Many-Sorted First-Order Model Theory
lecture 6

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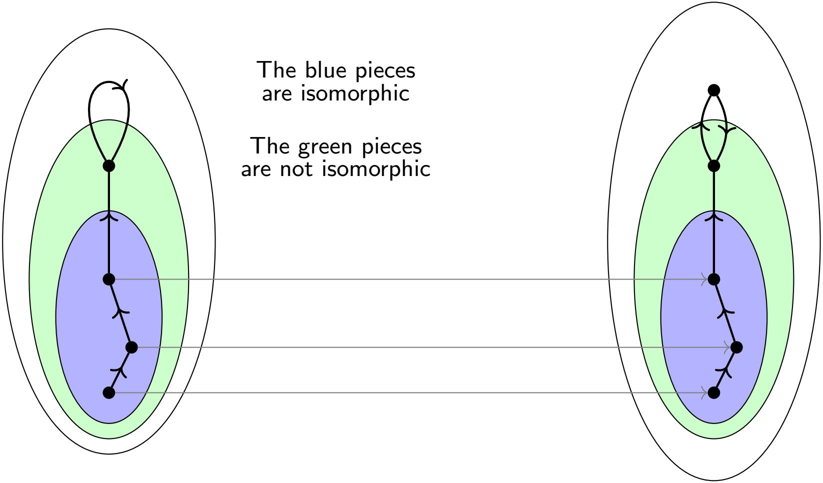
Comparing models piece by piece

Σ has only one unary function. Two Σ-models $\mathcal{A}$ and $\mathcal{B}$.



Unnested atomic sentences and partial isomorphisms

Shorthand notation for Definition 1. Let $\Sigma = (S, F, P)$ be a signature.

- $U = \{U_s\}_{s \in S'}$ and $V = \{V_s\}_{s \in S'}$ are S'-sorted sets, for some $S' \subseteq S$.
- $p = \{p_s \colon U_s \to V_s\}_{s \in S'}$ is an S'-sorted map.
- For $\pi \colon s_1, \ldots, s_k$, such that $\{s_1, \ldots, s_k\} \cap S' = \{s_{i_1}, \ldots, s_{i_m}\}$, the notation $\pi^{\mathcal{A}}|_U$ means $\pi^{\mathcal{A}} \cap U_{s_{i_1}} \times \cdots \times U_{s_{i_m}}$, and similarly for $\pi^{\mathcal{B}}|_V$.
- For $\sigma \colon s_1, \ldots, s_k \to s$, if $(u_1, \ldots, u_k) \in U_{s_1} \times \cdots \times U_{s_k}$ and $\sigma^{\mathcal{A}}(u_1, \ldots, u_k) \in U_s$, then $\sigma^{\mathcal{B}}(p(u_1), \ldots, p(u_k)) \in V$ and $p(\sigma^{\mathcal{A}}(u_1, \ldots, u_k)) = \sigma^{\mathcal{B}}(p(u_1), \ldots, p(u_k))$.

Definition 1

Let $\Sigma = (S, F, P)$ be a signature and let $\mathcal{A}$ and $\mathcal{B}$ be Σ-models. Let $U \subseteq |\mathcal{A}|$ and $V \subseteq |\mathcal{B}|$. A map $p \colon U \to V$ is a *partial isomorphism* if the following conditions hold:

1. p is bijective, with $\mathrm{dom}(p) = U$ and $\mathrm{ran}(p) = V$.
2. For any $u_1, \ldots, u_k \in U$, if $\sigma^{\mathcal{A}}(u_1, \ldots, u_k) \in U$, then $\sigma^{\mathcal{B}}(p(u_1), \ldots, p(u_k)) \in V$ and $p(\sigma^{\mathcal{A}}(u_1, \ldots, u_k)) = \sigma^{\mathcal{B}}(p(u_1), \ldots, p(u_k))$.
3. $p(\pi^{\mathcal{A}}|_U) = \pi^{\mathcal{B}}|_V$ for all $\pi \in P$

Unnested atomic sentences and partial isomorphisms

Lemma 2

Let $\mathcal{A}$ and $\mathcal{B}$ be Σ-structures, and let C be the set of all constants in Σ. The following are equivalent:

1. *$\mathcal{A}$ and $\mathcal{B}$ satisfy the same unnested atomic sentences.*
2. *The natural map $p\colon C^{\mathcal{A}} \to C^{\mathcal{B}}$, given by $c^{\mathcal{A}} \mapsto c^{\mathcal{B}}$ for each $c \in C$ is a partial isomorphism.*

Proof.

In diagrammatic sketch. RHS justifies (1) $\Rightarrow$ (2), LHS justifies (2) $\Rightarrow$ (1). For relations:

$$(c_1^{\mathcal{A}}, \ldots, c_k^{\mathcal{A}}) \in \pi^{\mathcal{A}} \overset{\text{by def}}{\Longleftrightarrow} \mathcal{A} \models \pi(c_1, \ldots, c_k)$$

by partial isomorphism $\Updownarrow$ $\qquad$ $\Updownarrow$ since $\pi(c_1,\ldots,c_k)$ is unnested atomic

$$(c_1^{\mathcal{B}}, \ldots, c_k^{\mathcal{B}}) \in \pi^{\mathcal{B}} \overset{\text{by def}}{\Longleftrightarrow} \mathcal{B} \models \pi(c_1, \ldots, c_k)$$

Unnested atomic sentences and partial isomorphisms

Proof.

