Ultraproducts
Filters and ultrafilters

We assume basic knowledge of Boolean algebras. In fact, we will only need powerset algebras, that is, Boolean algebras \( \mathcal{P}(X) \) for some \( X \).

**Definition 1**

A nonempty subset \( F \) of the universe of a Boolean algebra \( B \) is an **filter** if

1. \( a \in F \) and \( a \subseteq b \) implies \( b \in F \).
2. \( a, b \in F \) implies \( a \cap b \in F \).

▶ A filter \( F \) is **principal** if it is of the form \( \{ a : a \supseteq b \} \) for some \( b \in |B| \). We will write \( \uparrow b \) for principal filters.

▶ A maximal proper filter is called an **ultrafilter**.

**Exercise 1**

Let \( F \) be a filter on a Boolean algebra \( B \). The following are equivalent.

1. \( F \) is an ultrafilter.
2. If \( a \cup b \in F \) then \( a \in F \) or \( b \in F \).
3. \( \neg a \in F \) iff \( a \notin F \) (where \( \neg a \) is the complement of \( a \)).
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Finite intersection property

**Definition 2**

Let $X$ be a set, and $C$ a collection of subsets of $X$. Then, $C$ has **finite intersection property** if for every finite $C_1, \ldots, C_n \in C$ we have $C_1 \cap \cdots \cap C_n \neq \emptyset$.

**Lemma 3**

Let $X$ be a set, and $C$ a collection of subsets of $X$. If $C$ has finite intersection property, then it can be extended to an ultrafilter on $X$.

**Proof.**

- The family $F = \{Z \subseteq X : C \subseteq Z \text{ for some } C \in C\}$ is a proper filter (this is clear).
- Enumerate subsets of $X$ somehow. Let the enumeration be $\{Y_i : i < \gamma\}$ ($\gamma$ is an ordinal).
- For each $i < \gamma$ we will consider $Y_i$ and $\neg Y_i$ and extend $F$ to $F_i$ as follows.
- **Base.** If $Y_0 \in F$ or $\neg Y_0 \in F$, then $F_0 = F$. If $Y_0, \neg Y_0 \notin F$, then $F_0 = \uparrow\{Y_0 \cap Z : Z \in F\}$.
  
  Observe that $Y_0 \cap Z \neq \emptyset$ for all $Z \in F$; otherwise $\neg Y_0 \supseteq Z$ and so we would have $\neg Y_0 \in F$. It follows that $F_0$ is a filter containing $Y_0$. 

□
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Finite intersection property

Proof.

- **Step for successor.** Then $i = j + 1$, and $F_j$ is a filter. We proceed as in the base case.
- **Step for limit.** Then $F_j$ is a filter for all $j < i$, and $(F_j : j < i)$ is a chain of proper filters. Put $F_i = \bigcup_{j < i} F_j$. Then $F_i$ is a proper filter (**exercise**).
- At the end of the process, put $U = \bigcup_{i < \gamma} F_i$. Then $U$ is a proper filter.
- Moreover, since each subset of $X$ is $Y_i$ for some $i < \gamma$, by construction we have either $Y_i \in U$ or $\neg Y_i \in U$. So, by Exercise 1, $U$ is an ultrafilter.
- As $C \subseteq F \subseteq U$ the claim is proved.

**Exercise 2**

Prove that families of proper filters are closed under unions of chains.

**Remarks**

- Exercise 2 implies the existence of ultrafilters, by Zorn’s Lemma. But in fact it is strictly weaker. It is equivalent to Boolean Prime Ideal Theorem.
- It is instructive to compare the construction of ultrafilters to Lemma 5 of Lecture 4 (extending consistent sets of sentences to maximally consistent ones). The exact counterpart of Lemma 5 is known as Rasiowa-Sikorski Lemma.
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▶ At the end of the process, put $U = \bigcup_{i < \gamma} F_i$. Then $U$ is a proper filter.

▶ Moreover, since each subset of $X$ is $Y_i$ for some $i < \gamma$, by construction we have either $Y_i \in U$ or $\neg Y_i \in U$. So, by Exercise 1, $U$ is an ultrafilter.

▶ As $\mathcal{C} \subseteq F \subseteq U$ the claim is proved.

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Fréchet filter. Principal ultrafilters

Exercise 3

Let $C$ be the set of all cofinite subsets of $\mathbb{N}$.

- Prove that $C$ is a filter (it is known as Fréchet filter).
- Let $S$ be an infinite subset of $\mathbb{N}$. Prove that Fréchet filter can be extended to an ultrafilter containing $S$.

