Many-Sorted First-Order Model Theory lecture 2

Daniel Găină

Institute of Mathematics for Industry
Kyushu University

Notations

Let Σ be a signature.

- The set of sentences over Σ is denoted by $\mathtt{Sen}(\Sigma)$.
- The class of models over Σ is denoted by $|\mathtt{Mod}(\Sigma)|$.
- The class of homomorphisms over Σ is denoted by $\mathtt{Mod}(\Sigma)$.

Let $\Gamma \subseteq \mathtt{Sen}(\Sigma)$ and $\gamma \in \mathtt{Sen}(\Sigma)$.

- We write $\Gamma \models \gamma$ if $\mathcal{A} \models \Gamma$ implies $\mathcal{A} \models \gamma$ for all $\mathcal{A} \in |\mathtt{Mod}(\Sigma)|$.
  In this case, we say that γ is a semantic consequence of Γ.

Exercise 1

Let $\Sigma_{\mathtt{NAT+}}$ be the signature of natural numbers with addition and multiplication. Then

- $\forall x, y \cdot x + y = y + x \models \forall x \cdot x + 0 = 0 + x$.

Theories and presentations

Definition 2

- A *presentation* is a pair (Σ, Γ), where Σ is a signature and Γ is a set of Σ-sentences.
- $\Gamma^\bullet := \{\mathcal{A} \in |\mathtt{Mod}(\Sigma)| \mid \mathcal{A} \models_\Sigma \Gamma\}$ for all sets of sentences $\Gamma \subseteq \mathtt{Sen}(\Sigma)$;
- $\mathrm{M}^\bullet := \{\gamma \in \mathtt{Sen}(\Sigma) \mid \mathcal{A} \models_\Sigma \gamma$ for each $\mathcal{A} \in \mathrm{M}\}$ for all classes of models $\mathrm{M} \subseteq |\mathtt{Mod}(\Sigma)|$.
- A presentation (Σ, Γ) such that $\Gamma = \Gamma^{\bullet\bullet}$ is called a *theory*.
- A class of models $M \subseteq |\mathtt{Mod}(\Sigma)|$ such that $M^{\bullet\bullet} = M$ is called *elementary*.

Exercise 3

Let Σ be a signature, and $\Gamma \subseteq \mathtt{Sen}(\Sigma)$ a set of sentences.

1. *The set of semantic consequences of Γ is a theory, that is, $\{\gamma \mid \Gamma \models \gamma\}$ is a theory.*
2. *The class of all models satisfying Γ is elementary, that is, $|\mathtt{Mod}(\Sigma, \Gamma)|$ is elementary.*

Theories and presentations

Exercise 4

Let Σ *be a signature.*

1. $(\bigcup_{i\in I}\Gamma_i)^\bullet = \bigcap_{i\in I}\Gamma_i^\bullet$ *for all families of sets of* Σ*-sentences* $\{\Gamma_i\}_{i\in I}$.
2. $(\bigcup_{i\in I}\mathrm{M}_i)^\bullet = \bigcap_{i\in I}\mathrm{M}_i^\bullet$ *for all families of classes of* Σ*-models* $\{\mathrm{M}_i\}_{i\in I}$.

Exercise 5

The pair of functions $(\_)^\bullet$ *from Definition 2 forms a Galois connection, i.e. for all sets of* Σ*-sentences* $\Gamma_1, \Gamma_2, \Gamma$ *and all classes of* Σ*-models* $\mathrm{M}_1, \mathrm{M}_2, \mathrm{M}$, *we have:*

1. $\Gamma_1 \subseteq \Gamma_2$ *implies* $\Gamma_2^\bullet \subseteq \Gamma_1^\bullet$
2. $\mathrm{M}_1 \subseteq \mathrm{M}_2$ *implies* $\mathrm{M}_2^\bullet \subseteq \mathrm{M}_1^\bullet$
3. $\Gamma \subseteq \Gamma^{\bullet\bullet}$
4. $\mathrm{M} \subseteq \mathrm{M}^{\bullet\bullet}$
5. $\Gamma^\bullet = \Gamma^{\bullet\bullet\bullet}$
6. $\mathrm{M}^\bullet = \mathrm{M}^{\bullet\bullet\bullet}$

Signature morphisms

Definition 6 (Signature morphisms)

A signature morphism $\chi = (\chi^{st}, \chi^{op}, \chi^{rl}) \colon \Sigma \to \Sigma'$ consists of

- a function between the set of sorts $\chi^{st} \colon S \to S'$
- a family of functions between the sets of function symbols $\chi^{op} = \{\chi^{op} \colon F_{w \to s} \to F'_{\chi^{st}(w) \to \chi^{st}(s)}\}_{(w,s) \in S^* \times S}$,

  $$\sigma : s_1 \ldots s_n \to s \overset{\chi}{\longmapsto} \chi^{op}(\sigma) : \chi^{st}(s_1) \ldots \chi^{st}(s_n) \to \chi^{st}(s)$$
- a family of function between the sets of relation symbols $\chi^{rl} = \{\chi^{rl} \colon P_w \to P'_{\chi^{st}(w)}\}_{w \in S^*}$

  $$\pi : s_1 \ldots s_n \overset{\chi}{\longmapsto} \chi^{rl}(\pi) : \chi^{st}(s_1) \ldots \chi^{st}(s_n)$$

When there is no danger of confusion we may drop the superscripts *st*, *op* or *rl* from the above notations.

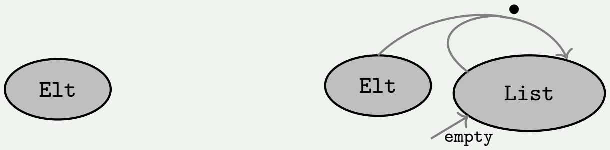
Examples of signature morphisms

Example 7

Signature extensions with variables $\Sigma \hookrightarrow \Sigma[X]$.

