Many-Sorted First-Order Model Theory
lecture 1

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Introduction

" Model theory = universal algebra + logic"

C.C. Chang, H. Jerome Keisler, Model Theory

"Model theory is about the classification of mathematical structures by means of logical formulas (...) large part of model theory is directly about constructions (of models) and only indirectly about classification"
"Model theory = algebraic geometry - fields"

W. Hodges, Shorter Model Theory

"Model theory = toolbox for the formal specification and verification of software, in which models are regarded as representations of reality"

Formal specification viewpoint

Contents

The course is divided in three parts:

1. Introduction of many-sorted first-order logic (models, sentences and satisfaction relation), and basic logical concepts (substitutions, reachable models and proof rules). The main result delivered here is Gödel's completeness — every semantic consequence has a formal proof.
   Lecturer: Daniel Găină
2. A characterization of elementary equivalence by Ehrenfeucht-Fraïssé games, commonly known as Fraïssé-Hintikka Theorem. Since finite games are quite intuitive and easy to describe, Fraïssé-Hintikka Theorem gives a better handle on elementary equivalence than Keisler-Shelah Theorem characterizing elementary equivalence via ultrapowers.
   Lecturer: Tomasz Kowalski
3. The relationship between gaining expressive power in extending first-order logic and losing some of its important properties. A paramount result in this direction is Lindström's theorem, which characterizes first-order logic among its extensions by two major properties: the Downward Löwenheim-Skolem Property and Compactness. In any proper extension of first-order logic at least one of the two fails.
   Lecturer: Guillermo Badia

Many-sorted sets

Definition 1 (Many-sorted set)

Let S be a set of elements called *sorts*. An *S-sorted set* A is an S-indexed family of sets denoted $\{A_s\}_{s \in S}$ or $\{A_s \mid s \in S\}$ or $(A_s)_{s \in S}$.

- A is *empty* if $A_s = \emptyset$ for all $s \in S$.
- A is *finite* if
  1. A_s is finite for all $s \in S$, and
  2. $\{s \in S \mid A_s \neq \emptyset\}$ is finite.

Example 2

Let $S = \{Bool, Nat, Int\}$ be a set of sorts. Let A be the S-sorted set defined as follows:

- $A_{Bool} = \{T, F\}$
- $A_{Nat} = \omega$ the set of natural numbers
- $A_{Int} = \mathbb{Z}$ the set of integers

Operations on many-sorted sets

Definition 3

Let $A = \{A_s\}_{s \in S}$ and $B = \{B_s\}_{s \in S}$.

- $A \cup B = \{A_s \cup B_s\}_{s \in S}$
- $A \cap B = \{A_s \cap B_s\}_{s \in S}$
- $A \times B = \{A_s \times B_s\}_{s \in S}$
- $A \setminus B = \{A_s \setminus B_s\}_{s \in S}$
- $A \uplus B = \{A_s \uplus B_s\}_{s \in S}$, where $A_s \uplus B_s = (A_s \times \{1\}) \cup (B_s \times \{2\})$
- $A \subseteq B$ iff $A_s \subseteq B_s$ for all $s \in S$
- $\mathtt{card}(A) = \mathtt{card}(\uplus_{s \in S} A_s)$

Let A be the sorted set defined in Example 2.
We have $\mathtt{card}(A) = \mathtt{card}(A_{Bool} \uplus A_{Nat} \uplus A_{Int}) = \omega$.

Operations on many-sorted sets

Example 4

Let $S = \{Bool, Nat, Int\}$. We define an S-sorted set B as follows:

- $B_{Bool} = \{N, F, T, B\}$
- $B_{Nat} = \{2n \mid n \in \omega\}$ the set of even natural numbers
- $B_{Int} = \mathbb{Q}$ the set of rational numbers

Let A be the set defined in Example 2 and B the set define in Example 4.
The disjoint union $A \uplus B$ is defined as follows:

- $(A \uplus B)_{Bool} = \{\langle T, 1\rangle, \langle F, 1\rangle, \langle T, 2\rangle, \langle F, 2\rangle, \langle N, 2\rangle, \langle B, 2\rangle\}$
- $(A \uplus B)_{Nat} = \{\langle n, 1\rangle \mid n \in \omega\} \cup \{\langle 2n, 2\rangle \mid n \in \omega\}$
- $(A \uplus B)_{Int} = \{\langle z, 1\rangle \mid z \in \mathbb{Z}\} \cup \{\langle q, 2\rangle \mid q \in \mathbb{Q}\}$

Many-sorted mappings

Definition 5 (Many-sorted function)

Let $A = \{A_s\}_{s \in S}$ and $B = \{B_s\}_{s \in S}$.
A *sorted function* $f : A \to B$ is an S-indexed family of functions $f = \{f_s : A_s \to B_s\}_{s \in S}$.