Lemma 4

An ultrafilter $U$ on $X$ is principal iff $U$ is of the form $\uparrow\{x\}$ for some $x \in X$.

Proof.

- If $U = \uparrow\{x\}$ for some $x \in X$, then $U$ is an ultrafilter (immediate).
- For converse, $U = \uparrow S$ for some $S \subset X$ by definition.
- Let $a \in S$. Then $\uparrow\{a\}$ in an ultrafilter, by previous part.
- Moreover, $U = \uparrow S \supseteq \uparrow\{a\}$ and since $U$ and $\uparrow\{a\}$ are ultrafilters, they must be equal. (Ultrafilters are either incomparable or equal.)
- Therefore, $U = \uparrow\{a\}$, as required.
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Reduced products and ultraproducts

Definition 5

Let \((\mathcal{A}_i)_{i \in I}\) be a family of similar models. Let \(F\) be a filter on \(I\). For \(a, b \in \prod_{i \in I} \mathcal{A}_i\), the equaliser of \(a\) and \(b\) is the set \(\{i \in I : a(i) = b(i)\}\). We denote equalisers by \([a = b]\). Define a binary relation putting

\[ a \sim_F b \quad \text{iff} \quad [a = b] \in F. \]

Lemma 6

The relation \(\sim_F\) is a congruence on \(\prod_{i \in I} \mathcal{A}_i\).

Proof.

- Reflexivity and symmetry of \(\sim_F\) is immediate. For transitivity, assume \([a = b] \in F\) and \([b = c] \in F\). Then \([a = b] \cap [b = c] \in F\), because \(F\) is a filter.
- But, \([a = b] \cap [b = c] \subseteq [a = c]\), so \([a = c] \in F\).
- Let \(f\) be a function, say, of two arguments. Assume \(a \sim b\) and \(c \sim d\).
- We have \([f(a, c) = f(b, d)] \supseteq [a = b] \cap [c = d]\); since \([a = b] \in F\) and \([c = d] \in F\) by assumption, we get \([f(a, c) = f(b, d)] \in F\). So \(f(a, c) \sim_F f(b, d)\) as required.
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Reduced products and ultraproducts

**Definition 7**

Let \((\mathfrak{A}_i)_{i \in I}\) be a family of similar models, let \(F\) be a filter on \(I\), and let \(\sim_F\) be the congruence relation of Lemma 6. The **reduced product** of \((\mathfrak{A}_i)_{i \in I}\) with respect to \(F\), denoted by \(\prod_{i \in I} \mathfrak{A}_i/F\), is the model defined as follows:

1. The universe of \(\prod_{i \in I} \mathfrak{A}_i/F\) and the interpretations of constant and function symbols are as in Definition 20 of Lecture 2 of the quotient structure \(\prod_{i \in I} \mathfrak{A}_i/\sim_F\).

2. Each relational symbol \(R(x_1, \ldots, x_n)\) is interpreted by

   \[ R_{\prod_{i \in I} \mathfrak{A}_i/F}(a_1/\sim_F, \ldots, a_n/\sim_F) \text{ iff } \{i \in I : R_{\mathfrak{A}_i}(a_1, \ldots, a_n)\} \in F. \]

- If \(\mathfrak{A}_i\) are copies of the same model \(\mathfrak{A}\), we write \(\mathfrak{A}^I/F\) and call it an **reduced power**.
- If \(F\) is an ultrafilter, \(\prod_{i \in I} \mathfrak{A}_i/F\) is called an **ultraproduct**, and \(\mathfrak{A}^I/F\) an **ultrapower**.

We write \(a/F\), instead of \(a/\sim_F\), for the elements of \(\prod_{i \in I} \mathfrak{A}_i/F\).
Comments on the definition

A fine point about homomorphisms and quotients

- Note that the definition of a homomorphism $f : \mathcal{A} \to \mathcal{B}$, for models $\mathcal{A}$ and $\mathcal{B}$, requires only that $R^\mathcal{A}(\bar{a})$ implies $R^\mathcal{B}(f(\bar{a}))$. So we can have a surjective homomorphism $f : \mathcal{A} \to \mathcal{B}$ with $R^\mathcal{A} = \emptyset$ and $R^\mathcal{B} \neq \emptyset$.

- Definition 20 of Lecture 2 requires something stronger: to have $R^\mathcal{B}(\bar{b})$ a witness must exist in the preimage of $\bar{b}$, that is, we must have $R^\mathcal{A}(\bar{a})$ for some $\bar{a}$ with $f(\bar{a}) = \bar{b}$.

