Many-Sorted First-Order Model Theory

Lecture 6

2\textsuperscript{nd} July, 2020
Easy halves: unions of chains

Theorem 1 (Chang-Łoś-Suszko: easy direction)

Let \((A_i)_{i<\gamma}\) be a family of structures that form a chain under embedding. That is, the index set \(\gamma\) is an ordinal and \(i \leq j\) implies \(A_i \leq A_j\). Put \(A = \bigcup_{i<\gamma} A_i\). Let \(\varphi(\bar{x})\) be a \(\Pi_2\) formula, and \(a_0\) a tuple from \(A_0\). If \(A_i \models \varphi(a_0)\) for every \(i < \gamma\), then \(A \models \varphi(a_0)\).

Proof.

- Write \(\varphi\) explicitly, as \(\forall y \cdot \exists z \cdot \psi(x, y, z)\).
- Take any tuple \(\bar{b}\) from \(A\). Since \(\bar{b}\) is finite it belongs to some \(A_i\).
- Since \(A_i \models \varphi(a_0)\) by assumption, we have \(A_i \models \exists z \cdot \psi(a_0, b, z)\).
- This is an existential formula with parameters from \(A_i\), and \(A_i \leq A\).
- By the easy half of the Łoś-Tarski Theorem (on substructures), we have \(A \models \exists z \cdot \psi(a_0, b, z)\).
- As \(\bar{b}\) was arbitrary, we have \(A \models \forall y \cdot \exists z \cdot \psi(a_0, y, z)\), as claimed. \(\square\)
Theorem 1 (Chang-Łoś-Suszko: easy direction)

Let \((A_i)_{i<\gamma}\) be a family of structures that form a chain under embedding. That is, the index set \(\gamma\) is an ordinal and \(i \leq j\) implies \(A_i \leq A_j\). Put \(A = \bigcup_{i<\gamma} A_i\). Let \(\varphi(\bar{x})\) be a \(\Pi_2\) formula, and \(\bar{a}_0\) a tuple from \(A_0\). If \(A_i \models \varphi(\bar{a}_0)\) for every \(i < \gamma\), then \(A \models \varphi(\bar{a}_0)\).

Proof.

- Write \(\varphi\) explicitly, as \(\forall \bar{y} \cdot \exists \bar{z} \cdot \psi(\bar{x}, \bar{y}, \bar{z})\).
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- As \(\bar{b}\) was arbitrary, we have \(A \models \forall \bar{y} \cdot \exists \bar{z} \cdot \psi(\bar{a}_0, \bar{y}, \bar{z})\), as claimed. \(\square\)
Unions of chains: algebraically closed fields

Example 2

Fix a prime $p$, and consider the chain

$$GF(p) \leq GF(p^2) \leq \cdots \leq GF(p^i) \leq \cdots$$

and let $F$ be its union. For any $n$ consider the sentence

$\varphi_n = \forall y \cdot \exists x \cdot y_n x^n + \ldots y_1 x + y_0 = 0$. Note that $\varphi_n$ is a $\Pi_2$ sentence.

- We have $GF(p^i) \models \varphi_n$ for $i \geq n$.
- Moreover, $F = \bigcup_{j \geq i} GF(p^j)$ for any $i \in \mathbb{N}$.
- It follows that $F \models \varphi_n$ for every $n$.
- Thus, $F$ is an algebraically closed field of characteristic $p$.

Exercise 1

Let $F$ be as above. Fill the gaps in the proof that $F$ is an algebraically closed field of characteristic $p$. 
Unions of chains: algebraically closed fields

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Fix a prime $p$, and consider the chain

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Let $F$ be as above. Fill the gaps in the proof that $F$ is an algebraically closed field of characteristic $p$. 
Exercises

Exercise 2

Let $\varphi$ be the sentence $\exists x, y \cdot \forall z \cdot \neg (x < z) \lor \neg (z < y)$ in the language of a binary relation $\prec$. Let $\psi$ be the conjunction of $\varphi$ with universal sentences stating that $\prec$ is a strict linear order. Construct a chain of models $(\mathcal{C}_n)_{n \in \mathbb{N}}$ such that $\mathcal{C}_n \models \psi$ but $\bigcup_{n \in \mathbb{N}} \mathcal{C}_n \not\models \psi$. Conclude that $\Sigma_2$ sentences are not preserved under unions of chains.

Exercise 3

Prove that positive formulas are preserved under onto homomorphisms.

Exercise 4 (Somewhat hard, but instructive)

Let $\{ \mathcal{A}_i : i < \gamma \}$ be a chain of similar structures, and let $\mathcal{A} = \bigcup_{i < \gamma} \mathcal{A}_i$. Prove that $\mathcal{A} \in \text{HSP}\{ \mathcal{A}_i : i < \gamma \}$.

Hint. Consider “eventually constant” sequences $(u_i : i < \gamma)$, i.e., such if $i > i_0$ (for some $i_0$), then $u_i = a_{i_0}$, where $a_{i_0} \in |\mathcal{A}_{i_0}|$. Next, consider the relation $\sim$ on these sequences, defined by $(u_i : i < \gamma) \sim (w_i : i < \gamma)$ if for all $i > j_0$ (for some $j_0$) we have $u_i = w_i$. 
Exercises

Exercise 2

Let $\varphi$ be the sentence $\exists x, y \cdot \forall z \cdot \neg (x < z) \vee \neg (z < y)$ in the language of a binary relation $\prec$. Let $\psi$ be the conjunction of $\varphi$ with universal sentences stating that $\prec$ is a strict linear order. Construct a chain of models $(C_n)_{n \in \mathbb{N}}$ such that $C_n \models \psi$ but $\bigcup_{n \in \mathbb{N}} C_n \not\models \psi$. Conclude that $\Sigma_2$ sentences are not preserved under unions of chains.