- If $f : A \to B$ and $g : B \to C$ are two S-sorted functions then the composition $f ; g = \{f_s ; g_s\}_{s \in S}$ is considered in diagrammatic order. [a]
- An S-sorted function $f = \{f_s : A_s \to B_s\}_{s \in S}$ is
  - *identity* if $f_s : A_s \to B_s$ is identity for all $s \in S$,
  - *inclusion* if $f_s : A_s \to B_s$ is inclusion for all $s \in S$,
  - *injection* if $f_s : A_s \to B_s$ is injection for all $s \in S$,
  - *surjection* if $f_s : A_s \to B_s$ is surjection for all $s \in S$.

[a] We prefer the notation $f ; g$ to $g \circ f$, which is used in computer science.

Signatures/Vocabularies

Notation

- If S is a set then we denote by S^* the set of strings over S.
- We let ε to denote the empty string
- If $S = \{a, b, c\}$ then $a, b, c, aa, ab, ac, ba, bb, bc, ca, cb, cc, aaa, \cdots \in S^*$.

Definition 6 (First-order signature)

A *many-sorted first-order signature* is a triple $\Sigma = (S, F, P)$, where

1. S is a set of *sorts*,
2. $F = \{F_{w \to s}\}_{(w,s) \in S^* \times S}$ is an $(S^* \times S)$-sorted set of *function symbols*, and
3. $P = \{P_w\}_{w \in S^*}$ is an S^*-sorted set of *relation symbols*.

If $\sigma \in F_{w \to s}$ then we call

- σ the *name* of the function symbol,
- w the *arity* of σ, and
- s the *sort* of σ.

Signatures/Vocabularies

- We may write $(\sigma : w \to s) \in F$ instead of $\sigma \in F_{w \to s}$.
  We regard the sorted set F as an ordinary set of elements of the form $\sigma : w \to s$.
- A similar remark holds for P too.
- If $P = \emptyset$ then we write $\Sigma = (S, F)$ instead of $\Sigma = (S, F, \emptyset)$.
  In this case, we say that Σ is an algebraic signature.

Notation

We let Σ, Σ' and Σ_i to range over signatures of the form (S, F, P), (S', F', P') and (S_i, F_i, P_i), respectively.

Single-sorted signatures

Example 7 (Natural numbers)

We define the signature of natural numbers NAT as follows:

- $\Sigma_{\mathtt{NAT}} = (S_{\mathtt{NAT}}, F_{\mathtt{NAT}})$
- $S_{\mathtt{NAT}} = \{\mathtt{Nat}\}$
- $F_{\mathtt{NAT}} = \{0 :\to \mathtt{Nat}, \mathtt{s\_} : \mathtt{Nat} \to \mathtt{Nat}\}$
- $P_{\mathtt{NAT}} = \{\_ < \_ : \mathtt{Nat\ Nat}\}$

Example 8 (Integers)

We define the signature of natural numbers NAT as follows:

- $\Sigma_{\mathtt{INT}} = (S_{\mathtt{INT}}, F_{\mathtt{INT}})$
- $S_{\mathtt{INT}} = \{\mathtt{Int}\}$
- $F_{\mathtt{INT}} = \{0 :\to \mathtt{Int}, \mathtt{s\_} : \mathtt{Int} \to \mathtt{Int}, \mathtt{p\_} : \mathtt{Int} \to \mathtt{Int}\}$

Example 9 (Groups)

- $\Sigma_{\mathtt{GROUP}} = (S_{\mathtt{GROUP}}, F_{\mathtt{GROUP}})$
- $S_{\mathtt{GROUP}} = \{\mathtt{Group}\}$
- $F_{\mathtt{GROUP}} = \{0 :\to \mathtt{Group}, \_ + \_ : \mathtt{Group\ Group} \to \mathtt{Group}, -\_ : \mathtt{Group} \to \mathtt{Group}\}$

Many-sorted signatures

Example 10 (Lists)

- $\Sigma_{\mathtt{LIST}} = (\mathtt{S}_{\mathtt{LIST}}, \mathtt{F}_{\mathtt{LIST}})$
- $\mathtt{S}_{\mathtt{LIST}} = \{\mathtt{Elem}, \mathtt{List}\}$
- $\mathtt{F}_{\mathtt{LIST}} = \{\mathtt{empty} : \to \mathtt{List},\ \mathtt{con} : \mathtt{Elem}\ \mathtt{List} \to \mathtt{List}\}$
- $\mathtt{P}_{\mathtt{LIST}} = \{\_\mathtt{in}\_ : \mathtt{Elem}\ \mathtt{List}\}$

Example 11 (Automata)

- $\Sigma_{\mathtt{AUTOM}} = (\mathtt{S}_{\mathtt{AUTOM}}, \mathtt{F}_{\mathtt{AUTOM}})$
- $\mathtt{S}_{\mathtt{AUTOM}} = \{\mathtt{Input}, \mathtt{Output}, \mathtt{State}\}$
- $\mathtt{F}_{\mathtt{AUTOM}} = \{\mathtt{init} : \to \mathtt{State},\ \mathtt{f} : \mathtt{Input}\ \mathtt{State} \to \mathtt{State},\ \mathtt{g} : \mathtt{State} \to \mathtt{Output}\}$

Example 12 (Graphs)

