}_n}\}$, *where*
- $\gamma_{E_i} = (\bigwedge_{e \in E_i} \exists \chi_i \cdot e) \wedge \; \forall \chi_i \cdot \bigvee E_i$ *for all* $i \in \{1, \ldots, n\}$.

The sentences in $\Theta_{\mathtt{tr}}$ will be referred to as *sentences in game-normal form*, or more briefly *game-normal sentences*.

Theorem 34 (Fraïssé-Hintikka Theorem). *Consider a syntactically structured institution* $\mathtt{I} = (\mathtt{Sig}, \mathtt{Sen}, \mathtt{Mod}, \models)$ *with the following parameters: (a)* $\mathtt{Sen}_b \colon \mathtt{Sig} \to \mathtt{Set}$ *is the sentence subfunctor which maps signatures to sets of 'unnested atomic' sentences, and (b)* $\mathtt{Q}$ *is the quantification space. Let* Σ *be a finitary signature in* $\mathtt{I}$.

T1) For each Σ*-model* $\mathfrak{A}$ *and all finite* $\mathtt{Q}$*-trees* $\mathtt{tr}$*, there exists a unique sentence* $\gamma_{(\mathfrak{A},\mathtt{tr})} \in \Theta_{\mathtt{tr}}$ *such that* $\mathfrak{A} \models \gamma_{(\mathfrak{A},\mathtt{tr})}$.

T2) $\mathfrak{A} \approx_{\mathtt{tr}} \mathfrak{B}$ *iff there exists a unique* $\rho \in \Theta_{\mathtt{tr}}$ *such that* $\mathfrak{A} \models \rho$ *and* $\mathfrak{B} \models \rho$*, for all* Σ*-models* $\mathfrak{A}, \mathfrak{B}$ *and finite* $\mathtt{Q}$*-trees* $\mathtt{tr}$ *with root* Σ.

T3) For all sentences $\rho \in \mathtt{Sen}(\Sigma)$*, there exist a finite chain of signature morphisms in* $\mathtt{Q}$ *and a subset* $\Gamma_\rho \subseteq \Theta_c$ *such that* $\rho \models\!\mid \bigvee \Gamma_\rho$.

Proof.

T1) We proceed by induction on the structure of $\mathtt{tr}$:

[$\mathtt{height}(\mathtt{tr}) = 0$] This case is straightforward.

[$\mathtt{height}(\mathtt{tr}) > 0$] Let $k = \mathtt{height}(\mathtt{tr})$, $\mathtt{tr} = \Sigma\{\overset{\chi_1}{\multimap} \mathtt{tr}_1, \ldots, \overset{\chi_n}{\multimap} \mathtt{tr}_n\}$, and $\Sigma_i = \mathtt{root}(\mathtt{tr}_i)$ for all $i \in \{1, \ldots, n\}$.

By the induction hypothesis, for each $i \in \{1, \ldots, n\}$ and any χ_i-expansion $\mathfrak{A}_i$ of $\mathfrak{A}$ there exists a unique sentence $\gamma_{(\mathfrak{A}_i, \mathtt{tr}_i)} \in \Theta_{\mathtt{tr}_i}$ such that $\mathfrak{A}_i \models \gamma_{(\mathfrak{A}_i, \mathtt{tr}_i)}$.

- Let $\Gamma_{(\mathfrak{A}, \mathtt{tr}, i)} \subseteq \Theta_{\mathtt{tr}_i}$ be the set of sentences consisting of all $\gamma_{(\mathfrak{A}_i, \mathtt{tr}_i)} \in \Theta_{\mathtt{tr}_i}$, where $\mathfrak{A}_i$ is some χ_i-expansion of $\mathfrak{A}$.
- Let $\gamma_{(\mathfrak{A}, \mathtt{tr})} = \bigwedge_{i=1}^{n} \{\gamma_{E_i} \mid E_i = \Gamma_{(\mathfrak{A}, \mathtt{tr}, i)}\}$.