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Let $\{A_i : i < \gamma\}$ be a chain of similar structures, and let $A = \bigcup_{i < \gamma} A_i$. Prove that $A \in \text{HSP}\{A_i : i < \gamma\}$.

Hint. Consider “eventually constant” sequences $(u_i : i < \gamma)$, i.e., such if $i > i_0$ (for some $i_0$), then $u_i = a_{i_0}$, where $a_{i_0} \in |A_{i_0}|$. Next, consider the relation $\sim$ on these sequences, defined by $(u_i : i < \gamma) \sim (w_i : i < \gamma)$ if for all $i > j_0$ (for some $j_0$) we have $u_i = w_i$. 

Exercises

Exercise 2

Let \( \varphi \) be the sentence \( \exists x, y \cdot \forall z \cdot \neg (x < z) \lor \neg (z < y) \) in the language of a binary relation \(<\). Let \( \psi \) be the conjunction of \( \varphi \) with universal sentences stating that \(<\) is a strict linear order. Construct a chain of models \( (C_n)_{n \in \mathbb{N}} \) such that \( C_n \models \psi \) but \( \bigcup_{n \in \mathbb{N}} C_n \not\models \psi \). Conclude that \( \Sigma_2 \) sentences are not preserved under unions of chains.

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Let \( \{A_i : i < \gamma\} \) be a chain of similar structures, and let \( A = \bigcup_{i < \gamma} A_i \). Prove that \( A \in \text{HSP}\{A_i : i < \gamma\} \).

Hint. Consider “eventually constant” sequences \( (u_i : i < \gamma) \), i.e., such if \( i > i_0 \) (for some \( i_0 \)), then \( u_i = a_{i_0} \), where \( a_{i_0} \in |A_{i_0}| \). Next, consider the relation \( \sim \) on these sequences, defined by \( (u_i : i < \gamma) \sim (w_i : i < \gamma) \) if for all \( i > j_0 \) (for some \( j_0 \)) we have \( u_i = w_i \).
Preservation under substructures

Theorem 3 (Łoś-Tarski)

Let \( T \) be a theory. If \( T \) is preserved under substructures, then \( T \) is equivalent to a set of \( \Pi_1 \) formulas.

Proof.

- Wlog, \( T \) is consistent. Let \( T_\forall \) be \( \{ \varphi \in \Pi_1 : T \models \varphi \} \).
- Let \( K = \{ A : A \leq B \text{ for some } B \in \text{Mod}(T) \} \).
  So \( K \) is the class of submodels of models of \( T \).
- As \( T \) is preserved by substructures, we have \( \text{Mod}(T) = K \subseteq \text{Mod}(T_\forall) \).
  We will show that \( \text{Mod}(T_\forall) \subseteq K \).
- Take \( A \models T_\forall \). Claim A: \( \text{diag}(A) \cup T \) is consistent.
  - Take a finite \( D_0 \subseteq \text{diag}(A) \) and a finite \( T_0 \subseteq T \).
    Put \( \delta(\bar{a}) = \wedge D_0 \), where \( \bar{a} \) are all diagram constants occurring in \( D_0 \).
    If \( D_0 \cup T_0 \) is inconsistent, then \( T_0 \models \neg \delta(\bar{a}) \).
    Note that \( T_0 \) does not mention \( \bar{a} \) at all. So \( T_0 \) entails \( \neg \delta(\bar{a}) \) for arbitrary \( \bar{a} \).
  - So, \( T_0 \models \forall \bar{x} \cdot \neg \delta(\bar{x}) \), and as \( \delta \) is quantifier free, we have \( \forall \bar{x} \cdot \neg \delta(\bar{x}) \in T_\forall \).
  - Thus, in particular, \( A \models \neg \delta(\bar{a}) \).
  - But \( \delta(\bar{a}) \in \text{diag}(A) \), so \( A \models \delta(\bar{a}) \). Contradiction.
- This proves Claim A.
Preservation under substructures

**Theorem 3 (Łoś-Tarski)**

Let $T$ be a theory. If $T$ is preserved under substructures, then $T$ is equivalent to a set of $\Pi_1$ formulas.