Example 8

$\Sigma_{\mathrm{ELT}} = (S_{\mathrm{ELT}}, F_{\mathrm{ELT}})$ $\Sigma_{\mathrm{LIST}} = (S_{\mathrm{LIST}}, F_{\mathrm{LIST}})$

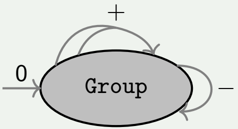


Let $\chi : \Sigma_{\mathrm{ELT}} \hookrightarrow \Sigma_{\mathrm{LIST}}$ be an inclusion.

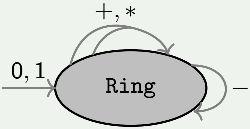
Examples of signature morphisms

Example 9

$\Sigma_{\texttt{GROUP}} = (S_{\texttt{GROUP}}, F_{\texttt{GROUP}})$



$\Sigma_{\texttt{RING}} = (S_{\texttt{RING}}, F_{\texttt{RING}})$



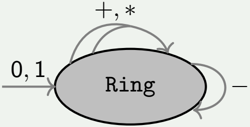
Let $\chi : \Sigma_{\texttt{GROUP}} \to \Sigma_{\texttt{RING}}$ be the signature morphism which renames the sort Group to Ring and it adds a new constant $(1 :\to \texttt{Ring})$ and a new binary function symbol $\_ * \_ : \texttt{Ring Ring} \to \texttt{Ring}$:

- $\texttt{Group} \mapsto \texttt{Ring}$,
- $(0 :\to \texttt{Group}) \mapsto (0 \mapsto \texttt{Ring})$
- $(-\_ : \texttt{Group} \to \texttt{Group}) \mapsto (-\_ : \texttt{Ring} \to \texttt{Ring})$
- $(\_ + \_ : \texttt{Group Group} \to \texttt{Group}) \mapsto (\_ + \_ : \texttt{Ring Ring} \to \texttt{Ring})$

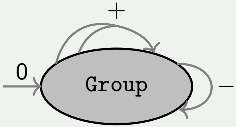
Examples of signature morphisms

Example 10

$\Sigma_{\texttt{RING}} = (S_{\texttt{RING}}, F_{\texttt{RING}})$



$\Sigma_{\texttt{GROUP}} = (S_{\texttt{GROUP}}, F_{\texttt{GROUP}})$



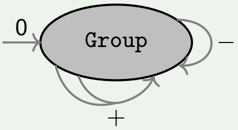
Let $\chi : \Sigma_{\texttt{RING}} \to \Sigma_{\texttt{GROUP}}$ be the the surjective signature morphism which maps:

- $\texttt{Ring} \mapsto \texttt{Group}$,
- $(1 :\to \texttt{Ring}) \mapsto (0 \mapsto \texttt{Group})$
- $(0 :\to \texttt{Ring}) \mapsto (0 \mapsto \texttt{Group})$
- $(-\_ : \texttt{Ring} \to \texttt{Ring}) \mapsto (-\_ : \texttt{Group} \to \texttt{Group})$
- $(\_ + \_ : \texttt{Ring Ring} \to \texttt{Ring}) \mapsto (\_ + \_ : \texttt{Group Group} \to \texttt{Group})$
- $(\_ * \_ : \texttt{Ring Ring} \to \texttt{Ring}) \mapsto (\_ + \_ : \texttt{Group Group} \to \texttt{Group})$

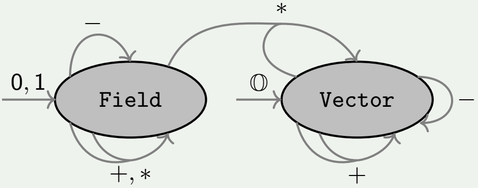
Examples of signature morphisms

Example 11

$\Sigma_{\texttt{GROUP}} = (S_{\texttt{GROUP}}, F_{\texttt{GROUP}})$



$\Sigma_{\texttt{VECTOR}} = (S_{\texttt{VECTOR}}, F_{\texttt{VECTOR}})$



Let $\chi : \Sigma_{\texttt{GROUP}} \to \Sigma_{\texttt{VECTOR}}$ be the injective signature morphism which maps:

- $\texttt{Group} \mapsto \texttt{Vector}$,
- $(0 :\to \texttt{Group}) \mapsto (\mathbb{O} \mapsto \texttt{Vector})$
- $(-\_ : \texttt{Group} \to \texttt{Group}) \mapsto (-\_ : \texttt{Vector} \to \texttt{Vector})$
- $(\_ + \_ : \texttt{Group Group} \to \texttt{Group}) \mapsto (\_ + \_ : \texttt{Vector Vector} \to \texttt{Vector})$

Model reducts

Definition 12 (Model reducts)

Given a signature morphism $\chi\colon \Sigma \to \Sigma'$ as in Definition 6,

1. the reduct $\mathcal{A}' \upharpoonright_\chi$ of a Σ'-model $\mathcal{A}'$ is a Σ-model defined as follows:
   - $(\mathcal{A}' \upharpoonright_\chi)_s = \mathcal{A}'_{\chi(s)}$ for all sorts $s \in S$,
   - $\sigma^{(\mathcal{A}' \upharpoonright_\chi)} = \chi(\sigma)^{\mathcal{A}'} : \mathcal{A}'_{\chi^{st}(w)} \to \mathcal{A}'_{\chi^{st}(s)}$ for all $(\sigma : w \to s) \in F$,
   - $\pi^{(\mathcal{A}' \upharpoonright_\chi)} = \chi(\pi)^{\mathcal{A}'}$ for all $(\pi : w) \in P$.
2. the reduct $h' \upharpoonright_\chi$ of a Σ'-homomorphism h' is a Σ-homomorphism $h' \upharpoonright_\chi = \{h'_{\chi(s)}\}_{s \in S}$.