- $\Sigma_{\mathtt{GRAPH}} = (\mathtt{S}_{\mathtt{GRAPH}}, \mathtt{F}_{\mathtt{GRAPH}})$
- $\mathtt{S}_{\mathtt{GRAPH}} = \{\mathtt{Node}, \mathtt{Edge}\}$
- $\mathtt{F}_{\mathtt{GRAPH}} = \{@_0 : \mathtt{Edge} \to \mathtt{Node}, @_1 : \mathtt{Edge} \to \mathtt{Node}\}$

First-order structures/Models

Definition 13 (Σ-models)

A Σ-*model* $\mathcal{A}$ interprets

- each $s \in S$ as a set $\mathcal{A}_s$, called the *carrier set* for the sort s
  $|\mathcal{A}| = \{\mathcal{A}_s\}_{s \in S}$ is called the *universe* of $\mathcal{A}$
- each $(\sigma \colon w \to s) \in F$ as a function $\sigma^{\mathcal{A}} \colon \mathcal{A}_w \to \mathcal{A}_s$,
  where $\mathcal{A}_w = \mathcal{A}_{s_1} \times \cdots \times \mathcal{A}_{s_n}$ if $w = s_1 \ldots s_n$
- each $(\pi : w) \in P$ as a relation $\pi^{\mathcal{A}} \subseteq \mathcal{A}_w$

Examples of models

Example 14

Let $\Sigma_{\mathtt{NAT}}$ be the signature of natural numbers defined in Example 7.
Some example of models over $\Sigma_{\mathtt{NAT}}$ are:

- $\mathbb{N}$ - natural numbers with the usual strict ordering
- $\mathbb{Z}$ - integers with the usual strict ordering
- $\mathbb{Z}_{\mathtt{n}}$ - integers modulo $\mathtt{n}$ with $\widehat{0} < \widehat{1} < \cdots < \widehat{n}$
- $\mathbb{N}'$ - natural numbers with two zeros
  - $\mathbb{N}'_{\mathtt{Nat}} = \omega \cup \{0'\}$
  - $\mathtt{s}\ \ 0' = 1$

  The ordering preserves the usual ordering on $\mathbb{N}$ and it adds $\{0' < n \mid n \in \omega \setminus \{0\}\}$.

Example 15

Let $\Sigma_{\mathtt{AUTOM}}$ be the signature defined in Example 11.
We define a model $\mathbb{A}$ over $\Sigma_{\mathtt{AUTOM}}$ as follows:

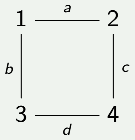
- $\mathbb{A}_{\mathtt{Input}} = \mathbb{A}_{\mathtt{Output}} = \mathbb{A}_{\mathtt{State}} = \omega$
- $\mathtt{init}^{\mathbb{A}} = 0$
- $\mathtt{f}^{\mathbb{A}}(m, n) = m + n$ for all $m, n \in \omega$
- $g^{\mathbb{A}}(n) = n + 1$ for all $n \in \omega$

Examples of models

Example 16

$\Sigma_{\mathtt{GRAPH}}$-model $\mathbb{G}$:

- $\mathbb{G}_{\mathtt{Node}} = \{1, 2, 3, 4\}$,
- $\mathbb{G}_{\mathtt{Edge}} = \{a, b, c, d\}$,
- $@_0^{\mathbb{G}}(a) = 1$, $@_1^{\mathbb{G}}(a) = 2$,
- $@_0^{\mathbb{G}}(b) = 1$, $@_1^{\mathbb{G}}(b) = 3$,
- $@_0^{\mathbb{G}}(c) = 2$, $@_1^{\mathbb{G}}(c) = 4$,
- $@_0^{\mathbb{G}}(d) = 3$, $@_1^{\mathbb{G}}(d) = 4$.



Initial term model

Definition 17 (Initial term model)

Let Σ be a signature. The initial *model of Σ-terms*, denoted T_Σ, is defined inductively:

- $\dfrac{}{c \in T_{\Sigma,s}}$ where $(c \to s) \in F$
- $\dfrac{t_1 \in T_{\Sigma,s_1} \ldots t_n \in T_{\Sigma,s_n}}{\sigma(t_1,\ldots,t_n) \in T_{\Sigma,s}}$ where $(\sigma : s_1 \ldots s_n \to s) \in F$
- the function $\sigma^{T_\Sigma} : T_{\Sigma,s_1} \times \cdots \times T_{\Sigma,s_n} \to T_{\Sigma,s}$ is defined by $\sigma^{T_\Sigma}(t_1,\ldots,t_n) = \sigma(t_1,\ldots,t_n)$ for all tuples $(t_1,\ldots,t_n) \in T_{\Sigma,s_1} \times \cdots \times T_{\Sigma,s_n}$
- $\pi^{T_\Sigma} = \emptyset$ for all $(\pi : s_1 \ldots s_n) \in P$

Initial term model

Remark 18

If there are different function symbols with the same name and the same arity, that is, $(\sigma : w \to s_1) \in F$ *and* $(\sigma : w \to s_2) \in F$ *such that* $s_1 \neq s_2$, *then we replace the first two rules from Definition 17 by the following:*

- $\dfrac{}{c_s \in T_{\Sigma,s}}$ *for all* $(c :\to s) \in F$
- $\dfrac{t_1 \in T_{\Sigma,s_1} \dots t_n \in T_{\Sigma,s_n}}{\sigma_s(t_1, \dots, t_n) \in T_{\Sigma,s}}$ *where* $(\sigma : s_1 \dots s_n \to s) \in F$

and keep the remaining rules as they are.

