Many-Sorted First-Order Model Theory
lecture 3

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Congruences

Exercise 1

$h(t^{\mathcal{A}}) = t^{\mathcal{B}}$ for all signatures Σ, all Σ-homomorphisms $h : \mathcal{A} \to \mathcal{B}$ and all Σ-terms t.

Definition 2 (Congruence)

Let Σ be a signature and $\mathcal{A}$ be a Σ-model. A *congruence* $\equiv = \{\equiv_s\}_{s \in S}$ on $\mathcal{A}$ is

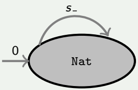
1. an *equivalence* on $|\mathcal{A}|$, i.e. an S-sorted relation $\equiv_s \subseteq \mathcal{A}_s \times \mathcal{A}_s$ for all $s \in S$ satisfying the following properties:
   - (*Reflexivity*) $\dfrac{}{a \equiv_s a}$ for all $s \in S$ and $a \in \mathcal{A}_s$
   - (*Symmetry*) $\dfrac{a_1 \equiv_s a_2}{a_2 \equiv_s a_1}$ for all $s \in S$ and $a_1, a_2 \in \mathcal{A}_s$
   - (*Transitivity*) $\dfrac{a_1 \equiv_s a_2 \quad a_2 \equiv_s a_3}{a_1 \equiv_s a_3}$ for all $s \in S$ and $a_1, a_2, a_3 \in \mathcal{A}_s$
2. *compatible with the function symbols* in Σ:
   - (*Congruence*) $\dfrac{a_1 \equiv_{s_1} a_1' \ldots a_n \equiv_{s_n} a_n'}{\sigma^{\mathcal{A}}(a_1, \ldots, a_n) \equiv_s \sigma^{\mathcal{A}}(a_1', \ldots, a_n')}$
     for all $(\sigma : s_1 \ldots s_n \to s) \in F$ and $a_i, a_i' \in \mathcal{A}_{s_i}$ for all $i \in \{1, \ldots, n\}$.

- We will drop the subscript s from $\equiv_s$ whenever it is clear from the context.
- We write $(a_1, \ldots, a_n) \equiv (a_1', \ldots, a_n')$ when $a_1 \equiv a_1'$ and ...and $a_n \equiv a_n'$.

Examples of congruences

Example 3

$\Sigma_{\mathtt{NAT}} = (S_{\mathtt{NAT}}, F_{\mathtt{NAT}})$



Let $\mathbb{N}$ be the $\Sigma_{\mathtt{NAT}}$-model of natural numbers which interprets all symbols in $\Sigma_{\mathtt{NAT}}$ in the usual way. We define the congruence $\equiv$ on $\mathbb{N}$ as follows:

$$n_1 \equiv n_2 \text{ if } (n_1 \bmod 2) = (n_2 \bmod 2) \text{ for all } n_1, n_2 \in \mathbb{N}.$$

Lemma 4

The relation $\equiv$ defined in Example 3 is a congruence.

Proof.

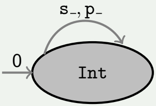
- *Reflexivity:* $n \equiv n$ iff $n \bmod 2 = n \bmod 2$
- *Symmetry:* $n_1 \equiv n_2$ iff $(n_1 \bmod 2) = (n_2 \bmod 2)$ iff $(n_2 \bmod 2) = (n_1 \bmod 2)$ iff $n_2 \equiv n_1$
- *Transitivity:* If $n_1 \equiv n_2$ and $n_2 \equiv n_3$ then $(n_1 \bmod 2) = (n_2 \bmod 2)$ and $(n_2 \bmod 2) = (n_3 \bmod 2)$, which implies $(n_1 \bmod 2) = (n_3 \bmod 2)$. Hence, $n_1 \equiv n_3$.
- *Compatibility with $F_{\mathtt{NAT}}$:*
  - $0^{\mathbb{N}} \equiv 0^{\mathbb{N}}$ iff $0 \bmod 2 = 0 \bmod 2$ iff $0 = 0$.
  - If $n_1 \equiv n_2$ then $(n_1 \bmod 2) = (n_2 \bmod 2)$ iff $(n_1 + 1) \bmod 2 = (n_2 + 1) \bmod 2$ iff $s^{\mathbb{N}} n_1 \bmod 2 = s^{\mathbb{N}} n_2 \bmod 2$ iff $s^{\mathbb{N}} n_1 \equiv s^{\mathbb{N}} n_2$.

□

Examples of congruences

Example 5

$\Sigma_{\mathtt{INT}} = (S_{\mathtt{INT}}, F_{\mathtt{INT}})$



Let $\Gamma := \{\mathtt{s}\ \mathtt{p}\ t = t \mid t \in T_{(\Sigma_{\mathtt{INT}})}\} \cup \{\mathtt{p}\ \mathtt{s}\ t = t \mid t \in T_{(\Sigma_{\mathtt{INT}})}\}$.
We define the congruence $\equiv_\Gamma$ on $T_{(\Sigma_{\mathtt{INT}})}$ as follows:

$$t_1 \equiv_\Gamma t_2 \text{ if } \Gamma \models t_1 = t_2 \text{ for all } t_1, t_2 \in T_{(\Sigma_{\mathtt{INT}})}$$

Lemma 6

The relation $\equiv_\Gamma$ defined in Example 5 is a congruence.

Proof.

- *Reflexivity:* $t \equiv_\Gamma t$ iff $\Gamma \models t = t$ iff $\mathcal{A} \models t = t$ for all $\mathcal{A} \in \Gamma^\bullet$ iff $t^{\mathcal{A}} = t^{\mathcal{A}}$ for all $\mathcal{A} \in \Gamma^\bullet$.
- *Symmetry:* $t_1 \equiv_\Gamma t_2$ iff $\Gamma \models t_1 = t_2$ iff $\mathcal{A} \models t_1 = t_2$ for all $\mathcal{A} \in \Gamma^\bullet$ iff $t_1^{\mathcal{A}} = t_2^{\mathcal{A}}$ for all $\mathcal{A} \in \Gamma^\bullet$ iff $t_2^{\mathcal{A}} = t_1^{\mathcal{A}}$ for all $\mathcal{A} \in \Gamma^\bullet$ iff $\mathcal{A} \models t_2 = t_1$ for all $\mathcal{A} \in \Gamma^\bullet$ iff $\Gamma \models t_2 = t_1$ iff $t_2 \equiv_\Gamma t_1$.