- If $\mathcal{A}' \upharpoonright_\chi = \mathcal{A}$ then (a) $\mathcal{A}'$ is called a χ-*expansion* of $\mathcal{A}$, and (b) $\mathcal{A}$ is called the χ-*reduct* of $\mathcal{A}'$.
- If $\chi : \Sigma \hookrightarrow \Sigma'$ is an inclusion and $\mathcal{A}'$ is a Σ'-model, we may write $\mathcal{A}' \upharpoonright_\Sigma$ instead of $\mathcal{A}' \upharpoonright_\chi$.

Examples of model reducts

Example 13

- Let $\Sigma_{\mathtt{NAT}+}$ be the signature of natural numbers with addition and multiplication.
- Let $\iota : \Sigma_{\mathtt{NAT}+} \hookrightarrow \Sigma_{\mathtt{NAT}+}[x, y]$ be an extension of $\Sigma_{\mathtt{NAT}+}$ with two variables x and y.
- Let $\mathbb{N}$ be the $\Sigma_{\mathtt{NAT}+}$-model of natural numbers interpreting all symbols in the usual way.
- Let $f : \{x, y\} \to |\mathbb{N}|$ be the evaluation defined by $f(x) = 2$ and $f(y) = 5$.

Then $\langle \mathbb{N}, f \rangle \upharpoonright_{\iota} = \mathbb{N}$.

Example 14

- Let $\Sigma_{\mathtt{ELT}} \hookrightarrow \Sigma_{\mathtt{LIST}}$ be the inclusion defined in Example 8.
- Let $\mathbb{L}$ be the structure consisting of lists of natural numbers:
  - $\mathbb{L}_{\mathtt{Elt}} = \omega$, the set of natural numbers, and
  - $\mathbb{L}_{\mathtt{List}} = \{\mathtt{empty} \bullet j_1 \bullet j_2 \bullet \cdots \bullet j_n \mid j_1, \ldots, j_n \in \omega \text{ and } n \in \omega\}$, the set of all lists of natural numbers.

Then $\mathbb{L} \upharpoonright_{\Sigma_{\mathtt{ELT}}}$ is the model that interprets the sort `Elt` as ω.

Remark 15

Notice that the reduct changes the universe of $\mathbb{L}$ by discarding the set corresponding to the sort `List`*.*

Examples of model reducts

Example 16

- Let $\chi : \Sigma_{\texttt{GROUP}} \to \Sigma_{\texttt{RING}}$ be the injective signature morphism of Example 9.
- Let $\mathbb{Z}$ be the $\Sigma_{\texttt{RING}}$-model of integers.

Then $\mathbb{Z}\upharpoonright_\chi$ is obtained from $\mathbb{Z}$ by discarding the interpretation of the unit and the multiplication:

- $(\mathbb{Z}\upharpoonright_\chi)_{\texttt{Group}} = \mathbb{Z}_{\texttt{Ring}}$, the set of integers
- $0^{\mathbb{Z}\upharpoonright_\chi} = 0^{\mathbb{Z}}$, since χ maps $(0 :\to \texttt{Group})$ to $(0 :\to \texttt{Ring})$
- $-^{(\mathbb{Z}\upharpoonright_\chi)} = -^{\mathbb{Z}}$, since χ maps $(-\_ : \texttt{Group} \to \texttt{Group})$ to $(-\_ : \texttt{Ring} \to \texttt{Ring})$
- $+^{(\mathbb{Z}\upharpoonright_\chi)} = +^{\mathbb{Z}}$, since χ maps $(\_ + \_ : \texttt{Group Group} \to \texttt{Group})$ to $(\_ + \_ : \texttt{Ring Ring} \to \texttt{Ring})$

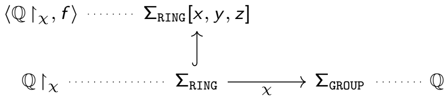
Examples of model reducts

Example 17

- Let $\chi : \Sigma_{\mathtt{RING}} \to \Sigma_{\mathtt{GROUP}}$ be the surjective signature morphism of Example 10.
- Let $\mathbb{Q}$ be the group of rationals defined over the signature $\Sigma_{\mathtt{GROUP}}$.

Then $\mathbb{Q}\upharpoonright_\chi$ is the model over $\Sigma_{\mathtt{RING}}$ which interprets

- both $0 :\to \mathtt{Ring}$ and $1 :\to \mathtt{Ring}$ as 0,
- both $\_ + \_ : \mathtt{Ring}\ \mathtt{Ring} \to \mathtt{Ring}$ and $\_ * \_ : \mathtt{Ring}\ \mathtt{Ring} \to \mathtt{Ring}$ as the addition.



Remark 18

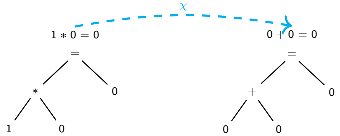
Notice that $\mathbb{Q}\upharpoonright_\chi$ is not a ring, since $\mathbb{Q}\upharpoonright_\chi$ doesn't satisfy the distributivity of multiplication over addition, that is, $\mathbb{Q}\upharpoonright_\chi \not\models \forall x, y, z \cdot x * (y + z) = (x * y) + (x * z)$. Define $f : \{x, y, z\} \to |\mathbb{Q}\upharpoonright_\chi|$ by $f(x) = f(y) = f(z) = 1$.