It is straightforward to check that $\mathfrak{A} \models \gamma_{(\mathfrak{A}, \mathtt{tr})}$.

If $\mathfrak{A} \models \bigwedge_{i=1}^{n} \gamma_{E_i} \in \Theta_{\mathtt{tr}}$ then we show that $E_i = \Gamma_{(\mathfrak{A}, \mathtt{tr}, i)}$ for all $i \in \{1, \ldots, n\}$:

1	let $i \in \{1, \ldots, n\}$ and note that $\gamma_{E_i} = (\bigwedge_{e \in E_i} \exists \chi_i \cdot e) \wedge \forall \chi_i \cdot \bigvee E_i$	
2	$E_i \subseteq \Gamma_{(\mathfrak{A}, \mathtt{tr}, i)}$	since $\mathfrak{A} \models \gamma_{E_i}$, we have $\mathfrak{A} \models \bigwedge_{e \in E_i} \exists \chi_i \cdot e$
3	$\Gamma_{(\mathfrak{A}, \mathtt{tr}, i)} \subseteq E_i$	since $\mathfrak{A} \models \gamma_{E_i}$, we have $\mathfrak{A} \models \forall \chi_i \cdot \bigvee E_i$
4	$\Gamma_{(\mathfrak{A}, \mathtt{tr}, i)} = E_i$	from 2 and 3

T2) We proceed by induction on the structure of $\mathtt{tr}$.

[$\mathtt{height}(\mathtt{tr}) = 0$] This case is straightforward.

[$\mathtt{height}(\mathtt{tr}) > 0$] Let $k = \mathtt{height}(\mathtt{tr})$, $\mathtt{tr} = \Sigma\{\overset{\chi_1}{\multimap} \mathtt{tr}_1, \ldots, \overset{\chi_n}{\multimap} \mathtt{tr}_n\}$, and $\Sigma_i = \mathtt{root}(\mathtt{tr}_i)$ for all $i \in \{1, \ldots, n\}$.

For the forward implication, assume that $\mathfrak{A} \approx_{\mathtt{tr}} \mathfrak{B}$. We show that $\gamma_{(\mathfrak{A}, \mathtt{tr})} = \gamma_{(\mathfrak{B}, \mathtt{tr})}$, which is equivalent to $\Gamma_{(\mathfrak{A}, \mathtt{tr}, i)} = \Gamma_{(\mathfrak{B}, \mathtt{tr}, i)}$ for all $i \in \{1, \ldots, n\}$:

- $\Gamma_{(\mathfrak{A}, \mathtt{tr}, i)} \subseteq \Gamma_{(\mathfrak{B}, \mathtt{tr}, i)}$:

1	let $e \in \Gamma_{(\mathfrak{A}, \mathtt{tr}, i)}$	
2	$\mathfrak{A}_i \models e$ for some χ_i-expansion $\mathfrak{A}_i$ of $\mathfrak{A}$	by the definition of $\Gamma_{(\mathfrak{A}, \mathtt{tr}, i)}$
3	$\mathfrak{A}_i \approx_{\mathtt{tr}_i} \mathfrak{B}_i$ for some χ_i-expansion of $\mathfrak{B}$	by Definition 25
4	$\mathfrak{A}_i \models \rho$ and $\mathfrak{B}_i \models \rho$ for some unique $\rho \in \Theta_{\mathtt{tr}_i}$	by the induction hypothesis
5	$\rho = e$	by statement T1 above
6	$e \in \Gamma_{(\mathfrak{B}, \mathtt{tr}, i)}$	since $\mathfrak{B}_i \models e$ and $\mathfrak{B}_i$ is a χ_i-expansion of $\mathfrak{B}$

- $\Gamma_{(\mathfrak{B}, \mathtt{tr}, i)} \subseteq \Gamma_{(\mathfrak{A}, \mathtt{tr}, i)}$: similarly as above.

For the converse implication, assume that $\mathfrak{A} \models \rho$ and $\mathfrak{B} \models \rho$ for some $\rho \in \Theta_{\mathtt{tr}}$.