**Proof.**

- Wlog, $T$ is consistent. Let $T_\forall$ be $\{\varphi \in \Pi_1 : T \models \varphi\}$.
- Let $K = \{\mathcal{A} : \mathcal{A} \leq \mathcal{B} \text{ for some } \mathcal{B} \in \text{Mod}(T)\}$. 
  *So $K$ is the class of submodels of models of $T$.*
- As $T$ is preserved by substructures, we have $\text{Mod}(T) = K \subseteq \text{Mod}(T_\forall)$. 
  *We will show that $\text{Mod}(T_\forall) \subseteq K$.*
- Take $\mathcal{A} \models T_\forall$. **Claim A:** $\text{diag}(\mathcal{A}) \cup T$ is consistent.
  - Take a finite $D_0 \subseteq \text{diag}(\mathcal{A})$ and a finite $T_0 \subseteq T$.
  - Put $\delta(\bar{a}) = \bigwedge D_0$, where $\bar{a}$ are all diagram constants occurring in $D_0$.
  - If $D_0 \cup T_0$ is inconsistent, then $T_0 \models \neg \delta(\bar{a})$.
    *Note that $T_0$ does not mention $\bar{a}$ at all. So $T_0$ entails $\neg \delta(\bar{a})$ for arbitrary $\bar{a}$.*
  - So, $T_0 \models \forall \bar{x} \cdot \neg \delta(\bar{x})$, and as $\delta$ is quantifier free, we have $\forall \bar{x} \cdot \neg \delta(\bar{x}) \in T_\forall$.
  - Thus, in particular, $\mathcal{A} \models \neg \delta(\bar{a})$.
  - But $\delta(\bar{a}) \in \text{diag}(\mathcal{A})$, so $\mathcal{A} \models \delta(\bar{a})$. Contradiction.
- This proves Claim A. 

□
Preservation under substructures

Proof continued.

- By Claim A $\text{diag}(\mathcal{A}) \cup T$ has a model, say, $\mathcal{M}$.
- Since $\mathcal{M} \models \text{diag}(\mathcal{A})$, by the diagram lemma $\mathcal{A} \leq \mathcal{M}$.
- So, $\mathcal{A} \in \mathcal{K}$ as claimed, finishing the proof.

The “arbitrary constant” trick is a formal version of a common practice of proving a general statement by picking some arbitrary elements.

Exercise 5 (Very easy, but instructive)

Formally, the arbitrary constant trick is the following statement.

- Let $\Sigma$ be a signature, and $C$ a set of new constants. Let $S$ be a set of $\Sigma[C]$-sentences in which no constant from $c$ occurs, or, which amounts to the same thing, a set of $\Sigma$-sentences in the signature $\Sigma[C]$. Let $\varphi(\bar{c})$ be a $\Sigma[C]$-sentence, with $\bar{c}$ a sequence of constants from $C$. Then, $S \models \varphi(\bar{c})$ implies $S \models \forall \bar{x} \cdot \varphi(\bar{x})$.

Prove it without recourse to completeness, soundness, or proof rules.
Proof continued.

By Claim A \( \text{diag}(\mathcal{A}) \cup T \) has a model, say, \( \mathcal{M} \).

Since \( \mathcal{M} \models \text{diag}(\mathcal{A}) \), by the diagram lemma \( \mathcal{A} \leq \mathcal{M} \).

So, \( \mathcal{A} \in \mathcal{K} \) as claimed, finishing the proof.

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Prove it without recourse to completeness, soundness, or proof rules.
Preservation under substructures

Proof continued.

- By Claim A \( \text{diag}(A) \cup T \) has a model, say, \( M \).
- Since \( M \models \text{diag}(A) \), by the diagram lemma \( A \leq M \).
- So, \( A \in K \) as claimed, finishing the proof.

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Prove it without recourse to completeness, soundness, or proof rules.
Arbitrary constant trick in our formal setting

The diagram on the left preserves $\Sigma$ (i.e. $i : \Sigma[c] \to \Sigma[x]$ is the identity on $\Sigma$) and $i$ maps $c$ to $x$.

1. In our formal setting a variable is a triple, so we have $x = (t, s, \Sigma)$. If one ignores the third component (which is what we do here, to keep things as classical as we can), then both $\text{Sen}(\iota_c) : \text{Sen}(\Sigma) \to \text{Sen}(\Sigma[c])$ and $\text{Sen}(\iota_x) : \text{Sen}(\Sigma) \to \text{Sen}(\Sigma[x])$ are inclusions.

2. Since we work under the assumption that the carrier sets of the models are non-empty sets, all injective signature morphisms are conservative; in particular, inclusions are conservative;

By the two remarks above, it is safe to ignore the signature $\Sigma$, and write, simply, (a) $T \cup \{S[x]\}$ instead of $\iota_c(T) \cup \{S[x]\}$, and (b) $T \cup \{S[x]\} \vdash \bot$ instead of $T \cup \{S[x]\} \vdash \Sigma[x] \bot$.

Now, if $T \cup \{S[c]\}$ is inconsistent, then since $i(T \cup \{S[c]\}) = T \cup \{S[x]\}$, by (Translation), $T \cup \{S[x]\}$ is inconsistent as well; since $T \cup S[x] \vdash \bot$, by (NegI), $T \vdash \neg S[x]$; by the rule of generalization, we get $T \vdash \forall x \cdot \neg S[x]$; by soundness, $T \models \forall x \cdot \neg S[x]$. 
Elementary substructures, elementary embeddings, and elementary equivalence
Definition 4

Let \( A \) and \( B \) be \( \Sigma \)-structures. Let \( f : A \rightarrow B \) be a homomorphism.

\( A \) is an **elementary substructure** of \( B \), (written \( A \preceq B \)) if \( A \preceq B \) and, for every formula \( \varphi(\overline{x}) \) and every tuple \( \overline{a} \) from \( |A| \), we have \( A \models \varphi(\overline{a}) \) iff \( B \models \varphi(\overline{a}) \).

\( f : A \rightarrow B \) is an **elementary embedding** (written \( f : A \preceq e B \)) if \( f \) is an embedding (written \( f : A \hookrightarrow B \)) and for every formula \( \varphi(\overline{x}) \) and every tuple \( \overline{a} \) from \( |A| \), we have \( A \models \varphi(\overline{a}) \) iff \( B \models \varphi(f(\overline{a})) \).