Exercise 19

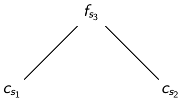
Using Remark 18, show that every term has a unique sort even if some function symbols are overloaded, that is, there exist $(\sigma : w_1 \to s_1) \in F$ *and* $(\sigma : w_2 \to s_2) \in F$ *such that* $(w_1, s_1) \neq (w_2, s_2)$.

Initial term model

Exercise 19 shows how to deal with overloaded function symbols

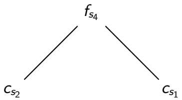
Example 20

Let $\Sigma = (S, F)$, $S = \{s_1, s_2, s_3, s_4\}$ and $F = \{c :\to s_1, c :\to s_2, f : s_1\ s_2 \to s_3, f : s_2\ s_1 \to s_4\}$.



$f_{s_3}(c_{s_1}, c_{s_2})$ is of sort s_3

$f : s_2\ s_2 \to s_3$ is used



$f_{s_4}(c_{s_2}, c_{s_1})$ is of sort s_4

$f : s_2\ s_1 \to s_4$ is used

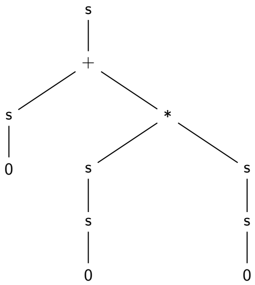
The top function symbol used can be decided from the context.

Initial term model

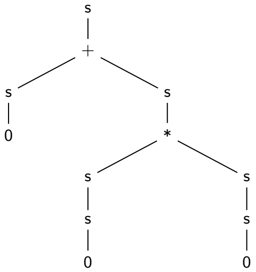
Example 21

Let $\Sigma_{\texttt{NAT+}}$ be the signature obtained from $\Sigma_{\texttt{NAT}}$ defined in Example 7 by adding $\_ + \_ : \texttt{Nat Nat} \to \texttt{Nat}$ and $\_ * \_ : \texttt{Nat Nat} \to \texttt{Nat}$.

Let $T_{\texttt{NAT+}}$ be the initial term model over $\Sigma_{\texttt{NAT+}}$ can be regarded as trees:



`s(s 0 + (s s 0 * s s 0))`



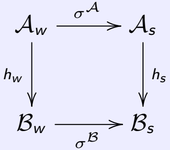
`s (s 0 + s(s s 0 * s s 0))`

Homomorphisms

Definition 22 (Σ-homomorphisms)

A Σ-*homomorphism* $h\colon \mathcal{A} \to \mathcal{B}$ is an S-sorted function $h\colon |\mathcal{A}| \to |\mathcal{B}|$ such that

1. the diagram on the right is commutative for all $(\sigma\colon w \to s) \in F$;
2. $h_w(\pi^{\mathcal{A}}) \subseteq \pi^{\mathcal{B}}$ for all $(\pi : w) \in P$.



h is *injective/surjective/bijective* if $h_s : A_s \to B_s$ is injective/surjective/bijective for all $s \in S$.

Exercise 23

Show that a composition of two Σ-homomorphisms (component-wise as many-sorted functions) is a Σ-homomorphism, and that the identity $1_{\mathcal{A}}$ on a Σ-structure $\mathcal{A}$ is a Σ-homomorphism.

Homomorphisms

Definition 24 (Isomorphism)

- A Σ-homomorphism $h : \mathcal{A} \to \mathcal{B}$ is a Σ-isomorphism iff there is another Σ-homomorphism $g : \mathcal{B} \to \mathcal{A}$ such that $h; g = 1_{\mathcal{A}}$ and $g; h = 1_{\mathcal{B}}$ (i.e., such that for each $s \in S$, $g_s(h_s(a)) = a$ for all $a \in \mathcal{A}_s$ and $h_s(g_s(b)) = b$ for all $b \in \mathcal{B}_s$).
- In this case, g is called the *inverse* of h, and is denoted h^{-1}.
- We write $\mathcal{A} \cong \mathcal{B}$ if there exists a Σ-isomorphism between $\mathcal{A}$ and $\mathcal{B}$.

Fact 25

The Σ-homomorphisms form a category under the composition.
The category of Σ-models is denoted by $\mathtt{Mod}(\Sigma)$, which means that

- *the class of Σ-models is denoted by $|\mathtt{Mod}(\Sigma)|$, and*
- *the class of Σ-homomorphisms is denoted by $\mathtt{Mod}(\Sigma)$.*

Examples of homomorphisms

Exercise 26

Prove that a homomorphism is an isomorphism iff it is bijective.

Example 27

Let $\Sigma_{\mathtt{NAT}}$ be the signature of Example 7.
Are the following mappings $\Sigma_{\mathtt{NAT}}$-homomorphisms?