RHS justifies (1) $\Rightarrow$ (2), LHS justifies (2) $\Rightarrow$ (1). For functions:

$$\begin{array}{ccc}
\sigma^{\mathcal{A}}(c_1^{\mathcal{A}}, \ldots, c_k^{\mathcal{A}}) = c^{\mathcal{A}} & \overset{\text{by def}}{\Longleftrightarrow} & \mathcal{A} \models \sigma(c_1, \ldots, c_k) = c \\
\text{by partial isomorphism} \Big\Updownarrow & & \Big\Updownarrow \text{since } \sigma(c_1,\ldots,c_k)=c \text{ is unnested atomic} \\
\sigma^{\mathcal{B}}(c_1^{\mathcal{B}}, \ldots, c_k^{\mathcal{B}}) = c^{\mathcal{B}} & \overset{\text{by def}}{\Longleftrightarrow} & \mathcal{B} \models \sigma(c_1, \ldots, c_k) = c
\end{array}$$

□

Exercise 1

Fill out the details of the proof. Not difficult, but long.

*** switch to drawing ***

Unnested atomic sentences and partial isomorphisms

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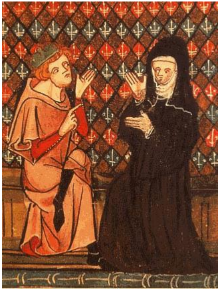
Ehrenfeucht-Fraïssé Games

Comparing models by games

A method conceptually due to Fraïssé (1950), and formulated in game theoretic terms by Ehrenfeucht (1961).

Two player game of perfect information

$\forall$	$\exists$
Spoiler	Duplicator
Abelard	Heloïse
$\forall$belard	$\exists$loise



Abelard and Heloïse as depicted in the 14th centure manuscript *Roman de la Rose*.

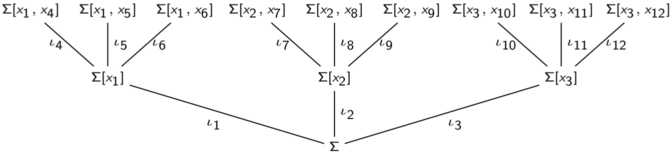
Playing the game

Let Σ be a signature and $\mathcal{A}$, $\mathcal{B}$ be Σ-models. The game is played as follows.

- ∀belard chooses an expansion of one structure to a signature $\Sigma[x]$ for some variable x (of some sort s).
- ∃loise responds by picking an expansion of *the other* structure.
- The moves are repeated k times, for some finite k chosen in advance.
- After the last move, we have $\Sigma[x_x, \dots, x_k]$-expansions $\mathcal{A}_k$ and $\mathcal{B}_k$.
- If they satisfy the same unnested atomic sentences, ∃loise wins this play of the game. Otherwise ∀belard wins.
- ∃loise wins the *game* if she can win every play regardless of ∀belard's moves. That is, if she has a *winning strategy*.

Since every move involves picking a sort, the entire game can be described as progressing up a tree whose nodes are signatures and branching corresponds to the set of sorts. We will call such trees *gameboard trees*.

Gameboard trees



Definition 3 (Gameboard trees)

A *gameboard tree* $\mathtt{tr}$ of height k is inductively defined as follows:

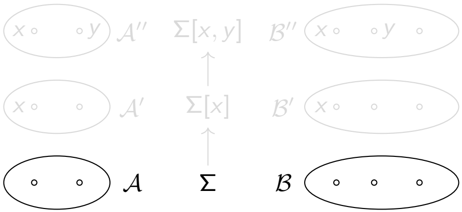
- $\boxed{k = 0}$ Any signature Σ is a gameboard tree with the root Σ and height 0.
- $\boxed{k \Rightarrow k+1}$ For any signature Σ with the set of sorts of cardinality at least λ, if
  - a. $\{\iota_i \colon \Sigma \hookrightarrow \Sigma[x_i]\}_{i<\lambda}$ is a family of signature inclusions, such that for any $i \neq j$, the variables x_i and x_j are of different sorts, and
  - b. $\{\mathtt{tr}_i\}_{i<\lambda}$ is a family of gameboard trees such that
    - i. the root of $\mathtt{tr}_i$ is $\Sigma[x_i]$ for all $i < \lambda$, and
    - ii. the height of $\mathtt{tr}_i$ is k for each $i < \lambda$,

  then $\Sigma\{\stackrel{\iota_i}{\hookrightarrow} \mathtt{tr}_i\}_{i<\lambda}$ is a gameboard tree with root Σ and height $k+1$.

Gameboard trees are *perfect*, that is, such that every node has λ descendants, and each leaf node is at the same height. But the set of sorts can be larger than the branching!

Playing the game: an example

- Signature $\Sigma = (\{\mathtt{any}\}, \emptyset)$: one sort, no function or relation symbols.
- Gameboard: 2 moves (one sort so no branching).
- $\mathcal{A}$ and $\mathcal{B}$: two models over Σ.
- Play: $\forall$belard chooses $\mathcal{A}'$ over $\Sigma[x]$, $\exists$loise responds by $\mathcal{B}'$;
  $\forall$belard chooses $\mathcal{B}''$ over $\Sigma[x, y]$, $\exists$loise responds by $\mathcal{A}''$.