Proof of Lemma 6.

- *Transitivity:* Assuming $t_1 \equiv_\Gamma t_2$ and $t_2 \equiv_\Gamma t_3$, we show $t_1 \equiv_\Gamma t_3$:

1	$\Gamma \models t_1 = t_2$ and $\Gamma \models t_2 = t_3$	by definition
2	$\mathcal{A} \models t_1 = t_2$ for all $\mathcal{A} \in \Gamma^\bullet$	since $\Gamma \models t_1 = t_2$
3	$t_1^{\mathcal{A}} = t_2^{\mathcal{A}}$ for all $\mathcal{A} \in \Gamma^\bullet$	by definition
4	$\mathcal{B} \models t_2 = t_3$ for all $\mathcal{B} \in \Gamma^\bullet$	since $\Gamma \models t_2 = t_3$
5	$t_2^{\mathcal{B}} = t_3^{\mathcal{B}}$ for all $\mathcal{B} \in \Gamma^\bullet$	since $\Gamma \models t_2 = t_3$
6	$t_1^{\mathcal{A}} = t_2^{\mathcal{A}}$ and $t_2^{\mathcal{A}} = t_3^{\mathcal{A}}$ for all $\mathcal{A} \in \Gamma^\bullet$	from 3 and 5
7	$t_1^{\mathcal{A}} = t_3^{\mathcal{A}}$ for all $\mathcal{A} \in \Gamma^\bullet$	since the equality is transitive
8	$\Gamma \models t_1 = t_3$	by definition
9	$t_1 \equiv_\Gamma t_3$	by definition

- *Compatibility with $F_{\mathtt{INT}}$:* Assuming that $t_1 \equiv_\Gamma t_2$ we show that $\mathtt{s}\ t_1 \equiv_\Gamma \mathtt{s}\ t_2$:

1	$\Gamma \models t_1 = t_2$	since $t_1 \equiv_\Gamma t_2$
2	$\mathcal{A} \models t_1 = t_2$ for all $\mathcal{A} \in \Gamma^\bullet$	since $\Gamma \models t_1 = t_2$
3	$t_1^{\mathcal{A}} = t_2^{\mathcal{A}}$ for all $\mathcal{A} \in \Gamma^\bullet$	by definition
4	$\mathtt{s}^{\mathcal{A}} t_1^{\mathcal{A}} = \mathtt{s}^{\mathcal{A}} t_2^{\mathcal{A}}$ for all $\mathcal{A} \in \Gamma^\bullet$	since $s^{\mathcal{A}} : \mathcal{A}_{\mathtt{Int}} \to \mathcal{A}_{\mathtt{Int}}$ is a function
5	$(\mathtt{s}\ t_1)^{\mathcal{A}} = (\mathtt{s}\ t_2)^{\mathcal{A}}$ for all $\mathcal{A} \in \Gamma^\bullet$	by definition
6	$\Gamma \models \mathtt{s}\ t_1 = \mathtt{s}\ t_2$	by definition
7	$\mathtt{s}\ t_1 \equiv_\Gamma \mathtt{s}\ t_2$	by definition

Similarly, one can show that $\mathtt{p}\ t_1 \equiv_\Gamma \mathtt{p}\ t_2$ assuming $t_1 \equiv_\Gamma t_2$.

□

Exercise 7

Let Γ be the set of sentences defined in Example 5. Prove that $T_{(\Sigma_{\mathtt{INT}})}/_{\equiv_\Gamma} \cong \mathbb{Z}$.

Congruence generated by equations

Theorem 8

Let Γ be a set of equations over Σ.
The relation $\equiv_\Gamma := \{(t_1, t_2) \mid \Gamma \models t_1 = t_2\}$ *is a congruence on* T_Σ.

Proof.

By noting that the arguments used in the proof of Lemma 6 can be used for any signature and any set of equations, not only for $\Sigma_{\mathtt{INT}}$ and Γ defined in Example 5. □

Kernel

Example 9

Let Σ be a signature and $h : \mathcal{A} \to \mathcal{B}$ a Σ-homomorphism. We define the congruence $\ker(h) = \{\ker(h)_s\}_{s \in S}$ on $\mathcal{A}$ as follows: $a \ \ker(h) \ b$ *if* $h(a) = h(b)$ for all sorts $s \in S$ and all elements $a, b \in \mathcal{A}_s$.

Lemma 10

The kernel of h, $\ker(h)$*, defined in Example 9 is a congruence on* $\mathcal{A}$*.*

Proof.

- *Reflexivity:* Obviously, $h(a) = h(a)$, which implies $(a, a) \in \ker(h)$.
- *Symmetry:* We assume $(a, b) \in \ker(h)$ and we show that $(b, a) \in \ker(h)$.
  We have: $a \ker(h) \ b$ iff $h(a) = h(b)$ iff $h(b) = h(a)$ iff $b \ \ker(h) \ a$ iff $(b, a) \in \ker(h)$.
- *Transitivity:* We assume $(a, b) \ker(h)$ and $(b, c) \in \ker(h)$, and we show that $(a, c) \in \ker(h)$.
  Since $(a, b) \in \ker(h)$ and $(b, c) \in \ker(h)$, we have $h(a) = h(b)$ and $h(b) = h(c)$.
  We obtain $h(a) = h(c)$. Hence, $(a, c) \in \ker(h)$.
- *Compatibility with F:* Let $(\sigma : w \to s) \in F$.
  We assume that $(a, b) \in \ker(h)_w$ and we prove that $(\sigma^{\mathcal{A}}(a), \sigma^{\mathcal{A}}(b)) \in \ker(h)_s$.
  Since $(a, b) \in \ker(h)_w$, we have $h_w(a) = h_w(b)$.
  It follows that $h_s(\sigma^{\mathcal{A}}(a)) = \sigma^{\mathcal{B}}(h_w(a)) = \sigma^{\mathcal{B}}(h_w(b)) = h_s(\sigma^{\mathcal{A}}(b))$. Hence, $(\sigma^{\mathcal{A}}(a), \sigma^{\mathcal{A}}(b)) \in \ker(h)_s$.