1. $h : \mathbb{N} \to \mathbb{Z}$, defined by $h(n) = n$ for all natural numbers $n \in |\mathbb{N}|$
   One can easily check that h defined above is a homorphism
2. $h : \mathbb{Z} \to \mathbb{N}$, defined by $h(z) = |z|$ for all integers $z \in |\mathbb{Z}|$
   h defined above is not a homomorphism,
   since $1 = h(s^{\mathbb{Z}}(-2)) \neq s^{\mathbb{N}}(h(-2)) = s^{\mathbb{N}}(2) = 3$
3. $h : \mathbb{Z} \to \mathbb{Z}_{\mathrm{n}}$, defined by $h(z) = \widehat{z}$ for all integers $z \in |\mathbb{Z}|$
   Notice that $h(0^{\mathbb{Z}}) = 0^{\mathbb{N}}$ and $h(s^{\mathbb{Z}}(z)) = \widehat{s^{\mathbb{Z}}(z)} = s^{\mathbb{Z}_n}(\widehat{z}) = s^{\mathbb{Z}_n}(h(z))$ for all $z \in |\mathbb{Z}|$.
   However, $n - 1 < n$ but $\widehat{n-1} \not<^{\mathbb{Z}_n} \widehat{n} = \widehat{0}$.
   Hence, h defined above is not a homomorphism.

Initiality of the term model

Theorem 28 (Initiality)

Given a signature Σ and any Σ-model $\mathcal{A}$, there is a unique Σ-homomorphism $T_\Sigma \to \mathcal{A}$.

Proof.

Existence We define $h : T_\Sigma \to \mathcal{A}$ by induction:

- $h(c) = c^{\mathcal{A}}$ for all $(c :\to s) \in F$.
- $h(\sigma(t_1, \ldots, t_n)) = \sigma^{\mathcal{A}}(h(t_1), \ldots, h(t_n))$ for all function symbols $(\sigma : s_1 \ldots, s_n \to s) \in F$ and all tuples of terms $(t_1, \ldots, t_n) \in T_{\Sigma,s_1} \times \cdots \times T_{\Sigma,s_n}$.

Uniqueness Assume another homomorphism $g : T_\Sigma \to \mathcal{A}$.
We show that $g(t) = h(t)$ for all terms $t \in T_\Sigma$.

- $g(c) = c^{\mathcal{A}} = h(c)$ for all $(c :\to s) \in F$.
- $g(\sigma(t_1, \ldots, t_n)) = \sigma^{\mathcal{A}}(g(t_1), \ldots, g(t_n)) \overset{IH}{=} \sigma^{\mathcal{A}}(h(t_1), \ldots, h(t_n))$ for all function symbols $(\sigma : s_1 \ldots, s_n \to s) \in F$ and all tuples of terms $(t_1, \ldots, t_n) \in T_{\Sigma,s_1} \times \cdots \times T_{\Sigma,s_n}$.

□

Notation 29

For all terms t over Σ, we call $h(t)$ the evaluation of t into $\mathcal{A}$ and we let $t^{\mathcal{A}}$ denote $h(t)$.

Evaluating terms

Example 30 (Evaluating terms over the naturals)

Let $\Sigma_{\mathtt{NAT}+}$ be the signature of Example 21. Let $\mathbb{N}$ be the model over $\Sigma_{\mathtt{NAT}+}$ in which

- the constant symbol 0 is interpreted as the number 0,
- the function symbol s is interpreted as the successor operation,
- the relation symbol $<$ is interpreted as usual strict ordering over the naturals,
- the function symbol $+$ is interpreted as the addition function, and
- $*$ as multiplication.

- Let $T_{\mathtt{NAT}+}$ be the initial term model over $\Sigma_{\mathtt{NAT}+}$ and let $h : T_{\mathtt{NAT}+} \to \mathbb{N}$ be the unique homomorphism given by Theorem 28.
- The evaluation of a term into $\mathbb{N}$ is given by the unique homomorphism h.
- For example, the evaluation of $t = \mathtt{s}\ (\mathtt{s}\ 0 + (\mathtt{s}\ \mathtt{s}\ 0 * \mathtt{s}\ \mathtt{s}\ 0))$ into $\mathbb{N}$ is $h(t) = h(\mathtt{s}\ (\mathtt{s}\ 0 + (\mathtt{s}\ \mathtt{s}\ 0 * \mathtt{s}\ \mathtt{s}\ 0))) = \mathtt{s}^{\mathbb{N}}(\mathtt{s}^{\mathbb{N}}0^{\mathbb{N}} + (\mathtt{s}^{\mathbb{N}}\mathtt{s}^{\mathbb{N}}0^{\mathbb{N}} *^{\mathbb{N}} \mathtt{s}^{\mathbb{N}}\mathtt{s}^{\mathbb{N}}0^{\mathbb{N}})) = 6$.