If there exists an elementary embedding of \( A \) into \( B \), we write \( A \preceq B \).

Example 5

Let \( A \) and \( B \) be infinite pure identity structures. Then

\( A \preceq B \) and \( B \preceq A \) imply \( A \cong B \) holds trivially (in fact, \( A = B \)).

\( A \preceq B \) and \( B \preceq A \) imply \( A \cong B \) is Schröder-Bernstein Theorem.
Elementary substructures and embeddings

Definition 4

Let \( \mathcal{A} \) and \( \mathcal{B} \) be \( \Sigma \)-structures. Let \( f : \mathcal{A} \to \mathcal{B} \) be a homomorphism.

- \( \mathcal{A} \) is an elementary substructure of \( \mathcal{B} \), (written \( \mathcal{A} \preceq \mathcal{B} \)) if \( \mathcal{A} \subseteq \mathcal{B} \) and, for every formula \( \varphi(\overline{x}) \) and every tuple \( \overline{a} \) from \( |\mathcal{A}| \), we have \( \mathcal{A} \models \varphi(\overline{a}) \text{ iff } \mathcal{B} \models \varphi(\overline{a}) \).

- \( f : \mathcal{A} \to \mathcal{B} \) is an elementary embedding (written \( f : \mathcal{A} \overset{e}{\rightarrow} \mathcal{B} \)) if \( f \) is an embedding (written \( f : \mathcal{A} \rightarrow \mathcal{B} \)) and for every formula \( \varphi(\overline{x}) \) and every tuple \( \overline{a} \) from \( |\mathcal{A}| \), we have \( \mathcal{A} \models \varphi(\overline{a}) \text{ iff } \mathcal{B} \models \varphi(f(\overline{a})) \).

- If there exists an elementary embedding of \( \mathcal{A} \) into \( \mathcal{B} \), we write \( \mathcal{A} \preceq \mathcal{B} \).

Example 5

Let \( \mathcal{A} \) and \( \mathcal{B} \) be infinite pure identity structures. Then

- “\( \mathcal{A} \preceq \mathcal{B} \) and \( \mathcal{B} \preceq \mathcal{A} \) imply \( \mathcal{A} \simeq \mathcal{B} \)” holds trivially (in fact, \( \mathcal{A} = \mathcal{B} \)).

- “\( \mathcal{A} \preceq \mathcal{B} \) and \( \mathcal{B} \preceq \mathcal{A} \) imply \( \mathcal{A} \simeq \mathcal{B} \)” is Schröder-Bernstein Theorem.
Lemma 6 (Tarski-Vaught test)

Let $\mathcal{A} \leq \mathcal{B}$ be similar structures. If for every formula $\varphi(x, \bar{a})$ with parameters $\bar{a}$ from $\mathcal{A}$, we have that $\mathcal{B} \models \exists x \cdot \varphi(x, \bar{a})$ implies $\mathcal{A} \models \exists x \cdot \varphi(x, \bar{a})$, then $\mathcal{A} \leq \mathcal{B}$ holds.

Proof.

- Let $\varphi(\bar{y})$ be a formula, and $\bar{a}$ be a tuple from $|\mathcal{A}|$. We will show that $(\star)$ $\mathcal{B} \models \varphi(\bar{a})$ iff $\mathcal{A} \models \varphi(\bar{a})$.
- Induction on the length of quantifier prefix in $\varphi$.
- Base: If $\varphi$ is quantifier free, then $(\star)$ holds by definition of satisfaction.
- Step for $\forall$: If $\varphi$ is $\forall x \cdot \psi(x, \bar{y})$, then $\mathcal{B} \models \forall x \cdot \psi(x, \bar{a})$ implies $\mathcal{A} \models \forall x \cdot \psi(x, \bar{a})$, as $\mathcal{A} \leq \mathcal{B}$.
- For converse, if $\mathcal{B} \not\models \forall x \cdot \psi(x, \bar{a})$, then $\mathcal{B} \models \neg \psi(b, \bar{a})$ for some $b \in |\mathcal{B}|$, and thus $\mathcal{B} \models \exists x \cdot \neg \psi(x, \bar{a})$. By assumption, we conclude $\mathcal{A} \models \exists x \cdot \neg \psi(x, \bar{a})$.
- Therefore $\mathcal{A} \models \neg \psi(c, \bar{a})$ for some $c \in |\mathcal{A}|$, and so $\mathcal{A} \not\models \forall x \cdot \psi(x, \bar{a})$.
- Step for $\exists$: If $\varphi$ is $\exists x \cdot \psi(x, \bar{y})$, then $\mathcal{A} \models \exists x \cdot \psi(x, \bar{y})$ implies $\mathcal{B} \models \exists x \cdot \psi(x, \bar{y})$, as $\mathcal{A} \leq \mathcal{B}$.
- The converse is precisely the assumption. □
Tarski-Vaught test

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Proof.