Unnested atomic sentences over $\Sigma[x, y]$ are $x = x$, $y = y$, $x = y$ and $y = x$, and we have

- $\mathcal{A}'' \models x = x, y = y$ (and $\mathcal{A}'' \not\models x = y, y = x$)
- $\mathcal{B}'' \models x = x, y = y$ (and $\mathcal{B}'' \not\models x = y, y = x$)

so the play is a win for $\exists$loise.

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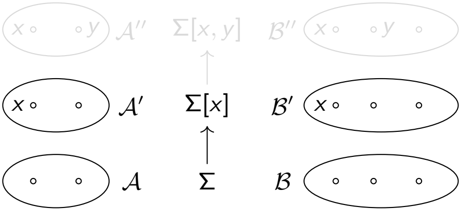

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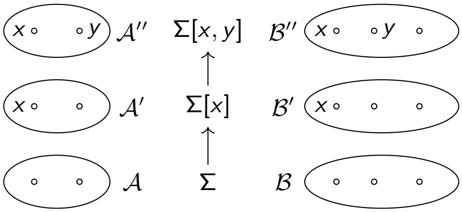

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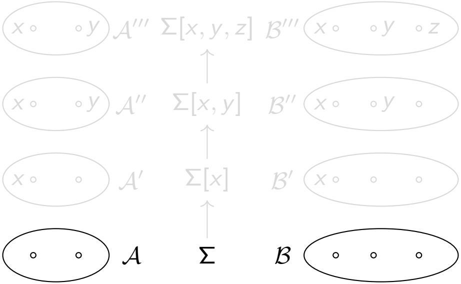
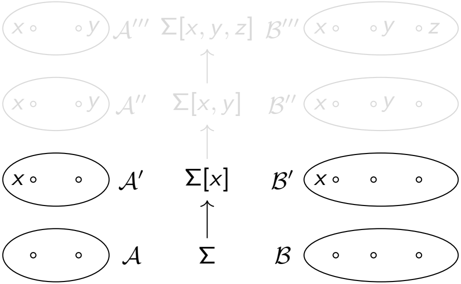


$\forall$belard's third move is $\mathcal{B}'''$. Now $\exists$loise has no response. Every expansion $\mathcal{A}'''$ of $\mathcal{A}''$ will satisfy $x = z$ or $y = z$. So $\exists$loise loses.

- $\exists$loise can win every play of the game over a tree (chain, in fact) of height 2. So she wins the game over such trees.
- But she has no winning strategy over any tree of height > 2.

*** switch to drawing ***

Playing the game: an example



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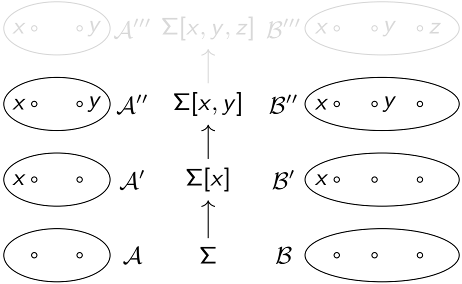

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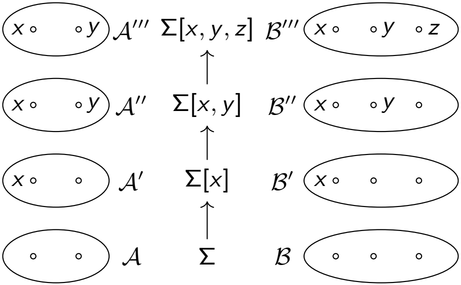

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Introduction

Definition 4

Atomic or negated atomic sentences are often call literals.

Definition 5

Models $\mathcal{A}$ and $\mathcal{B}$ over a signature Σ are called elementarily equivalent if $\mathcal{A}$ and $\mathcal{B}$ satisfy precisely the same Σ-sentences. We write $\mathcal{A} \equiv \mathcal{B}$ for elementary equivalence.

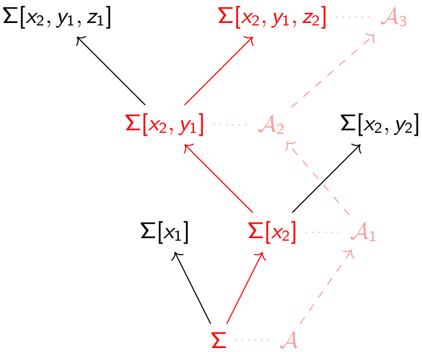
What we are after

Main goal

To show that ∃loise has a winning strategy in all games $EF_{\mathtt{tr}}(\mathcal{A}, B)$ iff $\mathcal{A} \equiv \mathcal{B}$.

- To do that, we will need to describe games by means of special sentences called game sentences.
- Game sentences describe how the game can proceed from the root to the leaves.
- Every model considered during any play of the game satisfies precisely one of these sentences.
- And the sentence says precisely what expansions the model can have.
- If $\mathcal{A}$ and $\mathcal{B}$ have matching expansions all along, then ∃loise can match every ∀belard's move.
- So ∃loise has a winning strategy in $EF_{\mathtt{tr}}(\mathcal{A}, B)$ if and only if $\mathcal{A}$ and $\mathcal{B}$ satisfy the same (unique) game sentence.
- And every sentence turns out to be equivalent to a disjunction of game sentences.