Evaluating terms

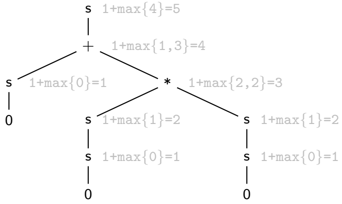
Example 31 (Depth of a term)

Given any first-order signature $\Sigma = (S, F, P)$, we define a Σ-model $\mathcal{M}$ as follows:

- $\mathcal{M}_s = \omega$ for all $s \in S$,
- $\mathcal{M}$ interprets each
  - constant $(c :\to s) \in F$ as 0, and
  - each function symbol $(\sigma : s_1 \dots s_n \to s) \in F$ as $\sigma^{\mathcal{M}} : \underbrace{\omega \times \cdots \times \omega}_{n\text{-times}} \to \omega$ defined by

$$\sigma^{\mathcal{M}}(i_1, \dots, i_n) = 1 + max\{i_1, \dots, i_n\} \text{ for all tuples } (i_1, \dots, i_n) \in \underbrace{\omega \times \cdots \times \omega}_{n\text{-times}}.$$

- $\mathcal{M}$ interprets each relation symbol as the empty set.

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d(s(s 0 + (s s 0 * s s 0))) = 5
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- Then the unique homomorphism $d : T_\Sigma \to \mathcal{M}$ computes the depth of Σ-terms, that is, the maximum amount of nesting in terms.
- Suppose $\Sigma = \Sigma_{\texttt{NAT+}}$ and $t = \texttt{s}\ (\texttt{s}\ 0 + (\texttt{s}\ \texttt{s}\ 0 * \texttt{s}\ \texttt{s}\ 0))$.
- Then $d(t) = 5$.

Exercise 32 (Size of a term)

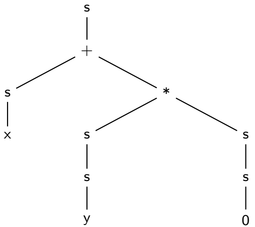
For any signature Σ*, define a model* $\mathcal{M}$ *over* Σ *such that the unique homomorphism* $\texttt{size} : T_\Sigma \to \mathcal{M}$ *maps each term* $t \in T_\Sigma$ *to the size of* t*. For example, if* $\Sigma = \Sigma_{\texttt{NAT}+}$ *of Example 21 and* $t = \texttt{s}\ (\texttt{s}\ 0 + (\texttt{s}\ \texttt{s}\ 0 * \texttt{s}\ \texttt{s}\ 0))$ *then* $\texttt{size}(t) = 11$*.*

Exercise 33

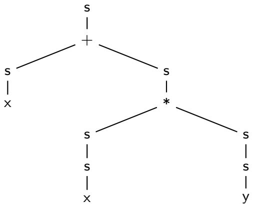
Use initiality to define an arrow from Σ*-terms which gives the number of interior (i.e., non-leaf) nodes in the corresponding tree.*

Variables

- Let $\{v_i \mid i \in \omega\}$ be a set of variable names.
- A *variable* for a signature $\Sigma = (S, F, P)$ is a triple $\langle v_i, s, \Sigma \rangle$, where
  - v_i is variable name, and
  - $s \in S$ is the sort of the variable.
- By definition, any variable $\langle v_i, s, \Sigma \rangle$ for Σ is different from the elements of Σ.
- Given an S-sorted set $X = \{X_s\}_{s \in S}$ of variables for Σ, we can obtained a new signature $\Sigma[X]$ by adding the variables in X as constants to the constants in F, that is, $\Sigma[X] = (S, F[X], P)$ and $F[X] = F \cup \{x :\to s \mid s \in S \text{ and } x \in X_s\}$.
- If there is no danger of confusion, we identify a variable only by its name and sort. A confusion arises when a variable $\langle v_i, s, \Sigma \rangle \in X$ is a constant in Σ, i.e., $\langle v_i, s \rangle \in F_{\to s}$.
- Moreover, if the signature Σ is single-sorted and no variable name occurs among its constants then we identify a variable only by its name.
- Let x and y be two variables for $\Sigma_{\texttt{NAT}+}$. Examples of terms with variables are given below.



$t_1[x, y] = \texttt{s}\ (\texttt{s}\ x + (\texttt{s}\ \texttt{s}\ y * \texttt{s}\ \texttt{s}\ 0))$



$t_2[x, y] = \texttt{s}\ (\texttt{s}\ x + \texttt{s}\ (\texttt{s}\ \texttt{s}\ x * \texttt{s}\ \texttt{s}\ y))$

Evaluating terms with variables

- Let $\Sigma_{\texttt{NAT+}}$ be the signature of Example 21 and let $\mathbb{N}$ be the $\Sigma_{\texttt{NAT+}}$-model of Example 30.