- Let $\varphi(y)$ be a formula, and $\bar{a}$ be a tuple from $|\mathcal{A}|$. We will show that $(\star)$ $\mathcal{B} \models \varphi(\bar{a})$ iff $\mathcal{A} \models \varphi(\bar{a})$.
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- **Step for $\forall$:** If $\varphi$ is $\forall x \cdot \psi(x, y)$, then $\mathcal{B} \models \forall x \cdot \psi(x, \bar{a})$ implies $\mathcal{A} \models \forall x \cdot \psi(x, \bar{a})$, as $\mathcal{A} \leq \mathcal{B}$.
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- Therefore $\mathcal{A} \models \neg \psi(c, \bar{a})$ for some $c \in |\mathcal{A}|$, and so $\mathcal{A} \not\models \forall x \cdot \psi(x, \bar{a})$.
- **Step for $\exists$:** If $\varphi$ is $\exists x \cdot \psi(x, y)$, then $\mathcal{A} \models \exists x \cdot \psi(x, y)$ implies $\mathcal{B} \models \exists x \cdot \psi(x, y)$, as $\mathcal{A} \leq \mathcal{B}$.
- The converse is precisely the assumption.
Unions of elementary chains

**Lemma 7**

Let \( \{ \mathcal{A}_i : i < \gamma \} \) be a family of similar structures such that \( \mathcal{A}_i \preceq \mathcal{A}_j \) for \( i \leq j \). Let \( \mathcal{C} = \bigcup_{i < \gamma} \mathcal{A}_i \). Then, \( \mathcal{A}_i \preceq \mathcal{C} \) holds for any \( i < \gamma \).

**Proof.**

- We use Tarski-Vaught test.
- Take a formula \( \varphi(x, \bar{a}) \), where \( \bar{a} \) is a tuple from \( \mathcal{A}_i \).
- Assume \( \mathcal{C} \models \exists x \cdot \varphi(x, \bar{a}) \).
- Then, \( \mathcal{C} \models \varphi(c, \bar{a}) \) for some \( c \in |\mathcal{C}| \). By definition of \( \mathcal{C} \) we have that \( c \in |\mathcal{A}_j| \).
- If \( j \leq i \), then \( c \in |\mathcal{A}_i| \) and so \( \mathcal{A}_i \models \exists x \cdot \varphi(x, \bar{a}) \).
- Assume \( i < j \). Then \( \mathcal{A}_i \preceq \mathcal{A}_j \) and \( \mathcal{A}_j \models \exists x \cdot \varphi(x, \bar{a}) \). Thus, \( \mathcal{A}_i \models \exists x \cdot \varphi(x, \bar{a}) \) as well. \( \square \)

**Exercise 6**

- Prove that \( \preceq \) is an ordering relation on classes of similar structures.
- Give counterexamples showing that \( \mathcal{A} \preceq \mathcal{B} \preceq \mathcal{C} \) and \( \mathcal{A} \preceq \mathcal{C} \) implies neither \( \mathcal{A} \preceq \mathcal{B} \) nor \( \mathcal{B} \preceq \mathcal{C} \).
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Let \( \{ \mathcal{A}_i : i < \gamma \} \) be a family of similar structures such that \( \mathcal{A}_i \preceq \mathcal{A}_j \) for \( i \leq j \). Let \( \mathcal{C} = \bigcup_{i < \gamma} \mathcal{A}_i \). Then, \( \mathcal{A}_i \preceq \mathcal{C} \) holds for any \( i < \gamma \).

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- Then, \( \mathcal{C} \models \varphi(c, \bar{a}) \) for some \( c \in |\mathcal{C}| \). By definition of \( \mathcal{C} \) we have that \( c \in |\mathcal{A}_j| \).
- If \( j \leq i \), then \( c \in |\mathcal{A}_i| \) and so \( \mathcal{A}_i \models \exists x \cdot \varphi(x, \bar{a}) \).
- Assume \( i < j \). Then \( \mathcal{A}_i \preceq \mathcal{A}_j \) and \( \mathcal{A}_j \models \exists x \cdot \varphi(x, \bar{a}) \). Thus, \( \mathcal{A}_i \models \exists x \cdot \varphi(x, \bar{a}) \) as well. \( \square \)

Exercise 6

- Prove that \( \preceq \) is an ordering relation on classes of similar structures.
- Give counterexamples showing that \( \mathcal{A} \preceq \mathcal{B} \preceq \mathcal{C} \) and \( \mathcal{A} \preceq \mathcal{C} \) implies neither \( \mathcal{A} \preceq \mathcal{B} \) nor \( \mathcal{B} \preceq \mathcal{C} \).
Unions of elementary chains

Lemma 7

Let \( \{ A_i : i < \gamma \} \) be a family of similar structures such that \( A_i \preceq A_j \) for \( i \leq j \). Let \( C = \bigcup_{i < \gamma} A_i \). Then, \( A_i \preceq C \) holds for any \( i < \gamma \).

Proof.

- We use Tarski-Vaught test.
- Take a formula \( \varphi(x, \bar{a}) \), where \( \bar{a} \) is a tuple from \( A_i \).
- Assume \( C \models \exists x \cdot \varphi(x, \bar{a}) \).
- Then, \( C \models \varphi(c, \bar{a}) \) for some \( c \in |C| \). By definition of \( C \) we have that \( c \in |A_j| \).
- If \( j \leq i \), then \( c \in |A_i| \) and so \( A_i \models \exists x \cdot \varphi(x, \bar{a}) \).
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- Prove that \( \preceq \) is an ordering relation on classes of similar structures.
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Elementary equivalence and isomorphism

Definition 8

Let $\mathcal{A}$ and $\mathcal{B}$ be $\Sigma$-structures. $\mathcal{A}$ and $\mathcal{B}$ are **elementarily equivalent** (written $\mathcal{A} \equiv \mathcal{B}$) if for every $\Sigma$-sentence $\varphi$ we have $\mathcal{A} \models \varphi$ iff $\mathcal{B} \models \varphi$.