$$
\begin{aligned}
(x * (y + z))^{\langle \mathbb{Q}\upharpoonright\chi, f\rangle} &= x^{\langle \mathbb{Q}\upharpoonright\chi, f\rangle} *^{\langle \mathbb{Q}\upharpoonright\chi, f\rangle} (y^{\langle \mathbb{Q}\upharpoonright\chi, f\rangle} +^{\langle \mathbb{Q}\upharpoonright\chi, f\rangle} z^{\langle \mathbb{Q}\upharpoonright\chi, f\rangle}) \\
&= f(x) +^{\mathbb{Q}} (f(y) +^{\mathbb{Q}} f(z)) = 3 \neq 4 \\
&= (f(x) +^{\mathbb{Q}} f(y)) +^{\mathbb{Q}} (f(x) +^{\mathbb{Q}} f(z)) \\
&= (x^{\langle \mathbb{Q}\upharpoonright\chi, f\rangle} *^{\langle \mathbb{Q}\upharpoonright\chi, f\rangle} y^{\langle \mathbb{Q}\upharpoonright\chi, f\rangle}) +^{\langle \mathbb{Q}\upharpoonright\chi, f\rangle} (x^{\langle \mathbb{Q}\upharpoonright\chi, f\rangle} *^{\langle \mathbb{Q}\upharpoonright\chi, f\rangle} z^{\langle \mathbb{Q}\upharpoonright\chi, f\rangle}) \\
&= ((x * y) + (x * z))^{\langle \mathbb{Q}\upharpoonright\chi, f\rangle}
\end{aligned}
$$

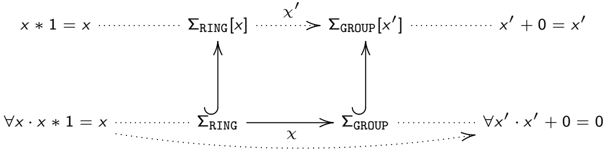
Sentence translations

- How do we translate $1 * 0 = 0$ along $\chi : \Sigma_{\mathtt{RING}} \to \Sigma_{\mathtt{GROUP}}$ defined in Example 10?

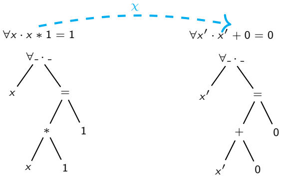
  Re: By replacing the symbols in $1 * 0 = 0$ according to χ, we obtain $0 + 0 = 0$.



- Let $x = \langle v_0, \mathtt{Ring}, \Sigma_{\mathtt{RING}} \rangle$ be a variable for $\Sigma_{\mathtt{RING}}$.
  How do we translate $\forall x \cdot x * 1 = x$ along $\chi : \Sigma_{\mathtt{RING}} \to \Sigma_{\mathtt{GROUP}}$?



Re: We define $x' := \langle v_0, \mathtt{Group}, \Sigma_{\mathtt{GROUP}} \rangle$, the translation of x along χ.
We define $\chi : \Sigma_{\mathtt{RING}}[x] \to \Sigma_{\mathtt{GROUP}}[x']$ the extension of χ that maps x to x'.
Then $\chi(\forall x \cdot x * 1 = x) := \forall x' \cdot x' + 0 = x'$.



Sentence translations

Definition 19 (Term translations)

Any signature morphism $\chi\colon \Sigma \to \Sigma'$ determines a function $\chi^{tm}\colon T_\Sigma \to T_{\Sigma'}$ inductively defined:

- $\chi^{tm}(c) = \chi^{op}(c)$ for all constants $(c :\to s) \in F$.
- $\chi^{tm}(\sigma(t_1, \ldots, t_n)) = \chi(\sigma)(\chi^{tm}(t_1), \ldots, \chi^{tm}(t_n))$, for all terms $\sigma(t_1, \ldots, t_n) \in T_\Sigma$.

We may drop the superscript *tm* from the above notations when there is no danger of confusion.

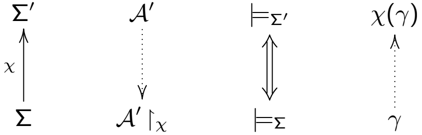
Definition 20 (Sentence translations)

Any signature morphism $\chi\colon \Sigma \to \Sigma'$ determines a function $\mathtt{Sen}(\chi)\colon \mathtt{Sen}(\Sigma) \to \mathtt{Sen}(\Sigma)$ which replaces the symbols from Σ with the symbols from Σ' according to χ:

- $\mathtt{Sen}(\chi)(t_1 = t_2) \coloneqq (\chi^{tm}(t_1) = \chi^{tm}(t_2))$
- $\mathtt{Sen}(\chi)(\pi(t_1, \ldots, t_n)) \coloneqq \chi^{op}(\pi)(\chi^{tm}(t_1), \ldots, \chi^{tm}(t_n))$
- $\mathtt{Sen}(\chi)(\neg\gamma) \coloneqq \neg\mathtt{Sen}(\chi)(\gamma)$
- $\mathtt{Sen}(\chi)(\bigvee \Gamma) \coloneqq \bigvee \mathtt{Sen}(\chi)(\Gamma)$
- $\mathtt{Sen}(\chi)(\exists X \cdot \gamma) \coloneqq \exists X' \cdot \mathtt{Sen}(\chi')(\gamma)$, where
  - $X' = \{\langle v_i, \chi(s), \Sigma'\rangle \mid \langle v_i, s, \Sigma\rangle \in X\}$, and
  - $\chi'\colon \Sigma[X] \to \Sigma'[X']$ is the extension of χ which maps each $\langle v_i, s, \Sigma\rangle \in X$ to $\langle v_i, \chi(s), \Sigma'\rangle \in X'$.

We denote $\mathtt{Sen}(\chi)$ simply by χ when there is no danger of confusion.