□

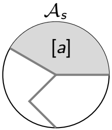
Quotient

Notation 11

Let $\equiv$ be a congruence on a Σ-model $\mathcal{A}$. Let $s \in S$ and $a \in \mathcal{A}_s$.
The class of a is $[a] \coloneqq \{a' \in \mathcal{A}_s \mid a \equiv a'\}$ sometimes denoted also by $a/_{\equiv}$.

Fact 12

Note that $[a] \subseteq \mathcal{A}_s$ for all $s \in S$ and $a \in \mathcal{A}_s$.
This means that $\equiv$ determines a partition of the universe $|\mathcal{A}|$.



Example 13

Consider the congruence defined in Example 3. Then:

- $[0] = \{0, 2, 4, 6, \dots\}$
- $[1] = \{1, 3, 5, 7, \dots\}$

Example 14

Consider the congruence defined in Example 5. Then:

- $[0] = \{0, \mathrm{p\ s}\ 0, \mathrm{s\ p}\ 0, \mathrm{p\ s\ p\ s}\ 0, \dots\}$
- $[\mathrm{p}\ 0] = \{\mathrm{p}\ 0, \mathrm{p\ s\ p}\ 0, \mathrm{s\ p\ p}\ 0, \mathrm{p\ s\ p\ s\ p}\ 0 \dots\}$

  $\vdots$

Notation 15

If $a = (a_1, \ldots, a_n) \in \mathcal{A}_{s_1} \times \cdots \times \mathcal{A}_{s_n}$ *then we let* $[a]$ *denote the tuple* $([a_1], \ldots, [a_n])$*.*

Definition 16 (Quotient structures)

Let $\equiv$ be a congruence on a Σ-model $\mathcal{A}$.
The *quotient structure of $\mathcal{A}$ modulo $\equiv$* is the Σ-structure $[\mathcal{A}]$ (also denoted $\mathcal{A}/_{\equiv}$) defined below:

- $[\mathcal{A}]_s = \{[a] \mid a \in \mathcal{A}_s\}$ for all sorts $s \in S$,
- for all function symbols $(\sigma : w \to s) \in F$,
  the function $\sigma^{[\mathcal{A}]} : [\mathcal{A}]_w \to [\mathcal{A}]_s$ is defined by $\sigma^{[\mathcal{A}]}([a]) = [\sigma^{\mathcal{A}}(a)]$ for all $a \in \mathcal{A}_w$;
- for all relation symbols $(\pi : w) \in P$,
  the relation $\pi^{[\mathcal{A}]}$ is defined by $\pi^{[\mathcal{A}]} = \{[a] \mid a \in \pi^{\mathcal{A}}\}$.

Lemma 17

$[\mathcal{A}]$ *is well-defined.*

Proof.

We show that $\sigma^{[\mathcal{A}]} : [\mathcal{A}]_w \to [\mathcal{A}]_s$ is a function: if $[a] = [b]$ then $a \equiv b$, which implies $\sigma^{\mathcal{A}}(a) \equiv \sigma^{\mathcal{A}}(b)$, and we get $\sigma^{[\mathcal{A}]}([a]) = [\sigma^{\mathcal{A}}(a)] = [\sigma^{\mathcal{A}}(b)] = \sigma^{[\mathcal{A}]}([b])$. □

Basic set of sentences

Definition 18 (Basic set of sentences)

A set of Σ-sentences Γ is *basic* if there exists a Σ-model $\mathcal{A}_\Gamma$ such that for all Σ-models $\mathcal{A}$,

$\mathcal{A} \models \Gamma$ iff there exists a homomorphism $\mathcal{A}_\Gamma \to \mathcal{A}$.

We say that $\mathcal{A}_\Gamma$ is a *basic model* of Γ. If in addition the homomorphism $\mathcal{A}_\Gamma \to \mathcal{A}$ is unique then the set Γ is called *epi-basic*.

Theorem 19

Any set of atomic sentences Γ is epi-basic basic.

Proof.

Let $\equiv_\Gamma := \{(t_1, t_2) \mid \Gamma \models t_1 = t_2\}$ be the congruence on T_Σ.
The basic model $\mathcal{A}_\Gamma$ is obtained from $T_\Sigma/_{\equiv_\Gamma}$ by interpreting each $(\pi : w) \in P$ as follows:

$$\pi^{(\mathcal{A}_\Gamma)} := \{[t] \mid \Gamma \models \pi(t)\}$$

We show that $\mathcal{A} \models \Gamma$ iff there exists a unique homomorphism $\mathcal{A}_\Gamma \to \mathcal{A}$.

Proof of Theorem 19.

$\boxed{\Rightarrow}$ Assuming that $\mathcal{A} \models \Gamma$ we show that there exists a unique $h : \mathcal{A}_\Gamma \to \mathcal{A}$.

We define $h : \mathcal{A}_\Gamma \to \mathcal{A}$ by $h([t]) = t^{\mathcal{A}}$ for all terms $t \in T_\Sigma$.

- We show that h is well-defined.
  - Assuming that $[t_1] = [t_2]$ we show that $h([t_1]) = h([t_2])$:

1	$h([t_1]) = t_1^{\mathcal{A}}$	by definition
2	$h([t_2]) = t_2^{\mathcal{A}}$	by definition
3	$\Gamma \models t_1 = t_2$	since $t_1 \equiv_\Gamma t_2$
4	$\mathcal{A} \models t_1 = t_2$	since $\mathcal{A} \models \Gamma$ and $\Gamma \models t_1 = t_2$
5	$t_1^{\mathcal{A}} = t_2^{\mathcal{A}}$	since $\mathcal{A} \models t_1 = t_2$
6	$h([t_1]) = h([t_2])$	by the definition of h

  - $h(\sigma^{\mathcal{A}_\Gamma}([t])) = h([\sigma(t)]) = \sigma(t)^{\mathcal{A}} = \sigma^{\mathcal{A}}(t^{\mathcal{A}}) = \sigma^{\mathcal{A}}(h([t]))$.
  - Assuming that $[t] \in \pi^{(\mathcal{A}_\Gamma)}$ we show that $h([t]) \in \pi^{\mathcal{A}}$:

1	$\Gamma \models \pi(t)$	by the definition of $\mathcal{A}_\Gamma$
2	$\mathcal{A} \models \pi(t)$	since $\Gamma \models \pi(t)$ and $\mathcal{A} \models \Gamma$
3	$t^{\mathcal{A}} \in \pi^{\mathcal{A}}$	by the definition of $\models$
4	$h([t]) \in \pi^{\mathcal{A}}$	by the definition of h

- We show that h is unique. Let $g : \mathcal{A}_\Gamma \to \mathcal{A}$ be another homomorphism. Then $g([t]) = g(t^{\mathcal{A}_\Gamma}) = t^{\mathcal{A}} = h(t^{\mathcal{A}_\Gamma}) = h([t])$ for all terms $t \in T_\Sigma$. Hence, $h = g$.

