RESEARCH ACHIEVEMENTS AND RESULTS

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My areas of research are differential and algebraic topology, particularly singularity theory of smooth maps, and also topology of stratified spaces. More specifically, my recent work relates the study of so-called fold maps to the theory of high-dimensional manifolds, especially to exotic spheres. Exploiting the topological impact of fold index constraints, I have introduced two new natural subgroup filtrations of the group of homotopy spheres. By employing an innovative palette of techniques ranging from parametrized Morse theory to symplectic geometry, I have shown that these filtrations capture plenty of essential phenomena of geometric topology. Among these are Milnor and Kervaire spheres studied in surgery theory, Banagl's positive TFTs, as well as the Gromoll filtration that connects to differential geometry.

Before discussing my various achievements, let me recall the notions of fold maps and homotopy spheres. A *fold map* can be defined in a catchy way as a map of smooth manifolds that looks locally like a family of Morse functions (i.e., differentiable functions with only non-degenerate critical points). Consequently, the singular locus of a fold map (its "folds") is always a submanifold on which the fold map restricts to a codimension one immersion. In analogy of the notion of Morse index, every fold component can be assigned an integer called (absolute) index which is either definite or indefinite. There is a long tradition in studying the following global problems about fold maps:

(a) *In what ways do fold maps reflect the topology of a space, e.g., in terms of invariants such as characteristic classes?* For instance, Levine [11] has characterized the existence of a fold map from certain manifolds into the plane in terms of their Euler characteristic. Eliashberg [6] studied fold maps between equidimensional manifolds.

(b) *Construct fold maps with desired properties such as prescribed boundary conditions, or constraints on the folds.* An important tool for the construction of fold maps is Eliashberg's folding theorem [7]. Roughly speaking, this sort of a "homotopy principle" produces up to homotopy from more algebraic data a fold map with prescribed singular locus, but it only works in the presence of all possible fold indices.

Homotopy spheres are compact differentiable manifolds having the homotopy type of the sphere of the same dimension. While being topologically spheres, homotopy spheres of dimension ≥ 5 do often admit non-standard ("exotic") differentiable structures – this was first realized by John Milnor in the 1950s with his revolutionary discovery of exotic 7-spheres. Subsequently, it was shown [10] that diffeomorphism classes of oriented differentiable structures on the topological sphere form a finite abelian group with group law induced by connected sum.

The following sections outline my achievements. In particular, my recent work [20, 21] on the standard filtration and the fold filtration – my new filtrations of the group of homotopy spheres – demonstrates the subtle interplay between fold maps and the study of exotic spheres.

THE STANDARD FILTRATION

The study of fold maps with only folds of definite index was initiated in 1974 by Burlet and de Rham [3] who coined the terminology *special generic map*. Ever since, special generic maps have extensively been studied by Saeki and others from the perspectives of classification problems [13, 14], cobordism theory [15, 14], and desingularization properties [17, 12].

In 1993 Saeki [14] posed the widely open problem to determine the set of integers p for which a given exotic sphere admits a special generic map into Euclidean space of dimension p. By invoking results of Weiss [18] from algebraic K-theory, I have recently [21] obtained an answer to Saeki's problem in case of the 7-dimensional Milnor sphere. After about 25 years this latest progress opens new perspectives for future research! My main insight, which is valid in any high dimension, relates my newly defined *standard filtration* strongly to the celebrated Gromoll filtration [9], thus building a bridge between global singularity theory and differential geometry. Let me describe these two filtrations in more detail.

The standard filtration is a new natural subgroup filtration of the group of homotopy spheres. Prior to the definition of the standard filtration I have introduced and studied so-called *standard* special generic maps. Roughly speaking, those are special generic maps from a homotopy sphere into some Euclidean space which factorize nicely over the closed unit ball. Their precise definition involves the important technique of Stein factorization [15]. Finally, the standard filtration of a given homotopy n-sphere is defined to be the greatest integer $p < n$ for which the homotopy sphere admits a standard special generic map into $\mathbb{R}^p$.

Homotopy spheres can be obtained from gluing two unit balls along their common boundary sphere. Measuring the complexity of the gluing map then leads naturally to the notion of Gromoll filtration [9]. The Gromoll filtration is still not fully understood even in dimension 7, where it contains a single interesting group which is only known to be a proper subgroup of the group of homotopy 7-spheres. While being far from computable, the Gromoll filtration plays a significant role in the field of Riemannian geometry in the context of curvature problems on exotic spheres. For instance, the related notion of Morse perfection [18] has implications on the existence of a Riemannian metric on a homotopy sphere whose sectional curvature satisfies the assumptions of the sphere theorem of Rauch, Berger and Klingenberg.

THE FOLD FILTRATION

Any homotopy sphere, be it exotic or not, is well-known to bound a *cobordism*, i.e. a compact oriented manifold of one dimension higher. In consequence, a rich source for invariants of homotopy spheres arises from (differential) topological properties of these cobordisms, a central example being Milnor's lambda-invariant. Saeki [15] has characterized the standard sphere among all homotopy spheres as the one which bounds a cobordism that admits a fold map into the plane with only folds of definite index (and with a nice behaviour near the boundary). Beyond Saeki's result I have incorporated interval-shaped constraints on the indefinite index of folds that arise naturally in Morse theory and relax Saeki's condition. Variation of the interval gives rise to the *fold filtration*, a new subgroup filtration of the group of homotopy spheres. My results [20] relate the fold filtration closely to another filtration that is defined in terms of topological connectedness of cobordisms, and can be characterized via Morse theory. A remarkable consequence is that indefinite folds are able to distinguish exotic Kervaire spheres from other exotic spheres in certain dimensions! The techniques I use to handle fold maps in this setting include Levine's elimination of cusps [11] paired with the complementary process of creating cusps, Cerf theory [4], Stein factorization [15], as well as the concept of forward handles that has been introduced by Gay and Kirby [8] in the context of symplectic geometry.

THE FOLD MAP TFT

In theoretical physics, the notion of cobordism enters significantly into Atiyah's axiomatic framework of topological field theories (TFTs). My research [20] about the fold filtration was motivated by the *fold map TFT*. This is a concrete example within Banagl's rigorous abstract framework of so-called *positive TFTs* [2], specified by systems of fields and action functionals. In contrast to Feynman's path integral positive TFTs avoid measure theoretic difficulties by invoking Eilenberg's concept of completeness of semirings from computer science. In [20] I have essentially contributed to computations of state sums within the fold map TFT. Apart from my results on the fold filtration that can be interpreted in terms of the aggregate invariant, I have eliminated a technical key condition on fields.

INTERSECTION SPACES

These are a homotopy theoretic approach of Banagl [1] towards the construction of a homology theory for stratified spaces that turns out to be non-isomorphic, but mirror symmetric in the context of string theory, to Goresky and MacPherson's intersection homology. My contribution [19] extends the applicability of the essential tool of Moore approximation from simply connected to arbitrary CW-complexes. This implies the existence of intersection spaces for all pseudomanifolds with isolated singularities. In addition, I give homotopy theoretic conditions on the links of the singularities under which functoriality of the intersection space construction holds.

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