# Many-Sorted First-Order Model Theory
# lecture 5

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## When are two models the same?

Let $\Sigma$ be a first-order signature. In Lecture 1 we saw that the class $\mathrm{Mod}(\Sigma)$ of all $\Sigma$-models is a category whose arrows (morphisms) are homomorphisms. Hence, it may be natural to say:

- $\Sigma$-models $\mathcal{A}$ and $\mathcal{B}$ are the same if there is an isomorphism $f : \mathcal{A} \to \mathcal{B}$, that is, if $\mathcal{A} \cong \mathcal{B}$.

This is a view from outside.

But there is also a view from inside.

- $\Sigma$-models $\mathcal{A}$ and $\mathcal{B}$ are the same if $\mathcal{A}$ and $\mathcal{B}$ satisfy precisely the same $\Sigma$-sentences, that is, if $\mathcal{A} \models \varphi$ iff $\mathcal{B} \models \varphi$ for every $\varphi \in \mathrm{Sen}(\Sigma)$.

**Definition 1**

Let $\Sigma$ be a signature, and let $\mathcal{A}$ and $\mathcal{B}$ be $\Sigma$-models. We say that $\mathcal{A}$ and $\mathcal{B}$ are *elementarily equivalent*, and write $\mathcal{A} \equiv \mathcal{B}$, whenever

$$\mathcal{A} \models \varphi \text{ iff } \mathcal{B} \models \varphi$$

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## When are two models of a theory the same?

Let $\Sigma$ be a first-order signature, and let $\Gamma$ be a set of $\Sigma$-sentences (in particular a theory, in particular empty). It is clear that every $(\Sigma, \Gamma)$-model is a $\Sigma$-model, and that every homomorphism between $(\Sigma, \Gamma)$-models is also a homomorphism between $\Sigma$-models. Hence $\mathsf{Mod}(\Sigma, \Gamma)$ is a subcategory of $\mathsf{Mod}(\Sigma)$.

**Definition 2**

Let $\Sigma$ be a signature, and let $\Gamma$ be a set of $\Sigma$-sentences. Let $\mathcal{A}, \mathcal{B} \in |\mathsf{Mod}(\Sigma, \Gamma)|$. We say that $\mathcal{A}$ and $\mathcal{B}$ are elementarily equivalent, and write $\mathcal{A} \equiv \mathcal{B}$, whenever

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So here as well we have the two views:

1. $\mathcal{A}, \mathcal{B} \in |\mathsf{Mod}(\Sigma, \Gamma)|$ are the same if $\mathcal{A} \cong \mathcal{B}$ (outside view).
2. $\mathcal{A}, \mathcal{B} \in |\mathsf{Mod}(\Sigma, \Gamma)|$ are the same if $\mathcal{A} \equiv \mathcal{B}$ (inside view).

**Exercise 1**

*Let $\mathcal{A}, \mathcal{B} \in |\mathsf{Mod}(\Sigma, \Gamma)|$. Prove that $\mathcal{A} \cong \mathcal{B}$ implies $\mathcal{A} \equiv \mathcal{B}$.*

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# Two examples from orders

$\Sigma_{\mathtt{ORD}} = (S_{\mathtt{ORD}}, F_{\mathtt{ORD}}, P_{\mathtt{ORD}})$ with $S_{\mathtt{ORD}} = \{\mathtt{Poset}\}$ and $F_{\mathtt{ORD}} = \emptyset$,
$P_{\mathtt{ORD}} = \{\_ \leq \_ : \mathtt{Poset\ Poset}\}$.

**Example 3**

Consider $\mathbb{N}$ and $\mathbb{Z}$ as $\Sigma_{\mathtt{ORD}}$-models.

- Are $\mathbb{N}$ and $\mathbb{Z}$ isomorphic? Clearly not.
- Are $\mathbb{N}$ and $\mathbb{Z}$ elementarily equivalent? Clearly not. For $\mathbb{N} \models \exists x \forall y \cdot x \leq y$, but $\mathbb{Z} \not\models \exists x \forall y \cdot x \leq y$.

**Example 4**

Consider $\mathbb{Q}$ and $\mathbb{R}$ as $\Sigma_{\mathtt{ORD}}$-models.

- Are $\mathbb{Q}$ and $\mathbb{R}$ isomorphic? NO, just because of cardinality.
- Are $\mathbb{Q}$ and $\mathbb{R}$ elementarily equivalent? YES. We will not prove it yet, but they are.