What we are after

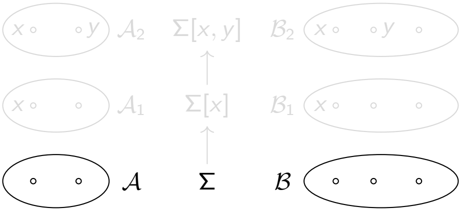


Given a tr such as above, we want a finite set of sentences $\Theta_{\mathtt{tr}}$ such that

- all moves of a Σ-model $\mathcal{A}$ from the root to any leaf of tr (e.g. $\mathcal{A} \to \mathcal{A}_1 \to \mathcal{A}_2 \to \mathcal{A}_3$) is described precisely by exactly one sentence $\gamma_{(\mathcal{A},\mathtt{tr})}$ in $\Theta_{\mathtt{tr}}$, that is, $\mathcal{A} \models \gamma_{(\mathcal{A},\mathtt{tr})}$ and $\mathcal{A} \not\models \Theta_{\mathtt{tr}} \setminus \{\gamma_{(\mathcal{A},\mathtt{tr})}\}$).
- if $\mathcal{B}$ can match all the moves of $\mathcal{A}$ from root to leaves then $\mathcal{B}$ satisfies the same sentence $\gamma_{(\mathcal{A},\mathtt{tr})} \in \Theta_{\mathtt{tr}}$.

Recall this example

- Signature $\Sigma = (\{\texttt{any}\}, \emptyset)$: one sort, no function or relation symbols.
- Gameboard: 2 moves (one sort so no branching, as it is redundant to consider branching in case of single sorted signatures).
- $\mathcal{A}$ and $\mathcal{B}$: two models over Σ.
- Play: $\forall$belard chooses $\mathcal{A}_1$ over $\Sigma[x]$, $\exists$loise responds by $\mathcal{B}_1$;
  $\forall$belard chooses $\mathcal{B}_2$ over $\Sigma[x, y]$, $\exists$loise responds by $\mathcal{A}_2$.



Unnested atomic sentences over $\Sigma[x, y]$ are $x = x$, $y = y$, $x = y$ and $y = x$, and we have

- $\mathcal{A}_2 \models x = x \wedge y = y \wedge x \neq y \wedge y \neq x$
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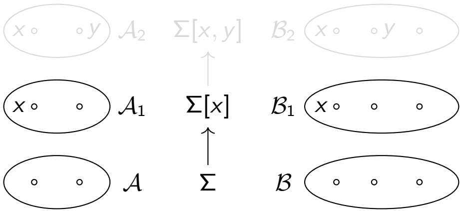

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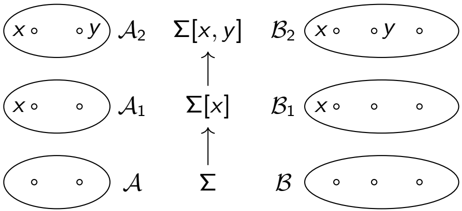

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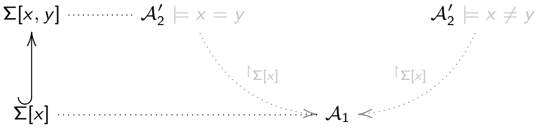
Game sentence (no rounds)

Consider the signature $\Sigma[x, y]$, which can be regarded as a tree of height 0.

- We have $\mathrm{Sen}_b(\Sigma[x, y]) = \{x = x, y = y, x = y, y = x\}$.
- Consider conjunctions of literals over $\mathrm{Sen}_b(\Sigma[x, y])$ in which all the members occur.
  - $\varphi_1 := x = x \wedge y = y \wedge x = y \wedge y = x$
  - $\varphi_2 := x = x \wedge y = y \wedge x \neq y \wedge y \neq x$
- These are the game sentences associated with the tree $\Sigma[x, y]$ of height 0. We denote this set by $\Theta_{\Sigma[x,y]}$.
- There are 16 (2^4) such sentences, 14 of them unsatisfiable.
- The two satisfiable ones are φ_1 and φ_2 defined above.
- By definition of satisfaction, every $\Sigma[x, y]$-model satisfies precisely one of them.
- Each model satisfies precisely one game sentence and ∃loise wins if $\mathcal{A}_2$ and $\mathcal{B}_2$ satisfy exactly the same game sentence.

Game sentences (one round)

Now consider the play along $\Sigma[x] \hookrightarrow \Sigma[x,y]$.



1. Assume that $\mathcal{A}_1$ has at least two elements.
   $\forall$belard has a choice: interpret y by the same element as x, or by a different element.
   How can we describe these to moves?
   a. (At least) These possible choices are described by $\exists y \cdot x = y \ \wedge\ \exists y \cdot x \neq y$.
   b. (At most) Moreover, these are all the choices available to him. This is described by $\forall y \cdot x = y \vee x \neq y$.

   Since $x = y \ \ \models\!\!\!\dashv\ \ \varphi_1$ and $x \neq y \ \ \models\!\!\!\dashv\ \ \varphi_2$, the following $\Sigma[x]$-sentence

$$\gamma_1 \coloneqq \underbrace{\exists y \cdot \varphi_1 \wedge \exists y \cdot \varphi_2}_{\text{at least}} \ \wedge\ \underbrace{\forall y \cdot (\varphi_1 \vee \varphi_2)}_{\text{at most}}$$

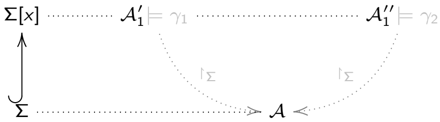
   describes all possible $\forall$belard's choices.