$\Sigma_{\texttt{NAT+}}[x, y]$ ············ $t_1 = \mathtt{s}(\mathtt{s}\ x + (\mathtt{s}\ \mathtt{s}\ y * \mathtt{s}\ \mathtt{s}\ 0))$ ············ $t_2 = \mathtt{s}(\mathtt{s}\ x + \mathtt{s}\ (\mathtt{s}\ \mathtt{s}\ x * \mathtt{s}\ \mathtt{s}\ \mathtt{s}\ y))$

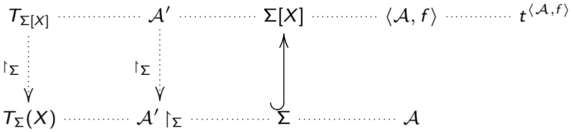
$\uparrow$

$\Sigma_{\texttt{NAT+}}$ ···································· $\mathbb{N}$

- In order to give an interpretation of the terms $t_1[x, y]$ or $t_2[x, y]$ into $\mathbb{N}$, we need an interpretation of x and y into $\mathbb{N}$.
- Interpretations of variables x and y in $\mathbb{N}$ are given as functions $f : \{x, y\} \to |\mathbb{N}|$.
- Notice that a pair $\langle \mathbb{N}, f \rangle$ can be regarded as a model over $\Sigma[x, y]$.
- By Theorem 28, $\mathtt{s}\ (\mathtt{s}\ x + (\mathtt{s}\ \mathtt{s}\ y * \mathtt{s}\ \mathtt{s}\ 0))$ and $\mathtt{s}\ (\mathtt{s}\ x + \mathtt{s}\ (\mathtt{s}\ \mathtt{s}\ x * \mathtt{s}\ \mathtt{s}\ \mathtt{s}\ y))$ have unique interpretations in $\langle \mathbb{N}, f \rangle$ given by the unique homomorphism $T_{\Sigma[X]} \to \langle \mathbb{N}, f \rangle$.
- If $f(x) = 3$ and $f(y) = 1$ then $t_1^{\langle \mathbb{N}, f \rangle} = 11$ and $t_2^{\langle \mathbb{N}, f \rangle} = 21$.

  In classical model theory $t_1^{\langle \mathbb{N}, f \rangle}$ and $t_2^{\langle \mathbb{N}, f \rangle}$ are denoted $t_1^{\mathbb{N}}[3, 1]$ and $t_2^{\mathbb{N}}[3, 1]$.

Evaluating terms with variables



- Let Σ be an arbitrary signature and X an arbitray set of variables for Σ.
- A model over $\Sigma[X]$ consists of a Σ-model $\mathcal{A}$ and a valuation $f : X \to |\mathcal{A}|$.
- By Theorem 28, any $\Sigma[X]$-term t has a unique interpretation in a Σ-model $\mathcal{A}$ via a valuation $f : X \to |\mathcal{A}|$, which is $t^{\langle \mathcal{A}, f \rangle}$.
- If $X = \{x_1, \ldots, x_n\}$ and $f(x_i) = a_i$ for all $i \in \{1, \ldots, n\}$ then we let $t^{\mathcal{A}}[a_1, \ldots, a_n]$ denote $t^{\langle \mathcal{A}, f \rangle}$.
- Conversely, given a model $\mathcal{A}'$ over $\Sigma[X]$, once can obtain Σ-model $\mathcal{A}' \upharpoonright_\Sigma$ from $\mathcal{A}'$ by discarding the interpretation of the variables in X.
- If $\mathcal{A}' = T_{\Sigma[X]}$ then we let $T_\Sigma(X)$ denote $T_{\Sigma[X]} \upharpoonright_\Sigma$.
- Let $\mathcal{A}$ be a Σ-model and $\mathcal{A}'$ be a $\Sigma[X]$-model such that $\mathcal{A}' \upharpoonright_\Sigma = \mathcal{A}$ (in other words $\mathcal{A}' = \langle \mathcal{A}, f \rangle$, where $f : X \to |\mathcal{A}|$ is defined by $f(x) = x^{\mathcal{A}'}$ for all $x \in X$).
  We say that $\mathcal{A}'$ is an *expansion* of $\mathcal{A}$ to the signature $\Sigma[X]$ and $\mathcal{A}$ is a *reduct* of $\mathcal{A}'$ to the signature Σ.

Sentences

Definition 34 (Σ-sentences)

For all signatures Σ, the set of Σ-sentences, denoted $\mathtt{Sen}(\Sigma)$, is constructed inductively:

- *Equations* $\dfrac{t_1, t_2 \in T_{\Sigma,s} \quad s \in S}{t_1 = t_2 \in \mathtt{Sen}(\Sigma)}$ and *Relations* $\dfrac{(\pi : s_1 \dots s_n) \in P \quad t_i \in T_{\Sigma,s_i}}{\pi(t_1, \dots, t_n) \in \mathtt{Sen}(\Sigma)}$
- *Disjunctions* $\dfrac{\Gamma \subseteq \mathtt{Sen}(\Sigma) \text{ is finite}}{\bigvee \Gamma \in \mathtt{Sen}(\Sigma)}$ and *Negations* $\dfrac{\gamma \in \mathtt{Sen}(\Sigma)}{\neg\gamma \in \mathtt{Sen}(\Sigma)}$
- *Quantifiers* $\dfrac{X \text{ is a finite set of variables for } \Sigma \quad \gamma' \in \mathtt{Sen}(\Sigma[X])}{\exists X \cdot \gamma' \in \mathtt{Sen}(\Sigma)}$
  where $\Sigma[X] = (S, F[X], P)$, and $F[X]$ is obtained from F by adding the elements of X as constants to F.