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## Two examples from groups

Consider the signature $\Sigma_{\text{GROUP}}$ Example 9 of Lecture 1. Let $\Gamma$ be the set

- $\forall x, y, z \cdot (x + y) + z = x + (y + z)$,
- $\forall x, y \cdot -x + x = 0$, $\forall x, y \cdot -x + x = 0$,
- $\forall x \cdot x + 0 = x$, $\forall x \cdot 0 + x = x$.

So $\mathrm{Mod}(\Sigma, \Gamma)$ is just the class of all groups, written additively, but not Abelian. Let $X$ be some set (of variables/generators). Take the initial term model $T_{\Sigma_{\text{GROUP}}}(X)$ and the congruence $\equiv_\Gamma := \{(t_1, t_2) \in T_{\Sigma_{\text{GROUP}}}(X) : \Gamma \models t_1 = t_2\}$ (see Lecture 3, Theorem 8). Then $T_{\Sigma_{\text{GROUP}}}(X)/\equiv_\Gamma$ is the free group generated by $X$. We will write $G(X)$ for it.

**Example 5**

Consider $G(\{a\})$ and $G(\{a, b\})$.

- Are $G(\{a\})$ and $G(\{a, b\})$ isomorphic? Certainly not. Probably the easiest way to see it, is to notice that
  - $G(\{a\}) \models \forall x, y \cdot x + y = y + x$, but
  - $G(\{a, b\}) \not\models \forall x, y \cdot x + y = y + x$.
- In fact, this shows that $G(\{a\}) \not\equiv G(\{a, b\})$.

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# Two examples from groups

## Example 6

Consider $G(\{a,b\})$ and $G(\{a,b,c\})$.

- Are $G(\{a,b\})$ and $G(\{a,b,c\})$ isomorphic? NO, but it requires a little longer proof.
- Are $G(\{a,b\})$ and $G(\{a,b,c\})$ elementarily equivalent? YES. Asked by Tarski in 1950s, answered about 2010 by Olga Kharlampovich and Alexei Miasnikov, and independently by Zlil Sela.

## Lemma 7

$G(\{x,y\}) \ncong G(\{a,b,c\})$.

## Proof.

- Suppose there is an isomorphism $f\colon G(\{x,y\}) \to G(\{a,b,c\})$.
- Then $c = f(u)$ for some $u \in G(\{x,y\})$, where $u = w/{\equiv_\Gamma}$ for some term $w = w(x,y) \in T_{\Sigma_{\mathrm{GROUP}}}(\{x,y\})$.
- As usual when working with free groups, we will ignore $\equiv_\Gamma$, and just write $w$ for the congruence class $w/{\equiv_\Gamma}$.
- Let $w_a = f^{-1}(a)$ and $w_b = f^{-1}(b)$. Since $G(\{x,y\})$ is a group, we have $x = u_a + w_a$ and $y = u_b + w_b$ for some $u_a$ and $u_b$.
- But then, $w = w(x,y) = w(u_a + w_a, u_b + w_b)$, so $w'(f^{-1}(a), f^{-1}(b))$, where $w'$ is the term $w(u_a + \_, u_b + \_)$.
- Therefore, $c = f(w) = f(w'(f^{-1}(a), f^{-1}(b)) = w'(a,b)$, as $f$ is a homomorphism .
- Hence $c$ is generated by $a$ and $b$, contradicting the assumption of $G(\{a,b,c\})$ being free. □

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# Unnested terms

## Definition 8 (Unnested terms)

For any first-order signature $\Sigma = (S, F, P)$, the set of unnested terms $T^u_\Sigma$ is defined as follows:

1. $c \in T^u_{\Sigma,s}$ for all constants $(c :\to s) \in F$ (in particular, for all variables), and
2. $\sigma(c_1, \ldots, c_n) \in T^u_{\Sigma,s}$ for all operation symbols $(\sigma\colon s_1 \ldots s_n \to s) \in F$ and all constants $(c_1 :\to s_1), \ldots, (c_n :\to s_n) \in F$.

Note: unnested terms are precisely the terms of depth at most 1.

## Example 9 (Unnested terms)

- For any signature without constants the set of unnested terms is empty.