Proof of Theorem 19.

⇐ Assuming a homomorphism $h : \mathcal{A}_\Gamma \to \mathcal{A}$ we prove $\mathcal{A} \models \Gamma$.

1. Assuming that $t_1 = t_2 \in \Gamma$ we show $\mathcal{A} \models t_1 = t_2$:

1	$t_1 \equiv_\Gamma t_2$	since $\Gamma \models t_1 = t_2$
2	$h([t_1]) = h([t_2])$	since $[t_1] = [t_2]$
3	$t_1^{\mathcal{A}} = t_2^{\mathcal{A}}$	since $h([t_i]) = t_i^{\mathcal{A}}$
4	$\mathcal{A} \models t_1 = t_2$	by the definition of $\models$

2. Assuming that $\pi(t) \in \Gamma$ we show $\mathcal{A} \models \pi(t)$:

1	$[t] \in \pi^{(\mathcal{A}_\Gamma)}$	since $\Gamma \models \pi(t)$
2	$h([t]) \in \pi^{\mathcal{A}}$	since h is a homomorphism
3	$t^{\mathcal{A}} \in \pi^{\mathcal{A}}$	since $h([t]) = t^{\mathcal{A}}$
4	$\mathcal{A} \models \pi(t)$	by the definition of $\models$

□

Exercise 20

Show that the basic models of epi-basic sentences are unique up to an isomorphism, that is, if Γ is epi-basic and $\mathcal{A}_\Gamma$ and $\mathcal{B}_\Gamma$ are basic models of Γ then $\mathcal{A}_\Gamma \cong \mathcal{B}_\Gamma$.

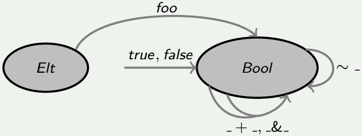
Convention

- For the sake of simplicity, we will identify a variable only by its name and sort provided that there is no danger of confusion.
- Using this convention, each inclusion $\iota\colon \Sigma \hookrightarrow \Sigma'$ is canonically extended to an inclusion of sentences $\iota\colon \mathtt{Sen}(\Sigma) \hookrightarrow \mathtt{Sen}(\Sigma')$, which corresponds to the approach of classical model theory.
- This convention simplifies the presentation greatly.
- A situation when we cannot apply this convention arises when translating a Σ-sentence $\forall X \cdot \gamma$ along the inclusion $\iota_X : \Sigma \hookrightarrow \Sigma[X]$.

Classical first-order reasoning

Example 21

$\Sigma_{\text{BOOL}} = (S_{\text{BOOL}}, F_{\text{BOOL}})$



Let Γ be a set of sentences over Σ_{BOOL} which consists of:

1. $\sim true = false$ and $\sim false = true$,
2. $\forall y \cdot y + \sim y = true$ and $\forall y \cdot y + y = y$,
3. $\forall y \cdot y \,\&\, \sim y = false$ and $\forall y \cdot y \,\&\, y = y$, and
4. $\forall x \cdot \sim foo(x) = foo(x)$.

By the ordinary rules of first-order deduction we get:

$$\begin{aligned} true &\overset{(2)}{=} foo(x) + \sim foo(x) \\ &\overset{(4)}{=} foo(x) + foo(x) \\ &\overset{(2)}{=} foo(x) \\ &\overset{(3)}{=} foo(x) \,\&\, foo(x) \\ &\overset{(4)}{=} foo(x) \,\&\, \sim foo(x) \\ &\overset{(3)}{=} false \end{aligned} \tag{1}$$

As a result of this deduction, one would expect that $true = false$ holds in all algebras satisfying Γ defined in Example 21. But this is not the case.

Classical first-order reasoning

Example 22

Let Σ_{BOOL} and Γ be the signature and the set of sentences defined in Example 21.
Let $\mathcal{A}$ be $T_{(\Sigma_{\mathrm{BOOL}})}/_{\equiv_\Gamma}$:

- $\mathcal{A}_{Elt} = \emptyset$ and $\mathcal{A}_{Bool} = \{true, false\}$,
- $\sim$ is interpreted as the negation, & as the conjunction and $+$ as the disjunction, and
- $foo^{\mathcal{A}}$ is the empty function.

- $\mathcal{A} \models_{(\Sigma_{\mathrm{BOOL}})} \forall x \cdot \sim foo(x) = foo(x)$, since there is no function from $\{x\}$ to $\mathcal{A}_{Elt} = \emptyset$.
- It follows that $\mathcal{A} \models_{(\Sigma_{\mathrm{BOOL}})} \Gamma$ but $\mathcal{A} \not\models_{(\Sigma_{\mathrm{BOOL}})} true = false$.
- This means that the usual first-order rules of deduction are not sound in the many-sorted case.
- This already shows that passing from unsorted to the many-sorted case is not straightforward as one would expect.