A fine point about the definition of reduced products

The definition of a reduced product requires only that the natural quotient map be a homomorphism: it is perfectly possible, indeed common, to have $\prod_{i \in I} \mathcal{A}_i/F \models R(a/F)$ for a relation that fails to hold in $\prod_{i \in I} \mathcal{A}_i$. It suffices that $R(a)$ fails at a single coordinate $i$, but still holds at a “large” set of coordinates to have $\{i \in I : R^{\mathcal{A}_i}(a)\} \in F$. 

### Footnotes

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Fundamental Theorem of Ultraproducts

Theorem 8 (Łoś)

Let $(\mathcal{A}_i)_{i \in I}/F$ be a reduced product of a family of similar models. Let $\varphi(\bar{x})$ be a primitive positive formula (i.e., an existentially quantified conjunction of atomic formulas). Then, for any tuple $\bar{a}$ from $\prod_{i \in I} \mathcal{A}_i$, the following are equivalent:

1. $\prod_{i \in I} \mathcal{A}_i/F \models \varphi(a_1/F, \ldots, a_n/F)$,
2. $\{ i \in I : \mathcal{A}_i \models \varphi(a_1(i), \ldots, a_n(i)) \} \in F$.

If $F$ is an ultrafilter, the above equivalence holds for every formula $\varphi(\bar{x})$.

Remarks

Note that every tuple $\bar{c}$ from $\prod_{i \in I} \mathcal{A}_i/F$ is of the form $\bar{a}/F$ for some $\bar{a}$ from $\prod_{i \in I} \mathcal{A}_i$, so the equivalence in the theorem is often expressed by:

- A formula holds in the reduced product if and only if the set of coordinates where it holds belongs to the filter.
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Łoś Theorem: preparing for proof

Notation

Let \((\mathcal{A}_i)_{i \in I}\) be a family of similar models, and \(F\) an ultrafilter on \(I\).

- The equaliser notation extends naturally to terms. Say, for terms \(t(x, y), s(x)\) and \(a, b \in \prod_{i \in I} \mathcal{A}_i\) we have
  
  \[
  \left[\left[ t(a, b) = s(a) \right]\right] = \{ i \in I : t(a, b)(i) = s(a)(i) \}.
  \]

- The notation extends further to arbitrary formulas. Let \(\varphi(x, y)\) be a formula. We put
  
  \[
  \left[\left[ \varphi(a, b) \right]\right] = \{ i \in I : \mathcal{A}_i \models \varphi(a(i), b(i)) \}.
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- This is a well-defined piece of notation, for if \(\varphi(x, y)\) is \(x = y\) and \(s, t\) are terms, we have
  
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  \left[\left[ \varphi(s, t) \right]\right] = \left[\left[ t = s \right]\right].
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The equivalence in Łoś Theorem, can then be succinctly written as

\[
\prod_{i \in I} \mathcal{A}_i/F \models \varphi(\bar{a}/F) \text{ iff } \left[\left[ \varphi(\bar{a}) \right]\right] \in F.
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\prod_{i \in I} \mathcal{A}_i / F \models \varphi(\bar{a} / F) \text{ iff } [[\varphi(\bar{a})]] \in F.
\]
Łoś Theorem: proof (primitive positive formulas)

Proof.

- **Induction on complexity of the formula** \( \varphi \).
- **Base**: If \( \varphi \) is atomic, then \( \prod_{i \in I} \mathcal{A}_i / F \models \varphi(\bar{a} / F) \) iff \( [[\varphi(\bar{a})]] \in F \) holds by definition.
- **Step for** \( \land \): If \( \varphi \) is \( \rho \land \tau \), we have
  \[
  \prod_{i \in I} \mathcal{A}_i / F \models (\rho \land \tau)(\bar{a} / F) \iff \prod_{i \in I} \mathcal{A}_i / F \models \rho(\bar{a} / F) \text{ and } \prod_{i \in I} \mathcal{A}_i / F \models \tau(\bar{a} / F)
  \]
  \[
  \iff [[\rho(\bar{a})]] \in F \text{ and } [[\tau(\bar{a})]] \in F
  \]
  \[
  \iff [((\rho \land \tau)(\bar{a}))] \in F \quad (\text{since } F \text{ is a filter})
  \]
- **Step for** \( \exists \): If \( \varphi \) is \( \exists z \cdot \rho \), we have
  \[
  \prod_{i \in I} \mathcal{A}_i / F \models \exists z \cdot \rho(z, \bar{a} / F) \iff \prod_{i \in I} \mathcal{A}_i / F \models \rho(c, \bar{a} / F) \text{ for some } c \in \prod_{i \in I} \mathcal{A}_i / F
  \]
  \[
  \iff \prod_{i \in I} \mathcal{A}_i / F \models \rho(a_0 / F, \bar{a} / F) \text{ for some } a_0 \in \prod_{i \in I} \mathcal{A}_i
  \]
  \[
  \iff [[\rho(a_0, \bar{a})]] \in F
  \]
  \[
  \iff [[\exists z \cdot \rho(z, \bar{a})]] \in F
  \]
- **In the last equivalence above, forward direction in immediate. For converse pick** \( a_0(i) \) **to be a witness for** \( \mathcal{A}_i \models \exists x \cdot \rho(x, \bar{a}(i)) \) **on** \( i \) **where it holds; on other coordinates pick** \( a_0(i) \) **arbitrarily.**
- This proves Łoś Theorem for primitive positive formulas and reduced products. \( \square \)
Łoś Theorem: proof (negation)