Satisfaction condition



Theorem 21 (Satisfaction is invariant w.r.t. change of notation)

For all signature morphisms $\chi\colon \Sigma \to \Sigma'$, *all* Σ'*-models* $\mathcal{A}'$, *and all* Σ*-sentences* γ, *we have*

$$\mathcal{A}' \models_{\Sigma'} \chi(\gamma) \textit{ iff } \mathcal{A}' \restriction_{\chi} \models_{\Sigma} \gamma$$

In order to prove Theorem 21, we need two preliminary results: Lemma 22 and Lemma 23.

Lemma 22

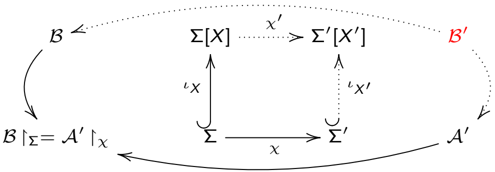
For all signature morphisms $\chi\colon \Sigma \to \Sigma'$*, all* Σ'*-models* $\mathcal{A}'$*, and all* Σ*-terms* t*, we have* $\chi(t)^{\mathcal{A}'} = t^{(\mathcal{A}'\upharpoonright_\chi)}$.

Proof.

We proceed by induction on the structure of terms:

- $\chi(c)^{\mathcal{A}'} \overset{def}{=} c^{(\mathcal{A}'\upharpoonright_\chi)}$ for all constants $(c :\to s) \in F$
- $\chi(\sigma(t_1, \ldots, t_n))^{\mathcal{A}'} =$
  $\chi(\sigma)(\chi(t_1), \ldots, \chi(t_n))^{\mathcal{A}'} =$
  $\chi(\sigma)^{\mathcal{A}'}(\chi(t_1)^{\mathcal{A}'}, \ldots, \chi(t_n)^{\mathcal{A}'}) \overset{IH}{=}$
  $\sigma^{(\mathcal{A}'\upharpoonright_\chi)}(t_1^{(\mathcal{A}'\upharpoonright_\chi)}, \ldots, t_n^{(\mathcal{A}'\upharpoonright_\chi)}) =$
  $\sigma(t_1, \ldots, t_n)^{(\mathcal{A}'\upharpoonright_\chi)}$.

□



Lemma 23

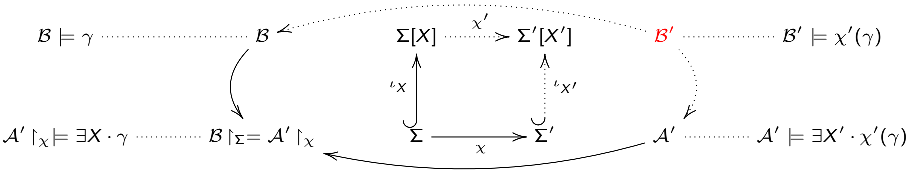
Assume a signature morphism $\chi\colon \Sigma \to \Sigma'$, a finite set of variables X for Σ, and a Σ'-model $\mathcal{A}'$, a $\Sigma[X]$-model $\mathcal{B}$ such that $\mathcal{A}'\upharpoonright_{\chi} = \mathcal{B}\upharpoonright_{\Sigma}$.
Then there exists a unique $\Sigma'[X']$-model $\mathcal{B}'$ such that $\mathcal{B}'\upharpoonright_{\chi'} = \mathcal{B}$ and $\mathcal{B}'\upharpoonright_{\Sigma'} = \mathcal{A}'$, where (a) X' is the translation of X along χ, (b) $\chi'\colon \Sigma[X] \to \Sigma'[X']$ is the extension of χ mapping each variable $\langle v_i, s, \Sigma\rangle \in X$ to $\langle v_i, \chi(s), \Sigma'\rangle \in X'$, and (c) $\iota_{X'}\colon \Sigma' \hookrightarrow \Sigma'[X']$ is an inclusion.

Proof.

We define $\mathcal{B}'$ as follows: (a) $\mathcal{B}'$ interprets each symbol in Σ' as $\mathcal{A}'$, and (b) $\langle v_i, \chi(s), \Sigma'\rangle^{\mathcal{B}'} = \langle v_i, s, \Sigma\rangle^{\mathcal{B}}$ for all $\langle x, \chi(s), \Sigma'\rangle \in X'$. It is straightforward to check that $\mathcal{B}'\upharpoonright_{\chi'} = \mathcal{B}$ and $\mathcal{B}'\upharpoonright_{\Sigma'} = \mathcal{A}'$.
For the uniqueness part, assume a $\Sigma'[X']$-model $\mathcal{C}$ such that $\mathcal{C}\upharpoonright_{\Sigma'} = \mathcal{A}'$ and $\mathcal{C}\upharpoonright_{\chi'} = \mathcal{B}$.

1. Since $\mathcal{C}\upharpoonright_{\Sigma'} = \mathcal{A}'$, $\mathcal{C}$ interprets all symbols in Σ' as $\mathcal{A}'$ and $\mathcal{B}'$.
2. Since $\mathcal{C}\upharpoonright_{\chi'} = \mathcal{B}$, we have $\langle v_i, \chi(s), \Sigma'\rangle^{\mathcal{C}} = \chi(\langle v_i, s, \Sigma\rangle)^{\mathcal{C}} = \langle v_i, s, \Sigma\rangle^{\mathcal{C}\upharpoonright_{\chi'}} = \langle v_i, s, \Sigma\rangle^{\mathcal{B}} = \langle v_i, s, \Sigma\rangle^{\mathcal{B}'\upharpoonright_{\chi'}} = \chi(\langle v_i, s, \Sigma\rangle)^{\mathcal{B}'} = \langle v_i, \chi(s), \Sigma'\rangle^{\mathcal{B}'}$, for all variables $\langle x, \chi(s), \Sigma'\rangle \in X'$.

By (1) and (2), $\mathcal{B}' = \mathcal{C}$. □



Proof of Theorem 21.