Entailment relations

(*Monotonicity*) $\dfrac{\Gamma' \subseteq \Gamma}{\Gamma \vdash_\Sigma \Gamma'}$	(*Transitivity*) $\dfrac{\Gamma \vdash_\Sigma \Gamma' \quad \Gamma' \vdash_\Sigma \Gamma''}{\Gamma \vdash_\Sigma \Gamma''}$
(*Unions*) $\dfrac{\Gamma \vdash_\Sigma \gamma' \text{ for all } \gamma' \in \Gamma'}{\Gamma \vdash_\Sigma \Gamma'}$	(*Translation*) $\dfrac{\Gamma \vdash_\Sigma \Gamma' \quad \chi\colon \Sigma \to \Sigma'}{\chi(\Gamma) \vdash_{\Sigma'} \chi(\Gamma')}$

Table: Entailment properties

Definition 23 (Entailment relation)

An *entailment relation* is a family of binary relations between sets of sentences indexed by signatures, that is, $\vdash := \{\vdash_\Sigma\}_{\Sigma \in |\mathtt{Sig}|}$ and $\vdash_\Sigma \subseteq \mathcal{P}(\mathtt{Sen}(\Sigma)) \times \mathcal{P}(\mathtt{Sen}(\Sigma))$ for all $\Sigma \in |\mathtt{Sig}|$, closed under the properties described in the table above.

Lemma 24

The satisfaction relation $\models$ is an entailment relation.

Proof.

Obviously $\models$ is monotonic, it is closed under unions and it is transitive. The closure of $\models$ under signature morphisms is a direct consequence of the satisfaction condition. □

Entailment relations

Definition 25 (Soundness)

An entailment relation $\vdash$ is *sound* if $\vdash \subseteq \models$.

Definition 26 (Completeness)

An entailment relation $\vdash$ is *complete* if $\models \subseteq \vdash$.

Definition 27 (Compactness)

An entailment relation $\vdash$ is *compact* if for each entailment $\Gamma \vdash_{\Sigma} E$ and each finite set $E_f \subseteq E$ there exists a finite signature $\Sigma_f \subseteq \Sigma$ and a finite set $\Gamma_f \subseteq \Gamma$ such that $\Gamma_f \vdash_{\Sigma_f} E_f$.

In Definition 27, both Γ_f and E_f are sets of sentences over the signature Σ_f.

Largest compact entailment relation

Lemma 28

For any entailment relation $\vdash$ there exists the largest compact entailment (sub)relation $\vdash^c \subseteq \vdash$:

- $\Gamma \vdash^c_\Sigma E$ *if for any finite* $E_f \subseteq E$ *there exist* $\Sigma_f \subseteq \Sigma$ *finite and* $\Gamma_f \subseteq \Gamma$ *finite such that* $\Gamma_f \vdash_{\Sigma_f} E_f$.

Proof.

Firstly, we show that $\vdash^c$ satisfies the entailment properties defined on page 16.

- *Monotonicity:* Assume $\Gamma \subseteq \Gamma'$. Let $\Gamma_f \subseteq \Gamma$ be a finite set.

1	$\Gamma_f \subseteq \mathtt{Sen}(\Sigma_f)$ for some finite signature $\Sigma_f \subseteq \Sigma$	since Γ_f is finite
2	$\Gamma_f \vdash_{\Sigma_f} \Gamma_f$	by (Monotonicity)
3	$\Gamma' \vdash^c_\Sigma \Gamma$	since $\Gamma_f \subseteq \Gamma \subseteq \Gamma'$

- *Transitivity:* Assume that $\Gamma \vdash^c_\Sigma \Gamma'$ and $\Gamma' \vdash^c_\Sigma \Gamma''$.

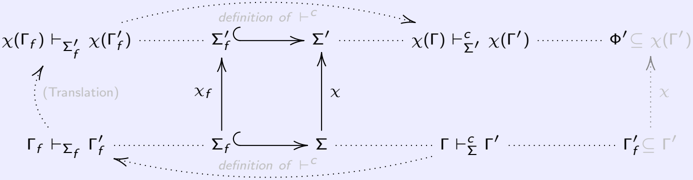
1	let $\Gamma''_f \subseteq \Gamma''$ be a finite subset	
2	$\Gamma'_f \vdash_{\Sigma_0} \Gamma''_f$ for some finite signature $\Sigma_0 \subseteq \Sigma$ and finite set $\Gamma'_f \subseteq \Gamma'$	since $\Gamma' \vdash^c_\Sigma \Gamma''$
3	$\Gamma_f \vdash_{\Sigma_1} \Gamma'_f$ for some finite signature $\Sigma_1 \subseteq \Sigma$ and finite set $\Gamma_f \subseteq \Gamma$	since $\Gamma \vdash^c_\Sigma \Gamma'$
4	$\Sigma_f := \Sigma_0 \cup \Sigma_1$ is finite	since Σ_0 and Σ_1 are finite
5	$\Gamma'_f \vdash_{\Sigma_f} \Gamma''_f$	from 2, by (Translation)
6	$\Gamma_f \vdash_{\Sigma_f} \Gamma'_f$	from 3, by (Translation)
8	$\Gamma_f \vdash_{\Sigma_f} \Gamma''_f$	from 5 and 6, by (Transitivity)
9	$\Gamma \vdash^c_\Sigma \Gamma''$	since Γ''_f is an arbitrary finite subset of Γ''

Proof of Lemma 28.

- *Union:* Assume that $\Gamma \vdash^c_\Sigma \varphi$ for all $\varphi \in \Phi$. Let $\Phi_f \subseteq \Phi$ be a finite subset.

1	for each $\varphi \in \Phi_f$ there exist $\Sigma_\varphi \subseteq \Sigma$ fine and $\Gamma_f \subseteq \Gamma$ finite such that $\Gamma_\varphi \vdash_{\Sigma_\varphi} \varphi$	since $\Gamma \vdash^c_\Sigma \varphi$ for all $\varphi \in \Phi_f$
2	$\Sigma_f := \bigcup_{\varphi \in \Phi_f} \Sigma_\varphi$ is finite	since Φ_f is finite and Σ_φ is finite for all $\varphi \in \Phi_f$
3	$\Gamma_\varphi \vdash_{\Sigma_f} \varphi$ for all $\varphi \in \Phi_f$	from 1, by (Translation)
4	$\Gamma_f := \bigcup_{\varphi \in \Phi_f} \Gamma_\varphi$ is finite	since Φ_f is finite and Γ_φ is finite for all $\varphi \in \Phi_f$
5	$\Gamma_f \vdash_{\Sigma_f} \varphi$ for all $\varphi \in \Phi_f$	from 3 and 4, by (Monotonicity) and (Transitivity)
6	$\Gamma_f \vdash_{\Sigma_f} \Phi_f$	by (Union)
7	$\Gamma \vdash^c_\Sigma \Phi$	since $\Phi_f \subseteq \Phi$ is an arbitrary finite subset of Φ

- *Translation:* Assume $\Gamma \vdash^c_\Sigma \Gamma'$ and $\chi : \Sigma \to \Sigma'$. Let $\Phi' \subseteq \chi(\Gamma')$ be a finite subset.