2. Assume that $\mathcal{A}_1$ has only one element.
   Then $\mathcal{A}_1$ has only one expansion $\mathcal{A}_2$ to $\Sigma[x,y]$ such that $\mathcal{A}_2 \models \varphi_1$ but $\mathcal{A}_2 \not\models \varphi_2$.
   In this case, $\mathcal{A}_1 \models \exists y \cdot \varphi_1 \ \wedge\ \forall y \cdot \varphi_1$.

$$\gamma_2 \coloneqq \exists y \cdot \varphi_1 \ \wedge\ \forall y \cdot \varphi_1$$

Game sentences (two rounds)

The game sentences for the tree $\Sigma[x] \hookrightarrow \Sigma[x, y]$ are γ_1, γ_2.
We construct the game sentences for the tree $\Sigma \hookrightarrow \Sigma[x] \hookrightarrow \Sigma[x, y]$.



1. Assume that $\mathcal{A}$ has at least two elements.
   Then any expansion $\mathcal{A}_1'$ of $\mathcal{A}$ to $\Sigma[x]$ satisfies γ_1.
   The unique game sentence that characterizes the moves along $\Sigma \hookrightarrow \Sigma[x] \hookrightarrow \Sigma[x, y]$ is
   $$\exists x \cdot \gamma_1 \wedge \forall x \cdot \gamma_1$$
2. Assume that $\mathcal{A}$ has one element.
   Then any expansion $\mathcal{A}_1''$ of $\mathcal{A}$ to $\Sigma[x]$ satisfies γ_2.
   The unique game sentence that characterizes the moves along $\Sigma \hookrightarrow \Sigma[x] \hookrightarrow \Sigma[x, y]$ is
   $$\exists x \cdot \gamma_2 \wedge \forall x \cdot \gamma_2$$
3. Assume that $\mathcal{A}$ has no elements.
   There is no expansion of $\mathcal{A}$ to $\Sigma[x]$.
   The unique game sentence which indicates there are no moves to make is
   $$(\wedge\emptyset) \wedge \forall x \cdot \vee \emptyset = \top \wedge \forall x \cdot \bot$$

Gameboard trees again

It will be convenient to have arbitrary gameboard trees (not necessarily perfect).

Definition 6 (Gameboard trees)

A *gameboard tree* $\mathtt{tr}$ of height k is inductively defined as follows:

- $\boxed{k = 0}$ Any signature Σ is a gameboard tree with the root Σ and height 0.
- $\boxed{k \Rightarrow k+1}$ For any signature Σ, and any finite n, if
  - (a) $\{\iota_i : \Sigma \hookrightarrow \Sigma[x_i]\}_{i<n}$ is a family of signature extensions with variables, and
  - (b) $\{\mathtt{tr}_i\}_{i<n}$ is a family of gameboard trees such that
    - (i) $root(\mathtt{tr}_i) = \Sigma[x_i]$ for all $i < n$, and
    - (ii) $height(\mathtt{tr}_i) = k$ for at least one $i < n$,

  then $\Sigma\{\stackrel{\iota_i}{\hookrightarrow} \mathtt{tr}_i\}_{i<n}$ is a gameboard tree with root Σ and height $k+1$.

It should be clear that the generalization is only a matter of convenience:

- Having a winning strategy over all trees obviously implies having a winning strategy over all perfect trees.
- But having a winning strategy over all perfect trees also implies having a winning strategy over all trees, since every tree is a subtree of a perfect one.

Game sentences defined

Definition 7 (Game sentences)

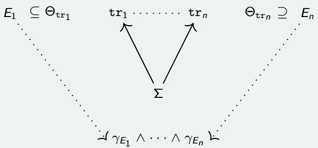
For all gameboard trees $\mathtt{tr}$ with finite root signature, the (finite) set of sentences $\Theta_{\mathtt{tr}}$ is defined as follows:

- $\mathtt{height}(\mathtt{tr}) = 0$. In this case, $\mathtt{tr}$ consists of a single root node, labelled by Σ, and

  $$\Theta_{\mathtt{tr}} = \{ \bigwedge_{e \in \mathrm{Sen}_b(\Sigma)} e^{f(e)} \mid f\colon \mathrm{Sen}_b(\Sigma) \to \{0,1\} \},$$

  where e^1 stands for e and e^0 stands for $\neg e$ for all $e \in \mathrm{Sen}_b(\Sigma)$.

- $\mathtt{height}(\mathtt{tr}) > 0$. Let $\mathtt{tr} = \Sigma\{\stackrel{\iota_1}{\hookrightarrow} \mathtt{tr}_1, \ldots, \stackrel{\iota_n}{\hookrightarrow} \mathtt{tr}_n\}$ be a tree and assume that the finite set of sentences Θ_{tr_i} has been defined for all $i \in \{1, \ldots, n\}$.



  $$\Theta_{\mathtt{tr}} = \{\gamma_{E_1} \wedge \ldots \wedge \gamma_{E_n} \mid E_1 \subseteq \Theta_{\mathtt{tr}_1}, \ldots, E_n \subseteq \Theta_{\mathtt{tr}_n}\},$$

  where $\gamma_{E_i} = (\bigwedge_{\varphi \in E_i} \exists x_i \cdot \varphi) \wedge \forall x_i \cdot \bigvee E_i$ for all $i \in \{1, \ldots, n\}$.