Proof.

- Now assume that \( F \) is an ultrafilter.
- **Step for \( \neg \):** If \( \varphi \) is \( \neg \rho \), then
  \[
  \prod_{i \in I} A_i / F \models \neg \rho(\bar{a}/F) \iff \prod_{i \in I} A_i / F \not\models \rho(\bar{a}/F)
  \]

  \[
  \iff [[\rho(\bar{a})]] \notin F
  \]

  \[
  \iff [[\neg \rho(\bar{a})]] \in F \quad (since \ F \ is \ an \ \text{ultrafilter})
  \]

- This proves Łoś Theorem for ultraproducts and arbitrary formulas.

Example 9 (Negation in reduced products)

Let \( \mathfrak{A} = \langle \mathbb{N}, \leq, S \rangle \), where \( S \) is a unary relation such that \( S(x) \) iff \( 13 \leq x \). Let \( F \) be the Fréchet filter on \( \omega \). Let \( \mathfrak{A} \) be the reduced power \( \mathbb{N}^\omega / F \).

- We have \( \mathfrak{A} \models S(\bar{a}/F) \) iff \( [[S(\bar{a})]] \in F \) iff \( [[S(\bar{a})]] \) is cofinite. Define \( \bar{a} \) by putting \( a(n) = 13 \) if \( n \) is even and \( a(n) = 12 \) if \( n \) is odd. Then \( [[S(\bar{a})]] \) is not cofinite, so \( \mathfrak{A} \not\models S(\bar{a}/F) \).
  Therefore, \( \mathfrak{A} \models \neg S(\bar{a}/F) \).

- But \( [[\neg S(\bar{a})]] \) is also not cofinite, so \( [[\neg S(\bar{a})]] \notin F \).

- Hence, \( \mathfrak{A} \models \neg S(\bar{a}/F) \) is not equivalent to \( [[\neg S(\bar{a})]] \in F \).
Łoś Theorem: proof (negation)

Proof.

- Now assume that $F$ is an ultrafilter.
- Step for $\neg$: If $\varphi$ is $\neg \rho$, then
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Example 9 (Negation in reduced products)

Let $\mathfrak{A} = \langle \mathbb{N}, \leq, S \rangle$, where $S$ is a unary relation such that $S(x)$ iff $13 \leq x$. Let $F$ be the Fréchet filter on $\omega$. Let $\mathfrak{A}$ be the reduced power $\mathbb{N}^\omega / F$.

- We have $\mathfrak{A} \models S(\bar{a} / F)$ iff $[[S(\bar{a})]] \in F$ iff $[[S(\bar{a})]]$ is cofinite. Define $\bar{a}$ by putting $a(n) = 13$ if $n$ is even and $a(n) = 12$ if $n$ is odd. Then $[[S(\bar{a})]]$ is not cofinite, so $\mathfrak{A} \not\models S(\bar{a} / F)$. Therefore, $\mathfrak{A} \models \neg S(\bar{a} / F)$.

- But $[[\neg S(\bar{a})]]$ is also not cofinite, so $[[\neg S(\bar{a})]] \notin F$.

- Hence, $\mathfrak{A} \models \neg S(\bar{a} / F)$ is not equivalent to $[[\neg S(\bar{a})]] \in F$. 
Compactness via ultraproducts

Theorem 10 (Compactness again)

Let $T$ be a theory. Suppose every finite $T' \subseteq T$ has a model. Then, $T$ has a model.

Proof sketch.