Let $\Sigma_{\mathrm{NAT}} = (S_{\mathrm{NAT}}, F_{\mathrm{NAT}})$, with $S_{\mathrm{NAT}} = \{\mathrm{Nat}\}$ and $F_{\mathrm{NAT}} = \{0 :\to \mathrm{Nat}, s\_ : \mathrm{Nat} \to \mathrm{Nat}\}$.

- Unnested terms over $\Sigma_{\mathrm{NAT}}$ are $0$ and $s(0)$.
- Unnested terms over $\Sigma_{\mathrm{NAT}}[x]$ are $0$, $x$, $s(0)$ and $s(x)$.

Let $\Sigma_{\mathrm{GROUP}} = (S_{\mathrm{GROUP}}, F_{\mathrm{GROUP}})$, with $S_{\mathrm{GROUP}} = \{\mathrm{Group}\}$ and
$F_{\mathrm{GROUP}} = \{0 :\to \mathrm{Group}, \_+\_ : \mathrm{Group}\ \mathrm{Group} \to \mathrm{Group}, -\_ : \mathrm{Group} \to \mathrm{Group}\}$.

- Unnested terms over $\Sigma_{\mathrm{GROUP}}$ are $0$, $-0$, $0+0$.
- Unnested terms over $\Sigma_{\mathrm{GROUP}}[x]$ are $0$, $-0$, $0+0$, $x$, $-x$, $x+0$, $0+x$, $x+x$.

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Let $\Sigma_{\mathrm{GROUP}} = (\mathrm{S}_{\mathrm{GROUP}}, \mathrm{F}_{\mathrm{GROUP}})$, with $\mathrm{S}_{\mathrm{GROUP}} = \{\mathtt{Group}\}$ and $\mathrm{F}_{\mathrm{GROUP}} = \{\mathtt{0} :\to \mathtt{Group}, \_ + \_ : \mathtt{Group}\ \mathtt{Group} \to \mathtt{Group}, -\_ : \mathtt{Group} \to \mathtt{Group}\}$.

- Unnested terms over $\Sigma_{\mathrm{GROUP}}$ are $\mathtt{0}$, $-\mathtt{0}$, $\mathtt{0} + \mathtt{0}$.
- Unnested terms over $\Sigma_{\mathrm{GROUP}}[x]$ are $\mathtt{0}$, $-\mathtt{0}$, $\mathtt{0} + \mathtt{0}$, $\mathtt{x}$, $-\mathtt{x}$, $\mathtt{x} + \mathtt{0}$, $\mathtt{0} + \mathtt{x}$, $\mathtt{x} + \mathtt{x}$.

# Unnested sentences

**Definition 10 (Unnested atomic sentences)**

For any signature $\Sigma$, an unnested atomic $\Sigma$-sentence is any atomic $\Sigma$-sentence of the form:

1. $t = c$, where $s \in S$, $t \in T^u_{\Sigma,s}$ and $(c :\rightarrow s) \in F$, or
2. $\pi(c_1, \ldots, c_n)$, where $(\pi\colon s_1 \ldots s_n) \in P$ and $(c_i :\rightarrow s_i) \in F$.

We write $\mathrm{Sen}_b(\Sigma)$ for the set of atomic unnested sentences over $\Sigma$.

**Lemma 11**

*Let $\Sigma$ be a finite signature, and $X$ a finite set of variables. Then $\mathrm{Sen}_b(\Sigma[X])$ is finite.*

**Proof.**

- Let $C$ be the set of all constants in $\Sigma$. Put $k = \mathrm{card}(C)$.
- Any function symbol $(\sigma : s_1 s_2 \cdots s_n \rightarrow s) \in S$ defines a function in $k^{k^n}$. As $S$ is finite, there are only finitely many unnested terms. Hence there are only finitely many unnested equations.
- As $P$ is finite, there are only finitely many unnested sentences of the form $\pi(c_1, \ldots, c_m)$.
- Therefore, there are only finitely many unnested atomic sentences. □

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# Unnested sentences

## Definition 12 (Unnested sentences)

An unnested $\Sigma$-sentence is any sentence obtained from an unnested atomic sentencs by applying Boolean connectives and quantification.

## Number of unnested sentences

Even in the finite signature, there are infinitely many unnested sentences. However, for any given quantifier depth, there are only finitely many unnested sentences in any finite signature.