Lemma 9

Let $\mathcal{A}$ and $\mathcal{B}$ be similar structures. Then, $\mathcal{A} \simeq \mathcal{B}$ implies $\mathcal{A} \preceq \mathcal{B}$ and $\mathcal{A} \preceq \mathcal{B}$ implies $\mathcal{A} \equiv \mathcal{B}$. The converses do not hold.

Exercise 7

Let $\mathcal{A}$ and $\mathcal{B}$ be infinite pure identity structures. Suppose $|\mathcal{A}|$ is a proper subset of $|\mathcal{B}|$, of strictly smaller cardinality.

- **Prove that** $\mathcal{A} \preceq \mathcal{B}$ (apply Tarski-Vaught test), but $\mathcal{A} \not\simeq \mathcal{B}$. **Conclude that** $\mathcal{B} \equiv \mathcal{A}$, but $\mathcal{B} \not\preceq \mathcal{A}$.

- **Show that** $\mathcal{A}$ and $\mathcal{B}$ cannot be finite. Develop it into a proof of the fact that for finite structures elementary equivalence and isomorphism coincide.
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Let $\mathcal{A}$ and $\mathcal{B}$ be similar structures. Then, $\mathcal{A} \cong \mathcal{B}$ implies $\mathcal{A} \preceq \mathcal{B}$ and $\mathcal{A} \preceq \mathcal{B}$ implies $\mathcal{A} \equiv \mathcal{B}$. The converses do not hold.

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Let $\mathcal{A}$ and $\mathcal{B}$ be infinite pure identity structures. Suppose $|\mathcal{A}|$ is a proper subset of $|\mathcal{B}|$, of strictly smaller cardinality.

- Prove that $\mathcal{A} \preceq \mathcal{B}$ (apply Tarski-Vaught test), but $\mathcal{A} \not\cong \mathcal{B}$. Conclude that $\mathcal{B} \equiv \mathcal{A}$, but $\mathcal{B} \not\preceq \mathcal{A}$.
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- **Show that** $\mathcal{A}$ and $\mathcal{B}$ cannot be finite. **Develop it into a proof of the fact that** for finite structures elementary equivalence and isomorphism **coincide**.
Preservation under unions of chains

Theorem 10 (Chang-Łoś-Suszko)

Let $T$ be a theory. If $T$ is preserved under unions of chains, then $T$ is equivalent to a set of $\Pi_2$ formulas.

Proof.

- Let $T_{\forall \exists}$ be $\{\varphi \in \Pi_2 : T \models \varphi\}$. Thus, every model of $T$ is a model of $T_{\forall \exists}$.

  We will show that every model of $T_{\forall \exists}$ is a model of $T$.

- Let $\mathcal{A} \models T_{\forall \exists}$. Expand the signature from $\Sigma$, to $\Sigma[\overline{a}]$, by naming all elements of $\mathcal{A}$.

- Let $D_\forall(\mathcal{A}) = \{\varphi \in \Pi_1 : (\mathcal{A}, \overline{a}) \models \varphi\}$. $D_\forall(\mathcal{A})$ is the set of all universal $\Sigma[\overline{a}]$-sentences which are true in $(\mathcal{A}, \overline{a})$.

- **Claim B:** $D_\forall(\mathcal{A}) \cup T$ is consistent.

  - Suppose the contrary. Then for some $D_0 \subseteq_{\text{fin}} D_\forall(\mathcal{A})$ and $T_0 \subseteq_{\text{fin}} T$, the set $D_0 \cup T_0$ is inconsistent. Put $\delta(\overline{a_0}) = \bigwedge D_0$, where $\overline{a_0}$ are the constants occurring in $D_0$.

    Written explicitly, $\delta(\overline{a_0})$ is of the form $\forall x \cdot \eta(\overline{a_0}, \overline{x})$.

  - Then, $T_0 \models \neg \delta(\overline{a_0})$, that is, $T_0 \models \neg \forall x \cdot \eta(\overline{a_0}, \overline{x})$.

  - Therefore, $T_0 \models \forall y \cdot \neg \forall x \cdot \eta(\overline{y}, \overline{x})$, that is $T_0 \models \forall y \cdot \exists x \cdot \neg \eta(\overline{y}, \overline{x})$.

    Arbitrary constant trick at work again.

  - Thus, $\forall y \cdot \exists x \cdot \neg \eta(\overline{y}, \overline{x}) \in T_{\forall \exists}$. Therefore, $\mathcal{A} \models \forall y \cdot \exists x \cdot \neg \eta(\overline{y}, \overline{x})$.

  - Thus, $(\mathcal{A}, \overline{a}) \models \exists x \cdot \neg \eta(\overline{a_0}, \overline{x})$. But $\exists x \cdot \neg \eta(\overline{a_0}, \overline{x})$ is $\neg \delta(\overline{a_0})$, so $(\mathcal{A}, \overline{a}) \models \neg \delta(\overline{a_0})$.