By Definition 33, we have $\rho = \bigwedge_{i=1}^{n} \gamma_{E_i}$, where $E_i \subseteq \Theta_{\mathtt{tr}_i}$ for all $i \in \{1, \ldots, n\}$.

- We show that for each $i \in \{1, \ldots, n\}$ and any χ_i-expansion $\mathfrak{A}_i$ of $\mathfrak{A}$ there exists a χ_i-expansion $\mathfrak{B}_i$ of $\mathfrak{B}$ such that $\mathfrak{A}_i \approx_{\mathtt{tr}_i} \mathfrak{B}_i$:

1	let $i \in \{1, \ldots, n\}$ and let $\mathfrak{A}_i$ be a χ_i-expansion of $\mathfrak{A}$	
2	$\mathfrak{A} \models \forall \chi_i \cdot \bigvee E_i$	since $\mathfrak{A} \models \gamma_{E_i}$
3	$\mathfrak{A}_i \models \bigvee E_i$	since $\mathfrak{A}_i$ is a χ_i-expansion of $\mathfrak{A}$

4	$\mathfrak{A}_i \models e$ for some $e \in E_i$	by the semantics of $\bigvee$
5	$\mathfrak{B} \models \bigwedge_{e' \in E_i} \exists \chi_i \cdot e'$	since $\mathfrak{B} \models \gamma_{E_i}$
6	$\mathfrak{B} \models \exists \chi_i \cdot e$	since $e \in E_i$
7	$\mathfrak{B}_i \models e$ for some χ_i-expansion $\mathfrak{B}_i$ of $\mathfrak{B}$	by the semantics of $\exists$
8	$\mathfrak{A}_i \approx_{\mathtt{tr}_i} \mathfrak{B}_i$	by the induction hypothesis, since $\mathfrak{A}_i \models e$ and $\mathfrak{B}_i \models e$

- Similarly, one can show that for each $i \in \{1, \ldots, n\}$ and any χ_i-expansion $\mathfrak{B}_i$ of $\mathfrak{B}$ there exists a χ_i-expansion $\mathfrak{A}_i$ of $\mathfrak{A}$ such that $\mathfrak{A}_i \approx_{\mathtt{tr}_i} \mathfrak{B}_i$

T3) Since $\mathtt{I}$ syntactically structured, it suffices to consider only sentences ρ obtained from the sentences in $\mathtt{I}_b$ by applying Boolean connectives and quantification over the signature morphisms in $\mathtt{Q}$. We proceed by induction on the structure of ρ:

[*For* $\rho \in \mathtt{Sen}_b(\Sigma)$] Let $\Gamma_\rho \subseteq \Theta_\Sigma$ be the subset of all sentences in Θ_Σ which contain ρ without negation. It is straightforward to show that $\rho \mathrel{|\!\!=\!\!|} \bigvee \Gamma_\rho$.

[*For* $\neg\rho$] By the induction hypothesis, $\rho \mathrel{|\!\!=\!\!|} \bigvee \Gamma_\rho$ for some $\mathtt{Q}$-chain c and subset $\Gamma_\rho \subseteq \Theta_c$. Let $\Gamma_{(\neg\rho)} = \Theta_c \setminus \Gamma_\rho$ be the complement of Γ_ρ in Θ_c. It is straightforward to show that $\neg\rho \mathrel{|\!\!=\!\!|} \bigvee \Gamma_{(\neg\rho)}$.

[*For* $\rho_1 \wedge \rho_2$] By the induction hypothesis, $\rho_i \mathrel{|\!\!=\!\!|} \bigvee \Gamma_{\rho_i}$ for some $\mathtt{Q}$-chain c and some subset $\Gamma_{\rho_i} \subseteq \Theta_c$. It is straightforward to show that $\rho_1 \wedge \rho_2 \mathrel{|\!\!=\!\!|} \bigvee (\Gamma_{\rho_1} \cap \Gamma_{\rho_2})$.