Each γ_{E_i} describe models whose expansions determine the subset of game sentences $E_i \subseteq \Theta_{\mathtt{tr}_i}$ as follows:

- at least part: for all sentences $\varphi \in E_i$ there is an expansion satisfying φ, and
- at most part: for all expansions there is a sentence $\varphi \in E_i$ satisfied by that expansion.

Fraïssé-Hintikka Theorem

Fraïsse-Hintikka Theorem (part 1)

For each gameboard tree $\mathtt{tr}$ *with finite root* Σ *and for each* Σ*-model* $\mathcal{A}$*, there exists a unique sentence* $\rho \in \Theta_{\mathtt{tr}}$ *such that* $\mathcal{A} \models \rho$*.*

We proceed by induction on $\mathtt{height}(\mathtt{tr})$.

$\boxed{\mathtt{height}(\mathtt{tr}) = 0}$ It is immediate by definition of satisfaction.

$\boxed{\mathtt{height}(\mathtt{tr}) > 0}$

- Let $\mathtt{tr} = \Sigma\{\overset{\iota_1}{\hookrightarrow} \mathtt{tr}_1, \ldots, \overset{\iota_n}{\hookrightarrow} \mathtt{tr}_n\}$ and let $\mathcal{A}$ be a Σ-model.
- By inductive hypothesis for each $\Sigma[x_i]$-expansion $\mathcal{A}_i$ of $\mathcal{A}$ there is a unique $\gamma_{(\mathcal{A}_i,\mathtt{tr}_i)} \in \Theta_{\mathtt{tr}_i}$ with $\mathcal{A}_i \models \gamma_{(\mathcal{A}_i,\mathtt{tr}_i)}$.
- Let $\Gamma_{(\mathcal{A},\mathtt{tr},i)} = \{\gamma_{(\mathcal{A}_i,\mathtt{tr}_i)} : \mathcal{A}_i \text{ a } \Sigma[x_i]\text{-expansion of } \mathcal{A}\}$.
- Let $\gamma_{(\mathcal{A},\mathtt{tr})} = \bigwedge_{i=1}^{n}((\bigwedge_{\varphi\in\Gamma_{(\mathcal{A},\mathtt{tr},i)}} \exists x_i \cdot \varphi) \wedge \ \forall x_i \cdot \bigvee \Gamma_{(\mathcal{A},\mathtt{tr},i)})$
- Then, $\mathcal{A} \models \gamma_{(\mathcal{A},\mathtt{tr})}$ by definitions.
- Now we need to show that $\gamma_{(\mathcal{A},\mathtt{tr})}$ is unique.

Fraïssé-Hintikka Theorem

- Suppose $\mathcal{A} \models \gamma_{E_1} \wedge \cdots \wedge \gamma_{E_n}$ for some $E_i \subseteq \Theta_{\mathtt{tr}_i}$ $(i = 1, \ldots, n)$.
- We show that $E_i = \Gamma_{(\mathcal{A},\mathtt{tr},i)}$ for each i.
- By definition, $\gamma_{E_i} = (\bigwedge_{\varphi \in E_i} \exists x_i \cdot \varphi) \wedge \; \forall x_i \cdot \bigvee E_i$.
- The first conjunct implies $E_i \subseteq \Gamma_{(\mathcal{A},\mathtt{tr},i)}$.
  - For if $\varphi \in E_i$ then some expansion $\mathcal{A}_i$ has $\mathcal{A}_i \models \varphi$.
  - By inductive hypothesis $\varphi = \gamma_{(\mathcal{A}_i,\mathtt{tr}_i)}$ (because $\varphi \in \Theta_{\mathtt{tr}_i}$ and so it is unique for $\mathcal{A}_i$).
  - So $\varphi \in \Gamma_{(\mathcal{A},\mathtt{tr},i)}$.
- The second conjunct implies $\Gamma_{(\mathcal{A},\mathtt{tr},i)} \subseteq E_i$.
  - For if $\gamma_{(\mathcal{A}_i,\mathtt{tr}_i)} \in \Gamma_{(\mathcal{A},\mathtt{tr},i)} \setminus E_i$, then the expansion $\mathcal{A}_i$ is not covered in E_i.
  - So $\mathcal{A} \not\models \forall x_i \cdot \bigvee E_i$, contradicting $\mathcal{A} \models \gamma_{E_i}$.
- So $E_i = \Gamma_{(\mathcal{A},\mathtt{tr},i)}$.

Fraïssé-Hintikka Theorem

Fraïsse-Hintikka Theorem (part 2)

For all gameboard trees $\texttt{tr}$ *with finite root* Σ *and all* Σ*-models* $\mathcal{A}$, $\mathcal{B}$, *the following are equivalent:*

1. $\mathcal{A} \approx_{\texttt{tr}} \mathcal{B}$.
2. *There exists a unique* $\rho \in \Theta_{\texttt{tr}}$ *such that* $\mathcal{A} \models \rho$ *and* $\mathcal{B} \models \rho$.

We proceed by induction on $\texttt{height}(\texttt{tr})$.

$\boxed{\texttt{height}(\texttt{tr}) = 0}$ By the definition of game sentences.

$\boxed{\texttt{height}(\texttt{tr}) > 0}$ Let $\texttt{tr} = \Sigma\{\overset{\iota_1}{\hookrightarrow} \texttt{tr}_1, \ldots, \overset{\iota_n}{\hookrightarrow} \texttt{tr}_n\}$ and let $\mathcal{A}$ and $\mathcal{B}$ be Σ-models.