We proceed by induction on the structure of sentences:

- $\mathcal{A}' \models_{\Sigma'} \chi(t_1 = t_2)$ iff $\mathcal{A}' \models_{\Sigma'} \chi(t_1) = \chi(t_2)$ iff $\chi(t_1)^{\mathcal{A}'} = \chi(t_2)^{\mathcal{A}'}$ iff (by Lemma 22) $t_1^{(\mathcal{A}' \restriction_\chi)} = t_2^{(\mathcal{A}' \restriction_\chi)}$ iff $\mathcal{A}' \restriction_\chi \models_\Sigma t_1 = t_2$.
  The case corresponding to $\pi(t_1, \ldots, t_n)$ is similar to the one above.
- $\mathcal{A}' \models_{\Sigma'} \chi(\neg\gamma)$ iff $\mathcal{A}' \models_{\Sigma'} \neg\chi(\gamma)$ iff $\mathcal{A}' \not\models_{\Sigma'} \chi(\gamma)$ iff $\mathcal{A}' \restriction_\chi \not\models_\Sigma \gamma$ iff $\mathcal{A}' \restriction_\chi \models_\Sigma \neg\gamma$.
  The case corresponding to $\bigvee \Gamma$ is similar to the one above.
- $\mathcal{A}' \models_{\Sigma'} \chi(\exists X \cdot \gamma)$ iff $\mathcal{A}' \models_{\Sigma'} \exists X' \cdot \chi'(\gamma)$ iff
  $\mathcal{B}' \models_{\Sigma'[X']} \chi'(\gamma)$ for some $\iota_{X'}$-expansion $\mathcal{B}'$ of $\mathcal{A}'$ iff (by the induction hypothesis)
  $\mathcal{B}' \restriction_{\chi'} \models_{\Sigma[X]} \gamma$ for some $\iota_{X'}$-expansion $\mathcal{B}'$ of $\mathcal{A}'$ iff ($\overset{Lemma\ 23}{\Longleftarrow}$)
  $\mathcal{B} \models_{\Sigma[X]} \gamma$ for some ι_X-expansion $\mathcal{B}$ of $\mathcal{A}' \restriction_\chi$ iff
  $\mathcal{A}' \restriction_\chi \models_\Sigma \exists X \cdot \gamma$.

□

First-order substitutions

Example 24

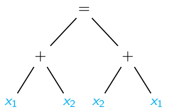
Let $\Sigma_{\mathtt{NAT+}}$ be the signature of natural numbers with addition and multiplication.
Let $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2, y_3\}$ be two sets of variables for $\Sigma_{\mathtt{NAT+}}$.
Let $\theta : X \to T_{\Sigma_{\mathtt{NAT+}}}(Y)$ be a function defined by $\theta(x_1) = y_1 + y_2$ and $\theta(x_2) = y_1 + (y_2 * y_3)$.

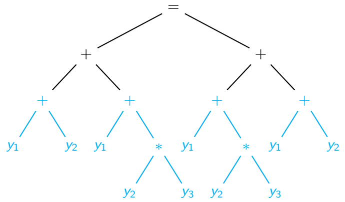
$\theta : X \to T_{\Sigma_{\mathtt{NAT+}}}(Y)$ determines a *reduct functor* $\upharpoonright_\theta : \mathtt{Mod}(\Sigma_{\mathtt{NAT+}}[Y]) \to \mathtt{Mod}(\Sigma_{\mathtt{NAT+}}[X])$:

- Let $\langle \mathbb{N}, f \rangle$ be a model over $\Sigma_{\mathtt{NAT+}}[Y]$, where
  - $\mathbb{N}$ is the model of natural numbers over $\Sigma_{\mathtt{NAT+}}$ and
  - $f : Y \to |\mathbb{N}|$ is defined by $f(y_1) = 1$, $f(y_2) = 3$ and $f(y_3) = 4$.
- The interpretation of x_i in to $\langle \mathbb{N}, f \rangle \upharpoonright_\theta$ is defined as the interpretation of $\theta(x_i)$ into $\langle \mathbb{N}, f \rangle$.
- $x_1^{\langle \mathbb{N}, f \rangle \upharpoonright_\theta} = \theta(x_1)^{\langle \mathbb{N}, f \rangle} = (y_1 + y_2)^{\langle \mathbb{N}, f \rangle} = 1 + 3 = 4$
  $x_2^{\langle \mathbb{N}, f \rangle \upharpoonright_\theta} = \theta(x_2)^{\langle \mathbb{N}, f \rangle} = (y_1 + (y_2 * y_3))^{\langle \mathbb{N}, f \rangle} = 1 + (3 * 4) = 13.$

$\theta : X \to T_{\Sigma_{\mathtt{NAT+}}}(Y)$ determines a *sentence translation* $\theta^{\mathtt{Sen}} : \mathtt{Sen}(\Sigma_{\mathtt{NAT+}}[X]) \to \mathtt{Sen}(\Sigma_{\mathtt{NAT+}}[Y])$:

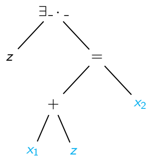
$$x_1 + x_2 = x_2 + x_1 \quad \xrightarrow{\theta} \quad \underbrace{(y_1 + y_2)}_{x_1} + \underbrace{(y_1 + (y_2 * y_3))}_{x_2} = \underbrace{(y_1 + (y_2 * y_3))}_{x_2} + \underbrace{(y_1 + y_2)}_{x_1}$$

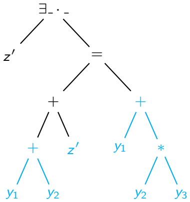




Let $z = \langle v_3, \mathtt{Nat}, \Sigma_{\mathtt{NAT+}}[X] \rangle$ be a variable for $\Sigma[X]$.
Let $z' := \langle v_3, \mathtt{Nat}, \Sigma_{\mathtt{NAT+}}[Y] \rangle$ be the translation of z along θ.