[*For* $\exists \chi \cdot \rho$, *where* $\chi\colon \Sigma \to \Sigma_1$] By the induction hypothesis, $\rho \mathrel{|\!\!=\!\!|} \bigvee \Gamma_\rho$ for some $\mathtt{Q}$-chain $c_1 = \{\chi_i\colon \Sigma_i \to \Sigma_{i+1}\}_{1 \leq i < k}$ and subset $\Gamma_\rho \subseteq \Theta_{c_1}$. We define the $\mathtt{Q}$-chain $c = \{\chi_i\colon \Sigma_i \to \Sigma_{i+1}\}_{i<k}$, where $\Sigma_0 = \Sigma$ and $\chi_0 = \chi$.
Let $\Gamma_{(\exists \chi \cdot \rho)} = \{\gamma_E \mid E \subseteq \Theta_{c_1} \text{ and } E \cap \Gamma_\rho \neq \emptyset\}$.

- We show that $\exists \chi \cdot \rho \models \bigvee \Gamma_{(\exists \chi \cdot \rho)}$:

1	assume that $\mathfrak{A} \models \exists \chi \cdot \rho$	
2	$\mathfrak{A}_1 \models \rho$ for some χ-exp. $\mathfrak{A}_1$ of $\mathfrak{A}$	by the semantics of $\exists$
3	$\mathfrak{A}_1 \models \bigvee \Gamma_\rho$	since $\rho \mathrel{|\!\!=\!\!|} \bigvee \Gamma_\rho$
4	$\mathfrak{A}_1 \models e$ for some $e \in \Gamma_\rho$	by the semantics of $\bigvee$
5	$e \in \Gamma_{(\mathfrak{A},c)} \cap \Gamma_\rho$	since $\Gamma_{(\mathfrak{A},c)}$ consists of all sentences in Θ_{c_1} satisfied by some expansion of $\mathfrak{A}$
6	$\gamma_{(\mathfrak{A},c)} = \gamma_{\Gamma_{(\mathfrak{A},c)}} \in \Gamma_{(\exists \chi \cdot \rho)}$	by the definition of $\Gamma_{(\exists \chi \cdot \rho)}$
7	$\mathfrak{A} \models \bigvee \Gamma_{(\exists \chi \cdot \rho)}$	since $\mathfrak{A} \models \gamma_{(\mathfrak{A},c)}$ and $\gamma_{(\mathfrak{A},c)} \in \Gamma_{(\exists \chi \cdot \rho)}$

- We show that $\bigvee \Gamma_{(\exists \chi \cdot \rho)} \models \exists \chi \cdot \rho$:

1	assume that $\mathfrak{A} \models \bigvee \Gamma_{(\exists \chi \cdot \rho)}$	
2	$\mathfrak{A} \models \gamma_E$ for some $E \subseteq \Theta_{c_1}$ such that $E \cap \Gamma_\rho \neq \emptyset$	by the definition of $\Gamma_{(\exists \chi \cdot \rho)}$
3	$\gamma_E = \gamma_{(\mathfrak{A},c)}$	by statement T2 above
4	$\Gamma_{(\mathfrak{A},c)} \cap \Gamma_\rho \neq \emptyset$	since $\Gamma_{(\mathfrak{A},c)} = E$ and $E \cap \Gamma_\rho \neq \emptyset$

5	$\mathfrak{A}_1 \models e$ for some χ-expansion $\mathfrak{A}_1$ of $\mathfrak{A}$ and some $e \in \Gamma_{(\mathfrak{A},c)} \cap \Gamma_\rho$	since $\Gamma_{(\mathfrak{A},c)}$ consists of all sentences in Θ_{c_1} satisfied by some χ-expansion of $\mathfrak{A}$
6	$\mathfrak{A}_1 \models \bigvee \Gamma_\rho$	since $e \in \Gamma_\rho$
7	$\mathfrak{A}_1 \models \rho$	since $\bigvee \Gamma_\rho \models\!\mid \rho$
8	$\mathfrak{A} \models \exists \chi \cdot \rho$	since $\mathfrak{A}_1$ is a χ-expansion of $\mathfrak{A}$