  - On the other hand, $\delta(\overline{a_0}) = \bigwedge D_0$, so $(\mathcal{A}, \overline{a}) \models \delta(\overline{a_0})$. Contradiction.
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  - Suppose the contrary. Then for some $D_0 \subseteq \text{fin } D_{\forall}(\mathcal{A})$ and $T_0 \subseteq \text{fin } T$, the set $D_0 \cup T_0$ is inconsistent. Put $\delta(\bar{a}_0) = \bigwedge D_0$, where $\bar{a}_0$ are the constants occurring in $D_0$.

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  - Thus, $\forall \bar{y} \cdot \exists \bar{x} \cdot \neg \eta(\bar{y}, \bar{x}) \in T_{\forall\exists}$. Therefore, $\mathcal{A} \models \forall \bar{y} \cdot \exists \bar{x} \cdot \neg \eta(\bar{y}, \bar{x})$.

  - Thus, $(\mathcal{A}, \bar{a}) \models \exists \bar{x} \cdot \neg \eta(\bar{a}_0, \bar{x})$. But $\exists \bar{x} \cdot \neg \eta(\bar{a}_0, \bar{x})$ is $\neg \delta(\bar{a}_0)$, so $(\mathcal{A}, \bar{a}) \models \neg \delta(\bar{a}_0)$.

  - On the other hand, $\delta(\bar{a}_0) = \bigwedge D_0$, so $(\mathcal{A}, \bar{a}) \models \delta(\bar{a}_0)$. Contradiction.
Preservation under unions of chains

Proof continued: alternating chains method.

- Let $\mathcal{B} \models D_\forall(\mathcal{A}) \cup T$.
- Since $D_\forall(\mathcal{A}) \supseteq \text{diag}(\mathcal{A})$, we have $\mathcal{A} \leq \mathcal{B}$.
- **Claim C:** $\text{diag}(\mathcal{B}) \cup \text{Th}(\mathcal{A}, \overline{a})$ is consistent.
  - **Exercise.** Prove it using the arbitrary constant trick. Note that any tuple $\overline{b}$ from $|\mathcal{B}|$ can be written as $(\overline{c}, \overline{d})$ with $\overline{c} \in |\mathcal{B}| \setminus |\mathcal{A}|$ and $\overline{d} \in |\mathcal{A}|$.
- Let $\mathcal{A}_1 \models \text{diag}(\mathcal{B}) \cup \text{Th}(\mathcal{A}, \overline{a})$.
- Then $\mathcal{A} \leq \mathcal{B} \leq \mathcal{A}_1$, and $\mathcal{B} \models T$; moreover, $\mathcal{A} \leq \mathcal{A}_1$.
  - The elementarity of the embedding follows from the fact that $\mathcal{A}_1 \models \text{Th}(\mathcal{A}, \overline{a})$.
- Continuing inductively, we get $\mathcal{A} \leq \mathcal{B} \leq \mathcal{A}_1 \leq \mathcal{B}_1 \leq \mathcal{A}_2 \leq \ldots$ **Draw a diagram!**
- Let $\mathcal{C}$ be the union of this chain.
- Then, $\mathcal{C}$ is also the union of $\mathcal{B} \leq \mathcal{B}_1 \leq \mathcal{B}_2 \leq \ldots$
- Since $\mathcal{B}_i \models T$ for every $i$ by construction, and $T$ is preserved under unions of chains, we get $\mathcal{C} \models T$.
- But $\mathcal{C}$ is also the union of $\mathcal{A} \leq \mathcal{A}_1 \leq \mathcal{A}_2 \leq \ldots$
- Since $\mathcal{A} \leq \mathcal{A}_1 \leq \mathcal{A}_2 \leq \ldots$, by Lemma 7 we get $\mathcal{A} \leq \mathcal{C}$.
  - **Exercise:** there is a hidden use of a part of Exercise 6 here. Find it.
- Therefore, $\mathcal{A} \models T$, as required.
Free algebras as basic/canonical models

Theorem 11 (Basic/Canonical model)

Let $E$ be a set of atomic sentences in a signature $\Sigma$. Then, there exists a $\Sigma$-structure $M$ such that

1. $M \models E$.

2. Every element of $|M|$ is of the form $t^M$ for a closed term $t$.

3. If $N$ is an $\Sigma$-structure and $N \models E$, then there exists a homomorphism $h : M \to N$. If $\Sigma$ is non-void, then this homomorphism is unique.

Proof.

By Theorem 26 from Lecture 2. We called $M$ “basic”. It is often called “canonical”.

Example 12

Let $G$ be the set of equations $x \cdot (y \cdot z) = (x \cdot y) \cdot z$, $x^{-1} \cdot x = e = x \cdot x^{-1}$, $x \cdot e = x = e \cdot x$, that is, the group axioms. Let $E$ be the set of all atomic sentences obtained from $G$ by replacing variables by closed terms in the signature expanded by constants from some set $X$ (here the “variables are constants” trick works very nicely). Then, the basic model $G$ of $X$ is the free group $G[X]$ generated by $X$. 

29 / 37
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Adding roots of polynomials to a field

Example 13

Let $K$ be a field. As usual we write $K[x]$ for the ring of polynomials in $x$ over $K$. We can think of $K[x]$ as a structure in the signature of rings with additional constants for each element of $K$, and one more for $x$. Take a polynomial $p(x)$ which is irreducible over $K$. Let $T$ be the set of all equations that are true in $K[x]$. Now consider $T \cup \{p(x) = 0\}$. This is a set of atomic sentences, so it has a basic model $M$.