$$\exists z \cdot x_1 + z = x_2 \quad \xrightarrow{\theta} \quad \exists z' \cdot \underbrace{(y_1 + y_2)}_{x_1} + z' = \underbrace{y_1 + (y_2 * y_3)}_{x_2}$$

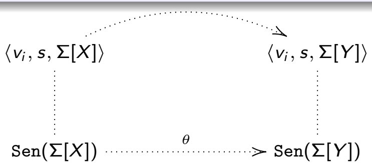




First-order substitutions

Definition 25 (Substitution)

Let Σ be a signature and X, Y two sets of variables for Σ.
A *Σ-substitution* between $\Sigma[X]$ and $\Sigma[Y]$ is a function $\theta : X \to T_\Sigma(Y)$.



A substitution $\theta\colon X \to T_\Sigma(Y)$ determine:

1. a sentence translation $\theta^{\mathtt{Sen}} : \mathtt{Sen}(\Sigma[X]) \to \mathtt{Sen}(\Sigma[Y])$ which is
   - the identity on the symbols in Σ, and
   - maps every variable $x \in X$ to $\theta(x) \in T_\Sigma(Y)$.
2. a model reduct $\upharpoonright_\theta\colon |\mathtt{Mod}(\Sigma[Y])| \to |\mathtt{Mod}(\Sigma[X])|$ defined by
   - $(\mathcal{B}\upharpoonright_\theta)_s = \mathcal{B}_s$ for all sorts $s \in S$,
   - $\sigma^{(\mathcal{B}\upharpoonright_\theta)} = \sigma^{\mathcal{B}}$ for all function symbols $\sigma \in F$,
   - $\pi^{(\mathcal{B}\upharpoonright_\theta)} = \pi^{\mathcal{B}}$ for all relation symbols $\sigma \in F$,
   - $x^{(\mathcal{B}\upharpoonright_\theta)} = \theta(x)^{\mathcal{B}}$ for all $x \in X$.

   for all models $\mathcal{B} \in |\mathtt{Mod}(\Sigma[Y])|$.

We drop the superscript $\mathtt{Sen}$ from $\theta^{\mathtt{Sen}}$ when there is no danger of confusion.

Theorem 26 (Satisfaction condition for substitutions)

Let $\theta : X \to T_\Sigma(Y)$ be a substitution. Then for all $\Sigma[Y]$-models $\mathcal{B}$ and all $\Sigma[X]$-sentences γ,

$$\mathcal{B} \models_{\Sigma[Y]} \theta^{\mathtt{Sen}}(\gamma) \text{ iff } \mathcal{B}\restriction_\theta \models_{\Sigma[X]} \gamma.$$

Similarly to the case of satisfaction condition for signature morphisms, we need two lemmas in order to prove Theorem 26.

Lemma 27

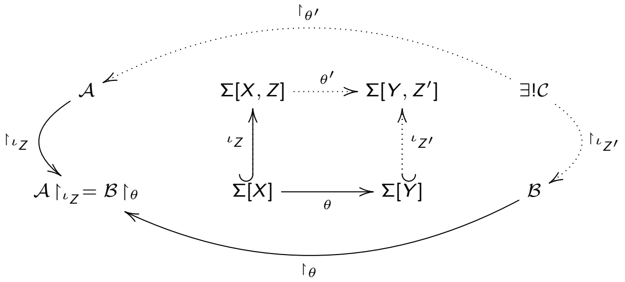
For all substitutions $\theta : X \to T_\Sigma(Y)$, all $\Sigma[X]$-terms t and all $\Sigma[Y]$-models $\mathcal{B}$, we have $t^{\mathcal{B}\restriction_\theta} = \theta(t)^{\mathcal{B}}$.

Proof.

We proceed by induction on the structure of terms:

- $x^{\mathcal{B}\restriction_\theta} = \theta(x)^{\mathcal{B}}$ for all variables $x \in X$;
- $\sigma(t_1, \ldots, t_n)^{\mathcal{B}\restriction_\theta} = \sigma^{\mathcal{B}\restriction_\theta}(t_1^{\mathcal{B}\restriction_\theta}, \ldots, t_n^{\mathcal{B}\restriction_\theta}) \stackrel{IH}{=} \sigma^{\mathcal{B}}(\theta(t_1)^{\mathcal{B}}, \ldots, \theta(t_n)^{\mathcal{B}}) = \theta(\sigma(t_1, \ldots, t_n))^{\mathcal{B}}$.

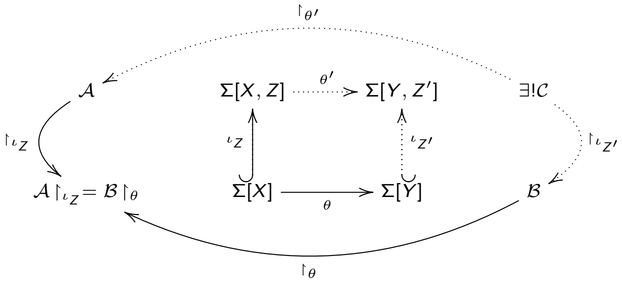
□



Lemma 28

Let $\theta : X \to T_\Sigma(Y)$ be a Σ-substitution and Z a set of variables for $\Sigma[X]$. For all $\Sigma[X, Z]$-models $\mathcal{A}$ and all $\Sigma[Y]$-models $\mathcal{B}$ such that $\mathcal{A}\upharpoonright_{\iota_Z} = \mathcal{B}\upharpoonright_{\theta}$ there exists a unique $\Sigma[Y, Z']$-model $\mathcal{C}$ such that $\mathcal{C}\upharpoonright_{\theta'} = \mathcal{A}$ and $\mathcal{C}\upharpoonright_{\iota_{Z'}} = \mathcal{B}$, where