We show (1) $\Rightarrow$ (2):

- Assume $\mathcal{A} \approx_{\texttt{tr}} \mathcal{B}$. We will show $\gamma_{(\mathcal{A},\texttt{tr})} = \gamma_{(\mathcal{B},\texttt{tr})}$.
- Equivalently, $\Gamma_{(\mathcal{A},\texttt{tr},i)} = \Gamma_{(\mathcal{B},\texttt{tr},i)}$ for all $i \in \{1, \ldots, n\}$. We show $\Gamma_{(\mathcal{A},\texttt{tr},i)} \subseteq \Gamma_{(\mathcal{B},\texttt{tr},i)}$:

1	let $\varphi \in \Gamma_{(\mathcal{A},\texttt{tr},i)}$	
2	$\mathcal{A}_i \models \varphi$ for some χ_i-expansion $\mathcal{A}_i$ of $\mathcal{A}$	by the definition of $\Gamma_{(\mathcal{A},\texttt{tr},i)}$
3	$\mathcal{A}_i \approx_{\texttt{tr}_i} \mathcal{B}_i$ for some χ_i-expansion of $\mathcal{B}$	since $\mathcal{A} \approx_{\texttt{tr}} \mathcal{B}$
4	$\mathcal{A}_i \models \rho$ and $\mathcal{B}_i \models \rho$ for some unique $\rho \in \Theta_{\texttt{tr}_i}$	by the induction hypothesis
5	$\rho = \varphi$	by the first part of the proof
6	$\varphi \in \Gamma_{(\mathcal{B},\texttt{tr},i)}$	since $\mathcal{B}_i \models \varphi$ and $\mathcal{B}_i$ is an expansion of $\mathcal{B}$ to $\Sigma[x_i]$

Fraïssé-Hintikka Theorem

We show that (2) $\Rightarrow$ (1):

- Assume $\mathcal{A} \models \rho$ and $\mathcal{B} \models \rho$ for a unique $\rho \in \Theta_{\mathtt{tr}}$.
- Then $\rho = \bigwedge_{i=1}^{n} \gamma_{E_i}$ for some $E_i \subseteq \Theta_{\mathtt{tr}_i}$, where $\gamma_{E_i} = (\bigwedge_{\varphi \in E_i} \exists x_i \cdot \varphi) \wedge \ \forall x_i \cdot \bigvee E_i$.
- Let $\mathcal{A}_i$ be a $\Sigma[x_i]$-expansion of $\mathcal{A}$.
- Then $\mathcal{A}_i \models e$ for some $e \in E_i$.
- Since $\mathcal{B} \models \gamma_{E_i}$ we have $\mathcal{B} \models \bigwedge\limits_{\varphi \in E_i} \exists x_i \cdot \varphi$.
- In particular, $\mathcal{B} \models \exists x_i \cdot e$.
- So $\mathcal{B}_i \models e$ for some expansion $\mathcal{B}_i$ of $\mathcal{B}$.
- By inductive hypothesis, $\mathcal{A} \approx_{\mathtt{tr}_i} \mathcal{B}$.
- Similarly, for every expansion $\mathcal{B}_i$ there is an expansion $\mathcal{A}_i$ such that $\mathcal{A}_i \approx_{\mathtt{tr}_i} \mathcal{B}_i$.
- So, $\mathcal{A} \approx_{\mathtt{tr}} \mathcal{B}$.

Fraïssé-Hintikka Theorem

Fraïsse-Hintikka Theorem (part 3)

For every unnested sentence ρ over a finite signature Σ there exists a gameboard tree $\mathtt{tr}$ *and a set $\Gamma_\rho \subseteq \Theta_{\mathtt{tr}}$ such that $\rho \Dashv\vDash \bigvee \Gamma_\rho$.*

We proceed by induction on the structure of ρ.

$\rho \in \mathsf{Sen}_b(\Sigma)$

- Let Γ_ρ be the set of all sentences in Θ_Σ in which ρ occurs only positively (i.e., without negation).
- Then $\rho \Dashv\vDash \bigvee \Gamma_\rho$.

$\neg\rho$

- let $\Gamma_{\neg\rho} = \Theta_{\mathtt{tr}} \setminus \Gamma_\rho$.
- By the inductive hypothesis and Part 1 we get the result.

$\rho_1 \wedge \rho_2$

- By the inductive hypothesis, for $i = 1, 2$ we have $\mathtt{tr}_i$ and $\Gamma_{\rho_i} \subseteq \Theta_{\mathtt{tr}_i}$ such that $\rho_i \Dashv\vDash \vee\Gamma_{\rho_i}$. Note that $\mathtt{tr}_1$ and $\mathtt{tr}_2$ both have root Σ.
- Identifying the root nodes of $\mathtt{tr}_1$ and $\mathtt{tr}_2$ we obtain a tree $\mathtt{tr}$.
- We put $\Gamma_{\rho_1\wedge\rho_2} = \{\gamma_1 \wedge \gamma_2 \mid \gamma_1 \in \Gamma_{\rho_1},\ \gamma_2 \in \Gamma_{\rho_2}\}$.
- Note that each $\gamma_1 \wedge \gamma_2 \in \Theta_{\mathtt{tr}}$.
- Then $\rho_1 \wedge \rho_2 \Dashv\vDash \bigvee \Gamma_{\rho_1\wedge\rho_2}$.