# Unnested sentences: examples

## Example 13 (Unnested sentences in $\Sigma_{\mathtt{NAT}}$)

- Unnested atomic sentences over $\Sigma_{\mathtt{NAT}}$ are $0 = 0$ and $\mathtt{s}(0) = 0$.
- Unnested atomic sentences over $\Sigma_{\mathtt{NAT}}[x]$ are $0 = 0$, $\mathtt{s}(0) = 0$, $\mathtt{x} = 0$, $\mathtt{s}(\mathtt{x}) = 0$, $0 = \mathtt{x}$, $\mathtt{s}(0) = \mathtt{x}$, $\mathtt{x} = \mathtt{x}$, $\mathtt{s}(\mathtt{x}) = \mathtt{x}$.
- Two non-atomic unnested sentences over $\Sigma_{\mathtt{NAT}}[y]$:
  - $\exists \mathtt{x} \cdot \mathtt{s}(0) = \mathtt{x} \wedge \mathtt{s}(\mathtt{x}) = \mathtt{y}$ and
  - $\forall \mathtt{x} \cdot \mathtt{s}(0) = \mathtt{x} \Rightarrow \mathtt{s}(\mathtt{x}) = \mathtt{y}$.

  Both are equivalent to $\mathtt{s}(\mathtt{s}(0)) = \mathtt{y}$.

## Example 14 (Unnested sentences in $\Sigma_{\mathtt{GROUP}}$)

- Unnested atomic sentences over $\Sigma_{\mathtt{GROUP}}$ are $0 = 0$, $-0 = 0$ and $0 + 0 = 0$.
- Some unnested atomic sentences over $\Sigma_{\mathtt{GROUP}}[x]$ are $\mathtt{x} = 0$, $-\mathtt{x} = \mathtt{x}$, $\mathtt{x} + 0 = 0$, $\mathtt{x} = \mathtt{x}$, $\mathtt{x} + \mathtt{x} = \mathtt{x}$, $0 + 0 = 0$.
- Two non-atomic unnested sentences over $\Sigma_{\mathtt{GROUP}}[x, y]$:
  - $\exists \mathtt{z} \cdot \mathtt{x} + \mathtt{y} = \mathtt{z} \wedge \mathtt{y} + \mathtt{x} = \mathtt{z}$ and
  - $\forall \mathtt{z} \cdot \mathtt{x} + \mathtt{y} = \mathtt{z} \Rightarrow \mathtt{y} + \mathtt{x} = \mathtt{z}$.

  Both are equivalent to $\mathtt{x} + \mathtt{y} = \mathtt{y} + \mathtt{x}$.

# Unnested sentences suffice for FOL

## Lemma 15

*For any first-order atomic sentence $\varphi$ there exists an unnested sentence $\varphi^u$ such that $\varphi$ is semantically equivalent to $\varphi^u$. Moreover, $\varphi^u$ can be chosen to be an existential or a universal sentence.*

## Proof by example (equations).

Let $\varphi$ be the sentence $(x+y)+z = x+(y+z)$ over $\Sigma_{\mathtt{GROUP}}[x,y,z]$ (just a signature; models do not need to be groups).

- Remove nesting by introducing new variables, $x+y = v$, $y+z = u$, $v+z = w$, $x+u = w$.
- Put the pieces together: $(x+y=v) \wedge (y+z=u) \wedge (v+z=w) \wedge (x+u=w)$ and get an unnested sentence over $\Sigma_{\mathtt{GROUP}}[x,y,z,u,v,w]$.
- Quantify the new variables away to form $\varphi^u$:
  - $\exists u,v,w \cdot (x+y=v) \wedge (y+z=u) \wedge (v+z=w) \wedge (x+u=w)$; or
  - $\forall u,v,w \cdot (x+y=v) \wedge (y+z=u) \wedge (v+z=w) \Rightarrow (x+u=w)$.

  Each is an unnested sentence over $\Sigma_{\mathtt{GROUP}}[x,y,z]$.
- By semantics of quantifiers, we have

$$\mathcal{A} \models \varphi \text{ iff } \mathcal{A} \models \varphi^u$$

  for any $\Sigma_{\mathtt{GROUP}}[x,y,z]$-model $\mathcal{A}$.

# Proof by example continued

## Proof by example (relations).

Let $\Sigma_{\mathtt{NAT}} = (\mathtt{S}_{\mathtt{NAT}}, \mathtt{F}_{\mathtt{NAT}}, \mathtt{P}_{\mathtt{NAT}})$, with $\mathtt{S}_{\mathtt{NAT}} = \{\mathtt{Nat}\}$, $\mathtt{F}_{\mathtt{NAT}} = \{0 :\to \mathtt{Nat}, \mathtt{s}\_ : \mathtt{Nat} \to \mathtt{Nat}\}$, and $\mathtt{P}_{\mathtt{NAT}} = \{\_ < \_ : \mathtt{Nat}\ \mathtt{Nat}\}$.