[*For* $\forall \chi \cdot \rho$, *where* $\chi\colon \Sigma \to \Sigma_1$] By the induction hypothesis, $\rho \models\!\mid \bigvee \Gamma_\rho$ for some $\mathtt{Q}$-chain $c_1 = \{\chi_i\colon \Sigma_i \to \Sigma_{i+1}\}_{1 \le i < k}$ and subset $\Gamma_\rho \subseteq \Theta_{c_1}$. We define the $\mathtt{Q}$-chain $c = \{\chi_i\colon \Sigma_i \to \Sigma_{i+1}\}_{i<k}$, where $\Sigma_0 = \Sigma$ and $\chi_0 = \chi$.

Let $\Gamma_{(\forall \chi \cdot \rho)} = \{\gamma_E \mid E \subseteq \Gamma_\rho\}$.

- We show that $\forall \chi \cdot \rho \models \bigvee \Gamma_{(\forall \chi \cdot \rho)}$:

1	assume that $\mathfrak{A} \models \forall \chi \cdot \rho$	
2	$\Gamma_{(\mathfrak{A},c)} \subseteq \Gamma_\rho$	by the following proof steps
2.1	let $e \in \Gamma_{(\mathfrak{A},c)}$	
2.2	$e = \gamma_{(\mathfrak{A}_1,c_1)}$ for some χ-expansion $\mathfrak{A}_1$ of $\mathfrak{A}$	by the definition of $\Gamma_{(\mathfrak{A},c)}$
2.3	$\mathfrak{A}_1 \models \rho$	since $\mathfrak{A} \models \forall \chi \cdot \rho$
2.4	$\mathfrak{A}_1 \models \bigvee \Gamma_\rho$	since $\rho \models\!\mid \bigvee \Gamma_\rho$
2.5	$\mathfrak{A}_1 \models e_1$ for some $e_1 \in \Gamma_\rho$	by the semantics of $\bigvee$
2.6	$e_1 = \gamma_{(\mathfrak{A}_1,c_1)}$	as $e_1 \in \Theta_{c_1}$ and $\mathfrak{A}_1$ satisfies a unique sentence in Θ_{c_1}
2.7	$e = \gamma_{(\mathfrak{A}_1,c_1)} = e_1 \in \Gamma_\rho$	from 2.2, 2.5 and 2.6
3	$\gamma_{(\mathfrak{A},c)} = \gamma_{\Gamma_{(\mathfrak{A},c)}} \in \Gamma_{(\forall \chi \cdot \rho)}$	by the definition of $\Gamma_{(\forall \chi \cdot \rho)}$
4	$\mathfrak{A} \models \bigvee \Gamma_{(\forall \chi \cdot \rho)}$	since $\mathfrak{A} \models \gamma_{(\mathfrak{A},c)}$ and $\gamma_{(\mathfrak{A},c)} \in \Gamma_{(\forall \chi \cdot \rho)}$

- We show that $\bigvee \Gamma_{(\forall \chi \cdot \rho)} \models \forall \chi \cdot \rho$:

1	assume that $\mathfrak{A} \models \bigvee \Gamma_{(\forall \chi \cdot \rho)}$	
2	$\mathfrak{A} \models \gamma_E$ for some $E \subseteq \Gamma_\rho$	by the definition of $\Gamma_{(\forall \chi \cdot \rho)}$
3	$\mathfrak{A} \models \forall \chi \cdot \bigvee E$	since $\gamma_E = (\bigwedge_{e \in E} \exists \chi \cdot e) \wedge \forall \chi \cdot \bigvee E$
4	$\mathfrak{A} \models \forall \chi \cdot \bigvee \Gamma_\rho$	since $E \subseteq \Gamma_\rho$
5	$\mathfrak{A} \models \forall \chi \cdot \rho$	since $\rho \models\!\mid \bigvee \Gamma_\rho$

□

Notice that Fraïssé-Hintikka Theorem is applicable to any syntactically structured institution $\mathtt{I}$. Therefore, it has a very wide range of applications. The following characterization of elementary equivalence is a corollary of the Fraïssé-Hintikka Theorem.