Fraïssé-Hintikka Theorem

$\exists x \cdot \rho$

- By inductive hypothesis we have $\mathtt{tr}_1$ and $\Gamma_\rho \subseteq \Theta_{\mathtt{tr}_1}$ such that $\rho \models\!\!\!\!\!\dashv \bigvee \Gamma_\rho$.
- Add a new root Σ and an expansion $\Sigma \hookrightarrow \Sigma[x]$ to get a new tree $\mathtt{tr}$.
- The following are equivalent:
  $\mathcal{A} \models \exists x \cdot \rho$ iff
  $\mathcal{A} \models \exists x \cdot \vee \Gamma_\rho$ iff
  $\mathcal{A}_1 \models \vee \Gamma_\rho$ for some expansion $\mathcal{A}_1$ of $\mathcal{A}$ to $\Sigma[x]$ iff
  $\Gamma_{(\mathcal{A},\mathtt{tr},1)} \cap \Gamma_\rho \neq \emptyset$.
- Define $\Gamma_{(\exists x \cdot \rho)} \coloneqq \{\gamma_E \mid E \subseteq \Theta_{\mathtt{tr}_1} \text{ and } E \cap \Gamma_\rho \neq \emptyset\}$.
- The following are equivalent:
  $\mathcal{A} \models \exists x \cdot \vee \Gamma_\rho$ iff
  $\Gamma_{(\mathcal{A},\mathtt{tr},1)} \cap \Gamma_\rho \neq \emptyset$ iff (by the definition of $\Gamma_{(\exists x \cdot \rho)}$)
  $\mathcal{A} \models \vee \Gamma_{(\exists x \cdot \rho)}$.
- Hence, $\exists x \cdot \rho \models\!\!\!\!\!\dashv \vee \Gamma_{(\exists x \cdot \rho)}$.

Fraïssé-Hintikka Theorem

Corollary 8

Let $\mathcal{A}$ and $\mathcal{B}$ be models over some signature Σ. The following are equivalent:

1. $\mathcal{A} \equiv \mathcal{B}$.
2. *$\exists$loise has a winning strategy for all games $EF_{\mathtt{tr}}(\mathcal{A}\restriction_{\Sigma_f}, \mathcal{B}\restriction_{\Sigma_f})$ for all finite subsignatures $\Sigma_f \subseteq \Sigma$ and all gameboard trees $\mathtt{tr}$ with root Σ_f.*
3. *$\mathcal{A}\restriction_{\Sigma_f} \approx_{\mathtt{tr}} \mathcal{B}\restriction_{\Sigma_f}$ for all finite subsignatures $\Sigma_f \subseteq \Sigma$ and all gameboard trees $\mathtt{tr}$ with root Σ_f.*

Equivalence of (2) and (3) was shown in Lecture 5 (Lemma 18).

(1) $\Rightarrow$ (2)

- Let $\Sigma_f \subseteq \Sigma$ be a finite signature and let $\mathtt{tr}$ be a gameboard tree.
- $\mathcal{A} \equiv \mathcal{B}$ implies $\mathcal{A}\restriction_{\Sigma_f} \equiv \mathcal{B}\restriction_{\Sigma_f}$.
- By Fraïssé-Hintikka Theorem part 1, we have $\gamma_{(\mathcal{A},\mathtt{tr})} = \gamma_{(\mathcal{B},\mathtt{tr})}$, as these sentences are unique and satisfied by both $\mathcal{A}$ and $\mathcal{B}$.
- By part 2, $\mathcal{A}\restriction_{\Sigma_f} \approx_{\mathtt{tr}} \mathcal{B}\restriction_{\Sigma_f}$.
- So $\exists$loise has a winning strategy for $EF_{\mathtt{tr}}(\mathcal{A}\restriction_{\Sigma_f}, \mathcal{B}\restriction_{\Sigma_f})$.

Fraïssé-Hintikka Theorem

$(2) \Rightarrow (1)$

- Assume $\exists$loise has a winning strategy for all games $EF_{\mathtt{tr}}(\mathcal{A}\restriction_{\Sigma_f}, \mathcal{B}\restriction_{\Sigma_f})$ for all finite signatures $\Sigma_f \subseteq \Sigma$ and all gameboard trees $\mathtt{tr}$ with root Σ_f.
- Let $\mathcal{A} \models \rho$ for some Σ-sentence ρ.
- Then ρ is a Σ_f-sentence for some finite signature $\Sigma_f \subseteq \Sigma$. By the satisfaction condition, $\mathcal{A}\restriction_{\Sigma_f} \models \rho$.
- By Fraïssé-Hintikka Theorem part 3, there is a set $\Gamma_\rho \subseteq \Theta_{\mathtt{tr}}$ such that $\rho \models\!\!\!\!\!\dashv \bigvee \Gamma_\rho$.
- Then $\mathcal{A}\restriction_{\Sigma_f} \models \bigvee \Gamma_\rho$.
- Since $\mathcal{A}\restriction_{\Sigma_f} \approx_{\mathtt{tr}} \mathcal{B}\restriction_{\Sigma_f}$, by parts 1 and 2, $\mathcal{A}\restriction_{\Sigma_f}$ and $\mathcal{B}\restriction_{\Sigma_f}$ satisfy precisely the same game sentence in Γ_ρ.
- So $\mathcal{B}\restriction_{\Sigma_f} \models \bigvee \Gamma_\rho$.
- By the satisfaction condition, $\mathcal{B} \models \rho$.
- By symmetry, it follows that $\mathcal{A} \equiv \mathcal{B}$.