1	$\chi(\Gamma'_f) = \Phi'$ for some $\Gamma'_f \subseteq \Gamma'$ finite	since Φ' is finite
2	$\Gamma_f \vdash_{\Sigma_f} \Gamma'_f$ for some $\Sigma_f \subseteq \Sigma$ finite and $\Gamma_f \subseteq \Gamma$ finite	by the definition of $\vdash^c$
3	$\chi(\Gamma_f) \vdash_{\Sigma'_f} \chi(\Gamma'_f)$	by (Translation), since $\vdash$ is an entailment relation
4	$\chi(\Gamma) \vdash^c_{\Sigma'} \chi(\Gamma')$	by the definition of $\vdash^c$

Secondly, by its definition, $\vdash^c$ is the largest compact entailment relation included in $\vdash$. □

Basic first-order proof rules

For the rest of the lecture, we restrict the sentences to

1. (ground) equations $t = t'$, and
2. (ground) relations $\pi(t_1, \ldots, t_n)$.

$$\text{(R)}\ \frac{}{\Gamma \vdash_\Sigma t = t} \qquad \text{(S)}\ \frac{\Gamma \vdash_\Sigma t = t'}{\Gamma \vdash_\Sigma t' = t} \qquad \text{(T)}\ \frac{\Gamma \vdash_\Sigma t = t' \quad \Gamma \vdash_\Sigma t' = t''}{\Gamma \vdash_\Sigma t = t''}$$

$$\text{(F)}\ \frac{\Gamma \vdash_\Sigma t_1 = t_1' \ldots \Gamma \vdash_\Sigma t_n = t_n'}{\Gamma \vdash_\Sigma \sigma(t_1, \ldots, t_n) = \sigma(t_1', \ldots, t_n')} \qquad \text{(P)}\ \frac{\Gamma \vdash_\Sigma \pi(t_1, \ldots, t_n) \quad \Gamma \vdash_\Sigma t_1 = t_1' \ldots \Gamma \vdash_\Sigma t_n = t_n'}{\Gamma \vdash_\Sigma \pi(t_1', \ldots, t_n')}$$

Table: Basic first-order proof rules

Definition 29 (Basic entailment relation)

The *basic entailment relation* $\vdash^b$ is the least binary relation on sets of sentences closed under

1. all entailment properties described on page 16 except (Translation), and
2. the basic proof rules described above.

Basic first-order entailment relation

Remark 30

According to Definition 29 $\vdash^b = \bigcup_{i\in\omega} \vdash^i$, *where the chain of relations* $\vdash^0, \vdash^1, \ldots$ *is defined inductively:*

(Monotonicity) $\dfrac{\Gamma \subseteq \Gamma'}{\Gamma' \vdash^0_\Sigma \Gamma}$

(Transitivity) $\dfrac{\Gamma \vdash^i_\Sigma \Gamma' \quad \Gamma' \vdash^i_\Sigma \Gamma''}{\Gamma \vdash^{i+1}_\Sigma \Gamma''}$

(Union) $\dfrac{\Gamma \vdash^i_\Sigma \varphi \text{ for all } \varphi \in \Phi}{\Gamma \vdash^{i+1}_\Sigma \Phi}$

(R) $\dfrac{}{\Gamma \vdash^0_\Sigma t = t}$

(S) $\dfrac{\Gamma \vdash^i_\Sigma t = t'}{\Gamma \vdash^{i+1}_\Sigma t' = t}$

(T) $\dfrac{\Gamma \vdash^i_\Sigma t = t' \quad \Gamma \vdash^i_\Sigma t' = t''}{\Gamma \vdash^{i+1}_\Sigma t = t''}$

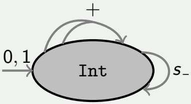
(F) $\dfrac{\Gamma \vdash^i_\Sigma t_1 = t'_1 \ldots \Gamma \vdash^i_\Sigma t_n = t'_n}{\Gamma \vdash^{i+1}_\Sigma \sigma(t_1, \ldots, t_n) = \sigma(t'_1, \ldots, t'_n)}$

(P) $\dfrac{\Gamma \vdash^i_\Sigma \pi(t_1, \ldots, t_n) \quad \Gamma \vdash^i_\Sigma t_1 = t'_1 \ldots \Gamma \vdash^i_\Sigma t_n = t'_n}{\Gamma \vdash^{i+1}_\Sigma \pi(t'_1, \ldots, t'_n)}$

Basic first-order reasoning

Example 31

$\Sigma = (S, F)$



Let Γ be the set of Σ-sentences which consists of:

- $s\ 0 = 1$
- $s\ s\ 0 = 0$
- $0 + 0 = 0$
- $s\ 0 + s\ 0 = 0$
- $0 + s\ 0 = s(0 + 0)$
- $s\ 0 + 0 = s(0 + 0)$

The following is a proof of the fact $\Gamma \vdash^b 1 + 1 = 0$, where Γ is defined in Example 31.

$$\text{(T)}\ \dfrac{\text{(F)}\ \dfrac{\Gamma \vdash^0 1 = s\ 0 \qquad \Gamma \vdash^0 1 = s\ 0}{\Gamma \vdash^1 1 + 1 = s\ 0 + s\ 0} \qquad \Gamma \vdash^0 s\ 0 + s\ 0 = 0}{\Gamma \vdash^2 1 + 1 = 0}$$

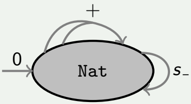
Exercise 32

Prove that $\Gamma \vdash^b (1 + 1) + 1 = 1$, where Γ is defined in Example 31.