- Let $\varphi$ be $\mathtt{s}(0) < \mathtt{s}(\mathtt{s}(\mathtt{s}(0)))$.
- Remove nesting: $s(0) = x$, $s(x) = y$, $s(y) = z$, $x < z$.
- Put the pieces together: $(s(0) = x) \wedge (s(x) = y) \wedge (s(y) = z) \wedge x < z$
- Quantify variables away:
  $\exists x, y, z \cdot (s(0) = x) \wedge (s(x) = y) \wedge (s(y) = z) \wedge x < z$, or
  $\forall x, y, z \cdot (s(0) = x) \wedge (s(x) = y) \wedge (s(y) = z) \Rightarrow x < z$

□

## Exercise 2 (see Lecture 1, Example 11)

*Let* $\Sigma_{\mathtt{AUTOM}} = (\mathtt{S}_{\mathtt{AUTOM}}, \mathtt{F}_{\mathtt{AUTOM}})$, *with*

- $\mathtt{S}_{\mathtt{AUTOM}} = \{\mathtt{Input}, \mathtt{Output}, \mathtt{State}\}$
- $\mathtt{F}_{\mathtt{AUTOM}} = \{\mathtt{init} :\to \mathtt{State},\ \mathtt{f} : \mathtt{Input}\ \mathtt{State} \to \mathtt{State},\ \mathtt{g} : \mathtt{State} \to \mathtt{Output}\}$

*What are the unnested terms of* $\Sigma_{\mathtt{AUTOM}}$*? What are the unnested atomic sentences?*

# Proof by example continued

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Let $\Sigma_{\mathtt{NAT}} = (\mathtt{S}_{\mathtt{NAT}}, \mathtt{F}_{\mathtt{NAT}}, \mathtt{P}_{\mathtt{NAT}})$, with $\mathtt{S}_{\mathtt{NAT}} = \{\mathtt{Nat}\}$, $\mathtt{F}_{\mathtt{NAT}} = \{0 :\to \mathtt{Nat}, \mathtt{s}\_ : \mathtt{Nat} \to \mathtt{Nat}\}$, and $\mathtt{P}_{\mathtt{NAT}} = \{\_ < \_ : \mathtt{Nat}\ \mathtt{Nat}\}$.

- Let $\varphi$ be $\mathtt{s}(0) < \mathtt{s}(\mathtt{s}(\mathtt{s}(0)))$.
- Remove nesting: $s(0) = x$, $s(x) = y$, $s(y) = z$, $x < z$.
- Put the pieces together: $(s(0) = x) \wedge (s(x) = y) \wedge (s(y) = z) \wedge x < z$
- Quantify variables away:
  $\exists x, y, z \cdot (s(0) = x) \wedge (s(x) = y) \wedge (s(y) = z) \wedge x < z$, or
  $\forall x, y, z \cdot (s(0) = x) \wedge (s(x) = y) \wedge (s(y) = z) \Rightarrow x < z$

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*What are the unnested terms of* $\Sigma_{\mathtt{AUTOM}}$*? What are the unnested atomic sentences?*

## Full proof: base case and equations

### Proof.

Induction on the depth of terms.

- Base case for equations is $c_1 = c_2$, which is unnested.
- Base case for relations it is $\pi(c_1, \ldots, c_n)$, which is unnested.
- For the inductive step, consider $\sigma(t_1, \ldots, t_n) = t$ and $\pi(t_1, \ldots, t_n)$, and assume the lemma holds for all equations between terms of depth less than the maximum of depths of $t_1, \ldots, t_n, t$.
- For equations. By semantics,

$$\sigma(t_1, \ldots, t_n) = t \models\!\!\!\dashv \forall x_1, \ldots, x_n, x \cdot \bigwedge_{i=1}^{n} (t_i = x_i) \wedge (t = x) \Rightarrow \sigma(x_1, \ldots, x_n) = x, \text{ and}$$

$$\sigma(t_1, \ldots, t_n) = t \models\!\!\!\dashv \exists x_1, \ldots, x_n, x \cdot \bigwedge_{i=1}^{n} (t_i = x_i) \wedge (t = x) \wedge \sigma(x_1, \ldots, x_n) = x.$$