- Let $I$ be the set of all finite subsets of $T$ and for each $i \in I$ let $\mathcal{A}_i$ be a model of $i$.
- For each $i \in I$, define $J_i = \{ j \in I : i \subseteq j \}$. Observe that $J_{i_1} \cap J_{i_2} = J_{i_1 \cup i_2}$, so the family $\mathcal{J} = (J_i)_{i \in I}$ has the finite intersection property.
- Take an ultrafilter $U$ on $I$ extending $\mathcal{J}$, and consider $\prod_{i \in I} \mathcal{A}_i/U$.
- For each $\varphi \in T$, we have $\{ \varphi \} \in I$ (say, $\{ \varphi \} = i_{\varphi}$), so $J_{i_{\varphi}} \in \mathcal{J}$.
- But $J_{i_{\varphi}} = \{ i \in I : \mathcal{A}_i \models \varphi \}$, so apply Łoś.
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- But $J_{i_\varphi} = \{i \in I : \mathcal{A}_i \models \varphi\}$, so apply Łoś.
Embedding into finitely generated submodels

Theorem 11

Every model is embeddable in an ultraproduct of its finitely generated submodels.

Proof sketch.

- Let \( I \) be the set of all finite subsets of \(|\mathcal{A}|\), and for each \( i \in I \) let \( \mathcal{A}_i \) be \( \langle i \rangle_{\mathcal{A}} \) (the submodel generated by \( i \)).
- For each \( i \in I \), define \( J_i = \{ j \in I : i \subseteq j \} \). Observe that \( J_{i_1} \cap J_{i_2} = J_{i_1 \cup i_2} \), so the family \( \mathcal{J} = (J_i)_{i \in I} \) has the finite intersection property.
- Take an ultrafilter \( U \) on \( I \) extending \( \mathcal{J} \), and consider \( \prod_{i \in I} \mathcal{A}_i/U \).
- Take the map \( \mu : \mathcal{A} \to \prod_{i \in I} \mathcal{A}_i/U \), defined by \( \mu(a) = (a_i : i \in I)/U \) where \( a_i = a \) if \( a \in |\mathcal{A}_i| \) and is arbitrary otherwise.
- \( \mu \) is the embedding we want.
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- Take an ultrafilter \( U \) on \( I \) extending \( \mathcal{J} \), and consider \( \prod_{i \in I} A_i / U \).

- Take the map \( \mu : A \to \prod_{i \in I} A_i / U \), defined by \( \mu(a) = (a_i : i \in I) / U \) where \( a_i = a \) if \( a \in |A_i| \) and is arbitrary otherwise.

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- Take an ultrafilter $U$ on $I$ extending $\mathcal{J}$, and consider $\prod_{i \in I} \mathcal{A}_i / U$.
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Elementary embedding into ultrapower

Notation

Let $\mathcal{A}$ be a model, $\mathcal{A}^I/U$ an ultrapower of $\mathcal{A}$, and $a$ an element from $\mathcal{A}$. We write $a/U$ for $(a(i): i \in I)/U$ where $a(i) = a$ for every $i \in I$.

Theorem 12

The map $a \mapsto a/U$ is an elementary embedding of $\mathcal{A}$ into $\mathcal{A}^I/U$.

Proof sketch.

- It is clearly an embedding.
- For elementarity, we use Tarski-Vaught test. Let $\varphi(x)$ be a formula with only $x$ free (forget about parameters, for simplicity).
- Suppose $\mathcal{A}^I/U \models \exists x \cdot \varphi(x)$. Then, for some $a/U = (a_i: i \in I)/U$, we have $\mathcal{A}^I/U \models \varphi(a/U)$.
- By Łoś, $[[\varphi(a)]] \in U$, so $\mathcal{A} \models \varphi(a)$.
- Thus, $\mathcal{A} \models \exists x \cdot \varphi(x)$ as required.
# Elementary embedding into ultrapower

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Exercises

Exercise 4

*Fill the details in proof sketches of Theorems 10, 11 and 12.*

Lemma 13

Let $\prod_{i \in I} A_i / U$ be an ultraproduct of some family of models, with $U$ a principal ultrafilter, say $U = \uparrow\{j\}$ for some $j \in I$. Then $\prod_{i \in I} A_i / U$ is isomorphic to $A_j$.

Proof.

- Take arbitrary $a / U$ and $b / U$ from $\prod_{i \in I} A_i / U$. Since $U = \uparrow\{j\}$, we have $[[a = b]] \in U$ iff $a(j) = b(j)$. Therefore, $a \sim_U b$ holds iff $a(j) = b(j)$.
- Thus, the quotient map from $\prod_{i \in I} A_i$ to $\prod_{i \in I} A_i / U$ is precisely the $j$-th projection.

Exercise 5

Let $I$ be a finite set. Prove that for any family $\{A_i : i \in I\}$ and any ultrafilter $U$ on $I$, we have $\prod_{i \in I} A_i / U \cong A_k$ for some $k \in K$. 
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