Basic first-order reasoning

Example 33

$\Sigma_{\mathtt{NAT}} = (S_{\mathtt{NAT}}, F_{\mathtt{NAT}})$



Let $\Gamma_{\mathtt{NAT}}$ be the set of $\Sigma_{\mathtt{NAT}}$-sentences which consists of:

- $0 + t = t$ for all $t \in T_{\Sigma_{\mathtt{NAT}}}$, and
- $s\ t_1 + t_2 = s(t_1 + t_2)$ for all $t_1, t_2 \in T_{\Sigma_{\mathtt{NAT}}}$.

$$\text{(T)}\ \dfrac{\text{(T)}\ \dfrac{\Gamma_{\mathtt{NAT}} \vdash^0 s\ s\ 0 + s\ 0 = s(s\ 0 + s\ 0) \qquad \text{(F)}\ \dfrac{\Gamma_{\mathtt{NAT}} \vdash^0 s\ 0 + s\ 0 = s(0 + s\ 0)}{\Gamma_{\mathtt{NAT}} \vdash^1 s(s\ 0 + s\ 0) = s\ s(0 + s\ 0)}}{\Gamma_{\mathtt{NAT}} \vdash^2 s\ s\ 0 + s\ 0 = s\ s(0 + s\ 0)} \qquad \text{(F)}\ \dfrac{\text{(F)}\ \dfrac{\Gamma_{\mathtt{NAT}} \vdash^0 0 + s\ 0 = s\ 0}{\Gamma_{\mathtt{NAT}} \vdash^1 s(0 + s\ 0) = s\ s\ 0}}{\Gamma_{\mathtt{NAT}} \vdash^2 s\ s(0 + s\ 0) = s\ s\ s\ 0}}{\Gamma_{\mathtt{NAT}} \vdash^3 s\ s\ 0 + s\ 0 = s\ s\ s\ 0}$$

Basic entailment relation is well-defined

Lemma 34 (Basic entailment relation is well-defined)

The basic entailment relation $\vdash^b = \bigcup_{i\in\omega} \vdash^i$ *is well-defined.*

Proof.

1. $\boxed{\vdash^i \subseteq \vdash^{i+1} \text{ for all } i \in \omega}$

   If $\Gamma \vdash^i_\Sigma \Gamma$ then by (Transitivity), $\Gamma \vdash^{i+1}_\Sigma \Gamma$. Hence, $\Gamma \vdash^i_\Sigma \Gamma$ for all $i \in \omega$.

   If $\Gamma \vdash^i_\Sigma \Gamma'$ then since $\Gamma' \vdash^i_\Sigma \Gamma'$, by (Transitivity), $\Gamma \vdash^{i+1}_\Sigma \Gamma'$.

   Hence, $\vdash^i \subseteq \vdash^{i+1}$ for all $i \in \omega$.

2. $\boxed{\vdash^b \text{ is closed under (Translation)}}$

   It is straightforward to show that $\vdash^b$ is closed under the basic proof rules and all the entailment properties except (Translation). In order to show that $\vdash^b$ is closed under (Translation) it suffices to show that $\chi(\vdash^i_\Sigma) \subseteq \vdash^i_{\Sigma'}$, *for all* $i \in \omega$:

   - *Monotonicity:* Assume that $\Gamma \subseteq \Gamma' \subseteq \mathtt{Sen}(\Sigma)$, which means $\Gamma \vdash^0_\Sigma \Gamma'$.
     Then $\chi(\Gamma) \subseteq \chi(\Gamma')$, which means $\chi(\Gamma) \vdash^0_{\Sigma'} \chi(\Gamma')$.
   - *R:* For all $\Gamma \subseteq \mathtt{Sen}(\Sigma)$ and all $t \in T_\Sigma$, we have $\Gamma \vdash^0_\Sigma t = t$.
     Let $\chi^{tm} : T_\Sigma \to T_{\Sigma'} \upharpoonright_\chi$ be the unique homomorphism.
     We have $\chi(\Gamma) \vdash^0_{\Sigma'} \chi^{tm}(t) = \chi^{tm}(t)$, which means $\chi(\Gamma) \vdash^0_{\Sigma'} \chi(t = t)$.

Proof of Lemma 34.

- *Transitivity:* Assume $\Gamma \vdash^i_\Sigma \Gamma'$ and $\Gamma' \vdash^i_\Sigma \Gamma''$, which by (Transitivity) means $\Gamma \vdash^{i+1}_\Sigma \Gamma''$.
  By the induction hypothesis, we have $\chi(\Gamma) \vdash^i_{\Sigma'} \chi(\Gamma')$ and $\chi(\Gamma') \vdash^i_{\Sigma'} \chi(\Gamma'')$.
  By (Transitivity), $\chi(\Gamma) \vdash^{i+1}_{\Sigma'} \chi(\Gamma'')$.
- *F:* Assume $\Gamma \vdash^i_\Sigma t_1 = t_1' \ldots \Gamma \vdash^i_\Sigma t_1 = t_1'$, which implies $\Gamma \vdash^{i+1}_\Sigma \sigma(t_1, \ldots, t_n) = \sigma(t_1', \ldots, t_n')$.
  By induction hypothesis, $\chi(\Gamma) \vdash^i_\Sigma \chi^{tm}(t_1) = \chi^{tm}(t_1') \ldots \chi(\Gamma) \vdash^i_\Sigma \chi^{tm}(t_n) = \chi^{tm}(t_n')$.
  By (F), $\chi(\Gamma) \vdash^{i+1}_{\Sigma'} \chi(\sigma)(\chi^{tm}(t_1), \ldots, \chi^{tm}(t_n)) = \chi(\sigma)(\chi^{tm}(t_1'), \ldots, \chi^{tm}(t_n'))$.
  Hence, $\chi(\Gamma) \vdash^{i+1}_{\Sigma'} \chi(\sigma(t_1, \ldots, t_n) = \sigma(t_1', \ldots, t_n'))$.

The remaining cases can be proved similarly.

3. $\vdash^b$ is the least entailment relation closed under the basic proof rules

Let $\vdash'$ be another entailment relation closed under the basic proof rules.
One can straightforwardly prove that $\vdash^i \subseteq \vdash'$ for all $i \in \omega$.
It follows that $\vdash^b \subseteq \vdash'$.

□

Basic soundness

Theorem 35 (Basic soundness)

The basic entailment relation $\vdash^b$ is sound.

Proof.