- $\iota_Z : \Sigma[X] \hookrightarrow \Sigma[X, Z]$ *is an inclusion,*
- $Z' = \{\langle v_i, s, \Sigma[Y]\rangle \mid \langle v_i, s, \Sigma[X]\rangle \in Z\}$ *is the translation of Z along θ,*
- $\iota_{Z'} : \Sigma[Y] \hookrightarrow \Sigma[Y, Z']$ *is an inclusion,*
- $\theta' : X \cup Z \to T_\Sigma(Y \cup Z')$ *is defined by*
  - $\theta'(\langle v_i, s, \Sigma\rangle) = \theta(\langle v_i, s, \Sigma\rangle)$ *for all* $\langle v_i, s, \Sigma\rangle \in X$,
  - $\theta(\langle z, s, \Sigma[X]\rangle) = \langle z, s, \Sigma[Y]\rangle$ *for all* $\langle z, s, \Sigma[X]\rangle \in Z$.



Proof of Lemma 28.

We define $\mathcal{C}$ as follows:

1. $\mathcal{C}$ interprets all symbols in $\Sigma[Y]$ as $\mathcal{B}$;
2. $\mathcal{C}$ interprets Z as $\mathcal{A}$, that is $\langle v_i, s, \Sigma[Y]\rangle^{\mathcal{C}} = \langle v_i, s, \Sigma[X]\rangle^{\mathcal{A}}$ for all $\langle v_i, s, \Sigma[Y]\rangle \in Z'$.

By (1), $\mathcal{C}\restriction_{\iota_{Z'}} = \mathcal{B}$. Since θ' extends θ, by (2), $\mathcal{C}\restriction_{\theta'} = \mathcal{A}$.
For the uniqueness part, assume another $\Sigma[Y, Z]$-model $\mathcal{D}$ such that $\mathcal{D}\restriction_{\theta'} = \mathcal{A}$ and $\mathcal{D}\restriction_{\iota_{Z'}} = \mathcal{B}$.

1. Since $\mathcal{D}\restriction_{\Sigma[Y]} = \mathcal{B} = \mathcal{C}\restriction_{\Sigma[Y]}$, that is, $\mathcal{D}$ interprets all symbols in $\Sigma[Y]$ as $\mathcal{B}$, and in particular, as the model $\mathcal{C}$;
2. for all $\langle v_i, s, \Sigma[Y]\rangle \in Z'$, since $\mathcal{D}\restriction_{\theta'} = \mathcal{A}$, we have $\langle v_i, s, \Sigma[Y]\rangle^{\mathcal{D}} = \theta'(\langle v_i, s, \Sigma[X]\rangle)^{\mathcal{D}} = \langle v_i, s, \Sigma[X]\rangle^{\mathcal{A}} = \theta'(\langle v_i, s, \Sigma[X]\rangle)^{\mathcal{C}} = \langle v_i, s, \Sigma[Y]\rangle^{\mathcal{C}}$.

By (1) and (2), we get $\mathcal{C} = \mathcal{D}$. □

Reachable models

Definition 29 (Reachable models)

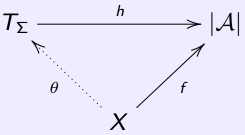
Let Σ be a signature and $\mathcal{A}$ be a Σ-model. $\mathcal{A}$ is *reachable* iff each element of $|\mathcal{A}|$ is the denotation of some term, i.e. the unique homomorphism $h : T_\Sigma \to \mathcal{A}$ is surjective.

Theorem 30

Let Σ be a signature, and X a set of variables for Σ. Then $\mathcal{A}$ is a reachable model iff for all sets of variables X for Σ, each expansion $\mathcal{B}$ of $\mathcal{A}$ to the signature $\Sigma[X]$ (i.e. a valuation $f : X \to |A|$) generates a substitution $\theta : X \to T_\Sigma$ such that $\mathcal{A}\restriction_\theta = \mathcal{B}$.

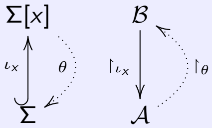
Proof of Theorem 30.

"$\Rightarrow$"
Let $f\colon X \to |\mathcal{A}|$ be a valuation. Since $h\colon T_\Sigma \to \mathcal{A}$ is surjective, there exists $\theta\colon X \to T_\Sigma$ such that $\theta; h = f$. We show that $\mathcal{A}\restriction_\theta = \langle \mathcal{A}, f\rangle = \mathcal{B}$:



- It suffices to show that $x^{(\mathcal{A}\restriction_\theta)} = \theta(x)^{\langle \mathcal{A}, f\rangle}$ for all $x \in X$.
- $x^{(\mathcal{A}\restriction_\theta)} = \theta(x)^{\mathcal{A}} = h(\theta(x)) = f(x) = x^{\langle \mathcal{A}, f\rangle}$ for all $x \in X$.

"$\Leftarrow$"
We show that each element in $|\mathcal{A}|$ is the denotation of some term. Let $s \in S$ and $a \in \mathcal{A}_s$. Let $x = \langle v_0, s, \Sigma\rangle$ and $\mathcal{B}$ the ι_x-expansion of $\mathcal{A}$ interpreting x as a, that is $x^{\mathcal{B}} = a$. By our assumptions, there exists a substitution $\theta\colon \{x\} \to T_\Sigma$ such that $\mathcal{A}\restriction_\theta = \mathcal{B}$. Hence, $\theta(x)^{\mathcal{A}} = x^{\mathcal{A}\restriction_\theta} = x^{\mathcal{B}} = a$.



□