- By diagram lemma, there is an onto homomorphism $h: K[x] \rightarrow M$. In particular $M$ is a ring.

- As $M \models p(x) = 0$, we have $h(a) = 0^M$ for every $a$ in the ideal $(p(x))$.

- Because $K[x]_{p(x)}$ satisfies $T \cup \{p(x) = 0\}$, we get that $K[x]_{p(x)}$ is a homomorphic image of $M$.

- In fact, $K[x]_{p(x)}$ and $M$ are isomorphic. For consider $K[x] \rightarrow M \rightarrow K[x]_{p(x)}$. Their composition is the quotient map.
Free algebras again

Let $\mathcal{K}$ be a class of similar algebras, and $X$ a set of variables. We write $\text{Eq}_X(\mathcal{K})$ for the set of all equations over $X$ true in $\mathcal{K}$. If $X$ is clear from context we write $\text{Eq}(\mathcal{K})$.

**Lemma 14 (Free algebras)**

Let $\mathcal{K}$ and $X$ be as above, and $E = \text{Eq}_X(\mathcal{K})$. Let $F[X]$ be the basic model for $E$. Then, $F[X] \in \text{SP}(\mathcal{K})$. Moreover, $\text{Eq}_X(\mathcal{K}) = \text{Eq}_X(F[X])$.

**Proof.**

- Let $\Phi = \{ \varphi_i : i < \gamma \}$ be the set of all homomorphisms $\varphi_i : F[X] \to K_i$, for some $K_i \in \mathcal{K}$. We take it up to identity of kernels, so it is a set.

- The map $\psi : F[X] \to \prod_{i < \gamma} K_i$, defined coordinatewise as $\psi = (\varphi_i : i < \gamma)$, is a homomorphism.

- By universal property of $F[X]$, for every $a, b \in |F[X]|$ with $a \neq b$, we have a homomorphism $\varphi_i$ such that $\varphi_i(a) \neq \varphi_i(b)$. Note that $a$ and $b$ are values of some terms $t$ and $s$ over $X$, so $t \approx s \notin E$.

- Thus, the image $\psi(F[X])$ is a substructure of $\prod_{i < \gamma} K_i$. Hence, $F[X] \in \text{SP}(\mathcal{K})$, as claimed.

- For the moreover part: $\text{Eq}_X(\mathcal{K}) \subseteq \text{Eq}_X(F[X])$ follows by preservation of equations by SP; $\text{Eq}_X(\mathcal{K}) \supseteq \text{Eq}_X(F[X])$ follows by the remark above in blue.
Free algebras again

Let $\mathcal{K}$ be a class of similar algebras, and $X$ a set of variables. We write $\text{Eq}_X(\mathcal{K})$ for the set of all equations over $X$ true in $\mathcal{K}$. If $X$ is clear from context we write $\text{Eq}(\mathcal{K})$.

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Birkhoff’s HSP Theorem

**Theorem 15 (Birkhoff)**

Let $\mathbb{K}$ be a class of similar algebras, and $X$ some countably infinite set of variables. If $HSP(\mathbb{K}) \subseteq \mathbb{K}$, then $\mathbb{K} = \text{Mod}(\text{Eq}_X(\mathbb{K}))$.

**Proof.**

- $\mathbb{K} \subseteq \text{Mod}(\text{Eq}_X(\mathbb{K}))$ trivially holds.
- Let $A \in \text{Mod}(\text{Eq}_X(\mathbb{K}))$. Fix some set $Y$, with $\text{card}(Y) = \max\{\text{card}(X), \text{card}(A)\}$.
- We have $\text{Eq}_X(\mathbb{K}) = \text{Eq}_Y(\mathbb{K})$ (*exercise*).
- Let $F[Y]$ be the basic model (free algebra) for $\text{Eq}_Y(\mathbb{K})$.
- By Lemma 14, we have $F[Y] \in \text{SP}(\mathbb{K})$.
- By universal property of $F[Y]$ we have $A \in H(F[Y])$ (*since $Y$ is large enough*).
- Hence, $A \in HSP(\mathbb{K})$, and so $A \in \mathbb{K}$, as required.

**Exercise 8**

Prove that for any infinite $X$ and $Y$ we have $\text{Eq}_X(\mathbb{K}) = \text{Eq}_Y(\mathbb{K})$. 

Birkhoff’s HSP Theorem

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Let $K$ be a class of similar algebras, and $X$ some countably infinite set of variables. If $\text{HSP}(K) \subseteq K$, then $K = \text{Mod}(\text{Eq}_X(K))$.

Proof.

- $K \subseteq \text{Mod}(\text{Eq}_X(K))$ trivially holds.
- Let $A \in \text{Mod}(\text{Eq}_X(K))$. Fix some set $Y$, with $\text{card}(Y) = \max\{\text{card}(X), \text{card}(A)\}$.
- We have $\text{Eq}_X(K) = \text{Eq}_Y(K)$ (exercise).
- Let $F[Y]$ be the basic model (free algebra) for $\text{Eq}_Y(K)$.
- By Lemma 14, we have $F[Y] \in \text{SP}(K)$.
- By universal property of $F[Y]$ we have $A \in H(F[Y])$ (since $Y$ is large enough).
- Hence, $A \in \text{HSP}(K)$, and so $A \in K$, as required.

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- Hence, $A \in \text{HSP}(\mathcal{K})$, and so $A \in \mathcal{K}$, as required.

Exercise 8

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