Corollary 35 (Fraïssé-Hintikka Theorem concretely)**.** *Let* $\mathtt{I}$ *be one of the institutions* FOL, HNK, PA, FOL^{pres}, HNK^{pres}, PA^{pres}, FOL_f *or* HOL. *Let* $\mathfrak{A}$ *and* $\mathfrak{B}$ *be two models over a signature* Σ *in* $\mathtt{I}$. *The following are equivalent:*

C1) $\mathfrak{A} \equiv \mathfrak{B}$;

C2) $\exists$*loise has a winning strategy for the game* $E_{\mathtt{tr}}(\mathfrak{A}\restriction_{\Sigma_f}, \mathfrak{B}\restriction_{\Sigma_f})$, *for all finite subsignatures* Σ_f *of* Σ *and all finite gameboard trees* $\mathtt{tr}$ *with the root* Σ_f.

C3) $\mathfrak{A}{\upharpoonright}_{\Sigma_f} \approx_c \mathfrak{B}{\upharpoonright}_{\Sigma_f}$ *for all finite sub-signatures* Σ_f *of* Σ, *and all finite chains* c *with the root* Σ_f.

Proof. By Lemma 17, FOL, PA and HNK are syntactically structured. By Corollary 21, FOL^{pres}, HNK^{pres} are PA^{pres} are syntactically structured. By Corollary 23, FOL_f and HOL are syntactically structured. It follows that Theorem 34 is applicable to the institution $\mathtt{I}$.

[C1 $\Rightarrow$ C2] Let Σ_f be a finite sub-signature of Σ and $\mathtt{tr}$ a finite gameboard tree with the root Σ_f. Since $\mathfrak{A} \equiv \mathfrak{B}$, we have $\mathfrak{A}{\upharpoonright}_{\Sigma_f} \equiv \mathfrak{B}{\upharpoonright}_{\Sigma_f}$. By Theorem 34 (T1), it follows that $\gamma_{(\mathfrak{A},\mathtt{tr})} = \gamma_{(\mathfrak{B},\mathtt{tr})} \in \Theta_{\mathtt{tr}}$. By Theorem 34 (T2), $\mathfrak{A}{\upharpoonright}_{\Sigma_f} \approx_{\mathtt{tr}} \mathfrak{B}{\upharpoonright}_{\Sigma_f}$. Therefore, ∃loise has a winning strategy for the game $E_{\mathtt{tr}}(\mathfrak{A}{\upharpoonright}_{\Sigma_f}, \mathfrak{B}{\upharpoonright}_{\Sigma_f})$.

[C2 $\Rightarrow$ C3] Every chain is a tree.

[C3 $\Rightarrow$ C1] We show that $\mathfrak{A} \models \rho$ implies $\mathfrak{B} \models \rho$ for all $\rho \in \mathtt{Sen}(\Sigma)$:

1	assume that $\mathfrak{A} \models \rho$	
2	$\rho = \chi(\rho_f)$ for some finite signature Σ_f, Σ_f-sentence ρ_f and inclusion $\chi\colon \Sigma_f \hookrightarrow \Sigma$	ρ is composed of a finite number of symbols
3	$\mathfrak{A}{\upharpoonright}_{\chi} \models \rho_f$	by the satisfaction condition
4	$\rho_f \models\!\mid \bigvee E$ for some $\mathtt{Q}$-chain c and subset $E \subseteq \Theta_c$.	by Theorem 34 (T3)
5	$\mathfrak{A}{\upharpoonright}_{\chi} \models \bigvee E$	by 3 and 4
6	$\mathfrak{B}{\upharpoonright}_{\chi} \models \bigvee E$	by Theorem 34 (T1 & T2), since $\mathfrak{A}{\upharpoonright}_{\chi} \approx_c \mathfrak{B}{\upharpoonright}_{\chi}$
7	$\mathfrak{B}{\upharpoonright}_{\chi} \models \rho_f$	since $\rho_f \models\!\mid \bigvee E$
8	$\mathfrak{B} \models \rho$	by the satisfaction condition