We show that $\models$ is closed under all basic proof rules defined on page 20.

- *R:* Obviously, $\Gamma \models_\Sigma t = t$.
- *S:* If $\Gamma \models_\Sigma t_1 = t_2$ then $\Gamma \models_\Sigma t_2 = t_1$.
- *T:* If $\Gamma \models_\Sigma t_1 = t_2$ and $\Gamma \models_\Sigma t_2 = t_3$ then $\Gamma \models_\Sigma t_1 = t_3$.
- *F:* If $\Gamma \models_\Sigma t_1 = u_1 \ldots \Gamma \models_\Sigma t_n = u_n$ then $\Gamma \models_\Sigma \sigma(t_1, \ldots, t_n) = \sigma(u_1, \ldots, u_n)$:

1	assume $\mathcal{A} \models \Gamma$	
2	$\mathcal{A} \models t_i = u_i$	since $\mathcal{A} \models \Gamma$ and $\Gamma \models t_i = u_i$
3	$\sigma^{\mathcal{A}}(t_1^{\mathcal{A}}, \ldots, t_n^{\mathcal{A}}) = \sigma^{\mathcal{A}}(u_1^{\mathcal{A}}, \ldots, u_n^{\mathcal{A}})$	since $t_i^{\mathcal{A}} = u_i^{\mathcal{A}}$ for all $i \in \{1, \ldots, n\}$
4	$\mathcal{A} \models \sigma(t_1, \ldots, t_n) = \sigma(u_1, \ldots, u_n)$	by definition
5	$\Gamma \models \sigma(t_1, \ldots, t_n) = \sigma(u_1, \ldots, u_n)$	since $\mathcal{A}$ was arbitrarily chosen

- *P:* Similar to the case above.

Since $\vdash^b$ is the least entailment relation closed under the basic proof rules, $\vdash^b \subseteq \models$. □

Basic compactness

Theorem 36 (Basic compactness)

The basic entailment relation $\vdash^b$ is compact.

Proof.

Let $\vdash^c$ be the largest compact entailment relation included in $\vdash^b$.
It is straightforward to show that $\vdash^c$ is closed under all basic proof rules:

> For example, consider the case corresponding to (T), and assume that $\Gamma \vdash^c_\Sigma t_1 = t_2$ and $\Gamma \vdash^c_\Sigma t_2 = t_3$. By definition, $\Gamma_0 \vdash^b_{\Sigma_0} t_1 = t_2$ and $\Gamma_1 \vdash^b_{\Sigma_1} t_2 = t_3$, for some finite $\Sigma_i \subseteq \Sigma$ and some finite $\Gamma_i \subseteq \Gamma$, where $i \in \{0, 1\}$. Let $\Sigma_f = \Sigma_0 \cup \Sigma_1$ and $\Gamma_f = \Gamma_0 \cup \Gamma_1$. By (*Translation*) and (*Monotonicity*), we get $\Gamma_f \vdash^b_{\Sigma_f} t_1 = t_2$ and $\Gamma_f \vdash^b_{\Sigma_f} t_2 = t_3$. By (T), $\Gamma_f \vdash^b_{\Sigma_f} t_1 = t_3$. Hence, $\Gamma \vdash^c_\Sigma t_1 = t_3$.

Since $\vdash^b$ is the least entailment relation closed under the basic proof rules, $\vdash^b \subseteq \vdash^c$.
By definition, $\vdash^c \subseteq \vdash^b$.
It follows that $\vdash^c = \vdash^b$.
Hence, $\vdash^b$ is compact. □

Basic completeness

Lemma 37

For all signatures Σ and all sets Γ of atomic sentences over Σ, the relation $\equiv_\Gamma := \{(t, t') \mid \Gamma \vdash^b t = t'\}$ is a congruence on T_Σ.

Proof.

Straightforward, by the basic proof rules (R), (S), (T) and (F). □

Lemma 38

Let Γ be a set of atomic sentences over a signature Σ and let $\equiv_\Gamma$ be the congruence defined in Lemma 37. Let $\mathcal{A}_\Gamma$ be the model obtained from $T_\Sigma/_{\equiv_\Gamma}$ by interpreting each $(\pi : w) \in P$ as $\{[t] \mid \Gamma \vdash^b \pi(t)\}$. Then $\mathcal{A}_\Gamma \models \Gamma$.

Proof.

Notice that $\pi^{\mathcal{A}_\Gamma}$ is well-defined:

if $\Gamma \vdash^b \pi(t)$ and $[t] = [t']$ then $\Gamma \vdash^b t = t'$, and by (P), $\Gamma \vdash^b \pi(t')$.

We show that $\mathcal{A}_\Gamma \models \Gamma$:

1. $\Gamma \vdash^b t = t'$ iff $[t] = [t']$ iff $\mathcal{A}_\Gamma \models t = t'$.
2. $\Gamma \vdash^b \pi(t)$ iff $[t] \in \pi^{\mathcal{A}_\Gamma}$ iff $\mathcal{A}_\Gamma \models \pi(t)$.

□

Basic completeness

Theorem 39 (Basic completeness)

For any set of atomic sentences Γ *over a signature* Σ*, the following are equivalent:*

(a) $\Gamma \models \rho$*, (b)* $\mathcal{A}_\Gamma \models \rho$ *and (c)* $\Gamma \vdash^b \rho$*,*

for all atomic sentences ρ *over* Σ*, where* $\mathcal{A}_\Gamma$ *is the model defined in Lemma 38.*

Proof.

$(a) \Rightarrow (b)$ Since $\mathcal{A}_\Gamma \models \Gamma$ and $\Gamma \models \rho$, we get $\mathcal{A}_\Gamma \models \rho$.

$(b) \Leftrightarrow (c)$

By the proof of Lemma 38:

1. $\Gamma \vdash^b t = t'$ iff $[t] = [t']$ iff $\mathcal{A}_\Gamma \models t = t'$.
2. $\Gamma \vdash^b \pi(t)$ iff $[t] \in \pi^{\mathcal{A}_\Gamma}$ iff $\mathcal{A}_\Gamma \models \pi(t)$.

$(c) \Rightarrow (a)$ By soundness, $\Gamma \vdash^b \rho$ implies $\Gamma \models \rho$. □