Sentences

- Note that the qualification of the variables by their signature context guarantees automatically, by a simple set theoretic argument, that when added as new constants to the signature they indeed do not clash with the already existing constants.
- Other sentence building operators are introduced using the classical definitions:
  - $\bot \coloneqq \bigvee \emptyset$
  - $\bigwedge \Gamma \coloneqq \neg(\bigvee_{\gamma \in \Gamma} \neg\gamma)$
  - $\gamma_1 \Rightarrow \gamma_2 \coloneqq \neg\gamma_1 \vee \gamma_2$
  - $\top \coloneqq \bigwedge \emptyset$
  - $\forall X \cdot \gamma' \coloneqq \neg\exists X \cdot \neg\gamma'$
  - $\gamma_1 \Leftrightarrow \gamma_2 \coloneqq (\gamma_1 \Rightarrow \gamma_2) \wedge (\gamma_2 \Rightarrow \gamma_1)$

Convention

Dealing with standard logical operators, we adopt the following convention about their binding strength:

- $\neg$ binds stronger than $\wedge$,
- which binds stronger than $\vee$,
- which binds stronger than $\Rightarrow$,
- which binds stronger than quantifiers;
- quantifiers $\exists$ and $\forall$ have the same binding strength.

Satisfaction relation

Definition 35 (Satisfaction relation)

The satisfaction between models and sentences is inductively defined:

- $\mathcal{A} \models_\Sigma t_1 = t_2$ iff $t_1^\mathcal{A} = t_2^\mathcal{A}$
- $\mathcal{A} \models_\Sigma \pi(t_1, \ldots, t_n)$ iff $(t_1^\mathcal{A}, \ldots, t_n^\mathcal{A}) \in \pi^\mathcal{A}$
- $\mathcal{A} \models_\Sigma \neg\gamma$ iff $\mathcal{A} \not\models_\Sigma \gamma$
- $\mathcal{A} \models_\Sigma \bigvee \Gamma$ iff $\mathcal{A} \models_\Sigma \gamma$ for some $\gamma \in \Gamma$
- $\mathcal{A} \models_\Sigma \exists X \cdot \gamma'$ iff $\mathcal{A}' \models_{\Sigma[X]} \gamma'$ for some expansion $\mathcal{A}'$ of $\mathcal{A}$ to the signature $\Sigma[X]$

We drop the subscript Σ from the notation $\models_\Sigma$ when there is no danger of confusion.

Satisfaction relation

Exercise 36

For any signature Σ and all Σ-models $\mathcal{A}$ we have:

- $\mathcal{A} \models_\Sigma \bigwedge \Gamma$ *iff* $\mathcal{A} \models_\Sigma \gamma$ *for all* $\gamma \in \Gamma$
- $\mathcal{A} \not\models_\Sigma \bot$ *and* $\mathcal{A} \models_\Sigma \top$
- $\mathcal{A} \models_\Sigma \gamma_1 \Rightarrow \gamma_2$ *iff* $\mathcal{A} \models \gamma_1$ *implies* $\mathcal{A} \models_\Sigma \gamma_2$
- $\mathcal{A} \models_\Sigma \forall X \cdot \gamma'$ *iff* $\mathcal{A}' \models_{\Sigma[X]} \gamma'$ *for all expansions of* $\mathcal{A}'$ *of* $\mathcal{A}$ *to* $\Sigma[X]$

Example 37

Let $\mathbb{N}$ be the $\Sigma_{\mathtt{NAT+}}$-model from Example 30.

- $\mathbb{N} \models \mathrm{s}\ 0 + \mathrm{s}\ 0 = \mathrm{s}\ \mathrm{s}\ 0$ iff $(\mathrm{s}\ 0 + \mathrm{s}\ 0)^{\mathbb{N}} = (\mathrm{s}\ \mathrm{s}\ 0)^{\mathbb{N}}$ iff $(1+1) = 2$
- $\mathbb{N} \models \forall x, y \cdot x + y = y + x$ iff
  $\mathbb{N}' \models x + y = y + x$ for all expansions $\mathbb{N}'$ of $\mathbb{N}$ to $\Sigma_{\mathtt{NAT+}}[x, y]$ iff
  $x^{\mathbb{N}'} +^{\mathbb{N}'} y^{\mathbb{N}'} = y^{\mathbb{N}'} +^{\mathbb{N}'} x^{\mathbb{N}'}$ for all expansions $\mathbb{N}'$ of $\mathbb{N}$ to $\Sigma_{\mathtt{NAT+}}[x, y]$

Definition 38

- $\mathcal{A} \models_\Sigma \Gamma$ iff $\mathcal{A} \models_\Sigma \gamma$ for all $\gamma \in \Gamma$.
- $\Gamma_1 \models_\Sigma \Gamma_2$ iff $\mathcal{A} \models_\Sigma \Gamma_1$ implies $\mathcal{A} \models_\Sigma \Gamma_2$ for all Σ-models $\mathcal{A}$.