Similarly, one can assume $\mathfrak{A} \models \rho$ and then show $\mathfrak{B} \models \rho$ for all $\rho \in \mathtt{Sen}(\Sigma)$. □

An analogous corollary holds for FOL_f^{pres} and HOL^{pres}, too. Subsystems of second-order arithmetic, mentioned in Example 8, are instances of FOL^{pres}.

5 Conclusions

Perhaps the classical hard-liner who insists that higher-order logic with Henkin semantics is nothing but a variant of first-order logic, still wonders what we have achieved in this article, apart from fancy terminology and notation. We believe the gain is manifold.

First and foremost, we provided a general proof of Fraïssé-Hintikka Theorem in the framework of institutions applicable to a wide range of logical systems. In addition to the applications given above, Theorem 34 applies also to the class of constructor-based logics which are obtained from some base logic by restricting the semantics to models reachable by some constructor operators [2]. This is an immediate corollary of the fact that Lemma 22 holds for these. Constructor-based logics are used in the area of formal methods [23, 22].

Secondly, our proof is what institution theorists call "structured". Such a proof reveals bare connections between the relevant properties, stripped of accidental paraphernalia. For example, our proof does not employ the machinery of quantifier rank and normal forms. It also applies straightforwardly to cases with empty sorts, with which the classical proof would struggle.

Thirdly, our proof applies directly to arbitrary signatures. The classical proof does the job, too, but in a roundabout way (via finite reducts).

Finally, we believe immodestly that by elucidating the fine structure of games in the institutional framework we have made game methods available to institution theorists.

References

[1] E. Astesiano, M. Bidoit, H. Kirchner, B. Krieg-Brückner, P. D. Mosses, D. Sannella, and A. Tarlecki. CASL: the Common Algebraic Specification Language. *Theoretical Computer Science*, 286(2):153–196, 2002.

[2] M. Bidoit, R. Hennicker, and A. Kurz. Observational logic, constructor-based logic, and their duality. *Theor. Comput. Sci.*, 3(298), 2003.

[3] A. Church. A Formulation of the Simple Theory of Types. *J. Symb. Log.*, 5(2):56–68, 1940.

[4] M. Codescu and D. Găină. Birkhoff completeness in institutions. *Logica Universalis*, 2(2):277–309, 2008.

[5] R. Diaconescu. Institution-independent Ultraproducts. *Fundamenta Informaticæ*, 55(3-4):321–348, 2003.

[6] R. Diaconescu. An Institution-independent proof of Craig Interpolation Theorem. *Studia Logica*, 77(1):59–79, 2004.

[7] R. Diaconescu. Elementary Diagrams in Institutions. *J. Log. Comput.*, 14(5):651–674, 2004.

[8] R. Diaconescu. Proof systems for institutional logic. *Journal of Logic and Computation*, 16(3):339–357, 2006.

[9] R. Diaconescu. *Institution-independent Model Theory.* Studies in Universal Logic. Birkhäuser, Basel, 1 edition, 2008.

[10] R. Diaconescu. Quasi-varieties and initial semantics for hybridized institutions. *Journal of Logic and Computation*, 26(3):855–891, 2016.

[11] R. Diaconescu and M. Petria. Saturated models in institutions. *Arch. Math. Log.*, 49(6):693–723, 2010.

[12] J. Goguen and R. Burstall. Institutions: Abstract Model Theory for Specification and Programming. *Journal of the Association for Computing Machinery*, 39(1):95–146, January 1992.

[13] D. Găină. Interpolation in logics with constructors. *Theoretical Computer Science*, 474:46–59, 2013.