- By induction hypothesis, there exist unnested sentences $\rho_i, \rho$ over $\Sigma[x_1, \ldots, x_n, x]$ such that $t_i = x_i \models\!\!\!\dashv \rho_i$ and $t = x \models\!\!\!\dashv \rho$. Thus,
- $\sigma(t_1, \ldots, t_n) = t \models\!\!\!\dashv \forall x_1, \ldots, x_n, x \cdot \bigwedge_{i=1}^{n} \rho_i \wedge \rho \Rightarrow \sigma(x_1, \ldots, x_n) = x$, and

$$\sigma(t_1, \ldots, t_n) = t \models\!\!\!\dashv \exists x_1, \ldots, x_n, x \cdot \bigwedge_{i=1}^{n} \rho_i \wedge \rho \wedge \sigma(x_1, \ldots, x_n) = x.$$

# Full proof: relations

## Proof.

- For relations. By semantics,

$$\pi(t_1, \dots, t_n) \models\!\dashv \forall x_1, \dots, x_n \cdot \bigwedge_{i=1}^{n} (t_i = x_i) \Rightarrow \pi(x_1, \dots, x_n), \text{ and}$$

$$\pi(t_1, \dots, t_n) \models\!\dashv \exists x_1, \dots, x_n \cdot \bigwedge_{i=1}^{n} (t_i = x_i) \wedge \pi(x_1, \dots, x_n).$$

- By induction hypothesis, there exist unnested sentences $\rho_i$ over $\Sigma[x_1, \dots, x_n]$ such that $t_i = x_i \models\!\dashv \rho_i$. Thus,

- $$\pi(t_1, \dots, t_n) \models\!\dashv \forall x_1, \dots, x_n, x \cdot \bigwedge_{i=1}^{n} \rho_i \Rightarrow \pi(x_1, \dots, x_n), \text{ and}$$

$$\pi(t_1, \dots, t_n) \models\!\dashv \exists x_1, \dots, x_n, x \cdot \bigwedge_{i=1}^{n} \rho_i \wedge \pi(x_1, \dots, x_n).$$

□

## A note about quantifiers:

The unnested equivalent $\varphi^u$ of an atomic sentence $\varphi$ is typically not atomic. It is either existential or universal.

## Full proof: relations

### Proof.

- For relations. By semantics,

$$\pi(t_1, \dots, t_n) \mathrel{\models\!\!\!\dashv} \forall x_1, \dots, x_n \cdot \bigwedge_{i=1}^{n} (t_i = x_i) \Rightarrow \pi(x_1, \dots, x_n), \text{ and}$$

$$\pi(t_1, \dots, t_n) \mathrel{\models\!\!\!\dashv} \exists x_1, \dots, x_n \cdot \bigwedge_{i=1}^{n} (t_i = x_i) \land \pi(x_1, \dots, x_n).$$

- By induction hypothesis, there exist unnested sentences $\rho_i$ over $\Sigma[x_1, \dots, x_n]$ such that $t_i = x_i \mathrel{\models\!\!\!\dashv} \rho_i$. Thus,

- $$\pi(t_1, \dots, t_n) \mathrel{\models\!\!\!\dashv} \forall x_1, \dots, x_n, x \cdot \bigwedge_{i=1}^{n} \rho_i \Rightarrow \pi(x_1, \dots, x_n), \text{ and}$$

$$\pi(t_1, \dots, t_n) \mathrel{\models\!\!\!\dashv} \exists x_1, \dots, x_n, x \cdot \bigwedge_{i=1}^{n} \rho_i \land \pi(x_1, \dots, x_n).$$

□

### A note about quantifiers:

The unnested equivalent $\varphi^u$ of an atomic sentence $\varphi$ is typically not atomic. It is either existential or universal.

# Unnested sentences suffice for FOL

**Theorem 16**

*Every first-order sentence is semantically equivalent to an unnested first-order sentence.*

Theorem 16 can also be stated as follows. Let $\mathrm{FOL}_u$ be the restriction of FOL to unnested sentences.

**Theorem 17**

$\mathrm{FOL}_u$ *is semantically equivalent to* FOL.

In other words, $\mathrm{FOL}_u$ has the same expressive power as FOL. Theorems of similar nature will be the focus of the third part of these lectures.