[14] D. Găină. Forcing, Downward Löwenheim-Skolem and Omitting Types Theorems, Institutionally. *Logica Universalis*, 8(3-4):469–498, 2014.

[15] D. Găină. Birkhoff style calculi for hybrid logics. *Formal Aspects of Computing*, 29(5):805–832, 2017.

[16] D. Găină. Downward Löwenheim-Skolem Theorem and interpolation in logics with constructors. *Journal of Logic and Computation*, 27(6):1717–1752, 2017.

[17] D. Găină. Forcing and Calculi for Hybrid Logics. *Journal of the Association for Computing Machinery*, 67(4):1–55, 2020.

[18] D. Găină and K. Futatsugi. Initial semantics in logics with constructors. *Journal of Logic and Computation*, 25(1):95–116, 2015.

[19] D. Găină and M. Petria. Completeness by forcing. *Journal of Logic and Computation*, 20(6):1165–1186, 2010.

[20] D. Găină and A. Popescu. An Institution-independent Generalization of Tarski's Elementary Chain Theorem. *Journal of Logic and Computation*, 16(6):713–735, 2006.

[21] D. Găină and A. Popescu. An Institution-Independent Proof of Robinson Consistency Theorem. *Studia Logica*, 85(1):41–73, 2007.

[22] D. Găină, I. Tutu, and A. Riesco. Specification and verification of invariant properties of transition systems. In *25th Asia-Pacific Software Engineering Conference, APSEC 2018, Nara, Japan, December 4-7, 2018*, pages 99–108. IEEE, 2018.

[23] D. Găină, M. Zhang, Y. Chiba, and Y. Arimoto. Constructor-based inductive theorem prover. In R. Heckel and S. Milius, editors, *Algebra and Coalgebra in Computer Science - 5th International Conference, CALCO 2013, Warsaw, Poland, September 3-6, 2013. Proceedings*, volume 8089 of *Lecture Notes in Computer Science*, pages 328–333. Springer, 2013.

[24] L. Henkin. Completeness in the theory of types. *J. Symb. Log.*, 15(2):81–91, 1950.

[25] K. J. J. Hintikka. Distributive normal forms in the calculus of predicates. *Acta Philos. Fenn.*, 6:71, 1953.

[26] W. Hodges. *Model theory*, volume 42 of *Encyclopedia of mathematics and its applications*. Cambridge University Press, 1993.

[27] M. Petria and R. Diaconescu. Abstract Beth definability in institutions. *Journal of Symbolic Logic*, 71(3):1002–1028, 2006.

[28] D. Sannella and A. Tarlecki, editors. *Recent Trends in Data Type Specification, 5th Workshop on Abstract Data Types*, volume 332 of *Lecture Notes in Computer Science*. Springer, 1987.

[29] S. G. Simpson. *Subsystems of second order arithmetic.* Perspectives in Logic. Cambridge University Press, Cambridge; Association for Symbolic Logic, Poughkeepsie, NY, second edition, 2009.

[30] A. Tarlecki. On the Existence of Free Models in Abstract Algebraic Institutions. *Theoretical Computer Science*, 37:269–304, 1985.

[31] A. Tarlecki. Bits and pieces of the theory of institutions. In K. Drosten, H. Ehrich, M. Gogolla, and U. W. Lipeck, editors, *Proceedings of the 4st Workshop on Abstract Data Type, 1986. University of Braunschweig, Germany*, 1986.

[32] A. Tarlecki. Quasi-varieties in Abstract Algebraic Institutions. *Journal of Computer and System Sciences*, 33(3):333–360, 1986.

[33] W. Thomas. On the Ehrenfeucht-Fraïssé game in theoretical computer science. In J. J. Gaudel M.C., editor, *TAPSOFT'93: Theory and Practice of Software Development. CAAP 1993*, volume 668 of *Lecture Notes in Computer Science*. Springer, Berlin, Heidelberg, 1993.