The Milnor 7-Sphere does not admit a Special Generic Map into $\mathbb{R}^3$

Dominik Wrazidlo
Kyushu University

Introduction: Problem of Existence of Special Generic Maps

Let M^n denote a connected closed smooth manifold of dimension $n \geq 5$.

A *special generic function* on M^n is . . .

. . . a Morse function $M^n \to \mathbb{R}$ with exactly 2 critical points (equivalently, with only minima and maxima).

Theorem. *(Morse theoretic characterization of homotopy n-spheres)*
M^n is homeomorphic to S^n $\Leftrightarrow$ M^n admits a special generic function.

Special generic functions generalize naturally to maps with higher-dimensional codomain as follows.

A *special generic map* on M^n is . . .

. . . a smooth map $M^n \to \mathbb{R}^p$ ($p \in \{1, \dots, n\}$) all of whose critical points have the local normal form

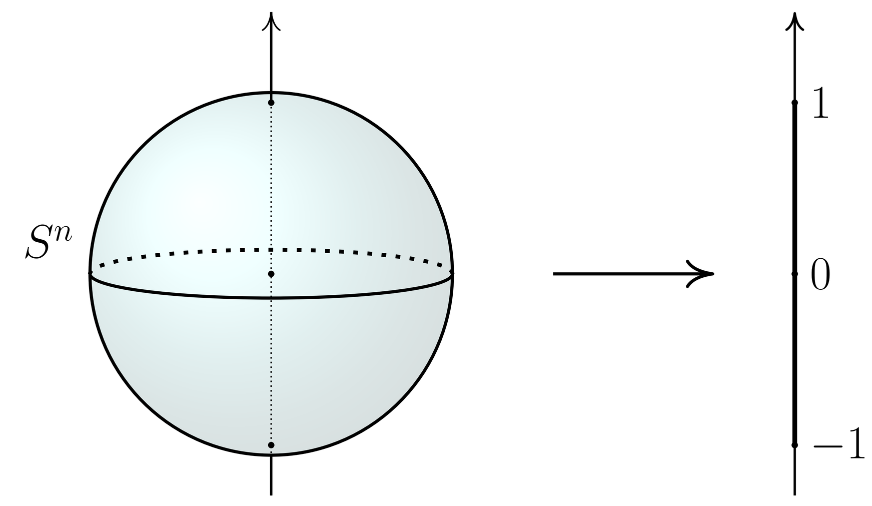
$$(x_1, \dots, x_n) \mapsto (x_1, \dots, x_{p-1}, x_p^2 + \dots + x_n^2).$$

Problem. *(O. Saeki, 1993)* Study the set

$$\mathcal{S}(M^n) := \{p \in \{1, \dots, n\} \mid \exists \text{ special generic map } M^n \to \mathbb{R}^p\}.$$

Example: Special Generic Maps on S^n

The canonical height function on S^n is a special generic function:



More generally, the composition of the inclusion $S^n \hookrightarrow \mathbb{R}^{n+1}$ with any orthogonal projection $\mathbb{R}^{n+1} \to \mathbb{R}^p$ yields a s.g.m. $S^n \to \mathbb{R}^p$.

Theorem. *(O. Saeki, 1993)* For any exotic n-sphere Σ^n, $n \geq 7$,

$$\{1, 2, n\} \subset \mathcal{S}(\Sigma^n) \subset \{1, 2, \dots, n-4, n\}.$$

Gromoll Filtration & Morse Perfection

Let Θ_n denote the well-known group of homotopy n-spheres of dimension $n \geq 6$.

The group Γ^n of *twisted spheres* consists of . . .

. . . equivalence classes γ represented by orientation preserving diffeomorphisms $D^{n-1} \cong D^{n-1}$ that are the identity near S^{n-2}, and considered up to isotopy through such maps.

Notation. For $a > b$ let $\pi^a_b \colon \mathbb{R}^a \to \mathbb{R}^b$ denote the projection to the *last* b coordinates. Let S^{n-1}_+ (resp. S^{n-1}_-) be the hemisphere of S^{n-1} with *first* component ≥ 0 (resp. ≤ 0).
Convention. For any $[g] \in \Gamma^n$ let $\varphi_g \colon S^{n-1} \cong S^{n-1}$ denote the diffeomorphism uniquely determined by requiring that φ_g extends $\mathrm{id}_{S^{n-1}_+}$, and that $\pi^n_{n-1} \circ \varphi_g = g \circ \pi^n_{n-1}$ on S^{n-1}_-.

Theorem. *(Twisted spheres)* The map $\Gamma^n \to \Theta_n$ which assigns to $\gamma \in \Gamma^n$ the homotopy n-sphere $\Sigma(\gamma)$ obtained by gluing two copies D_1 and D_2 of D^n via $\varphi_g \colon \partial D_1 \to \partial D_2$, and oriented such that $D_1 \hookrightarrow \Sigma(\gamma)$ is orientation preserving, is an isomorphism.

The *Gromoll filtration* of Γ^n is . . .

. . . a subgroup filtration of the form $0 = \Gamma^n_{n-1} \subset \dots \subset \Gamma^n_1 = \Gamma^n$, where

$$\gamma \in \Gamma^n_{p+1} \quad :\Leftrightarrow \quad \gamma = [g] \text{ such that } \pi^{n-1}_p \circ g = \pi^{n-1}_p.$$

The *Morse perfection* of M^n is . . .

. . . the greatest integer $k \geq -1$ for which there exists a smooth map $\eta \colon S^k \times M^n \to \mathbb{R}$ such that $\eta_{-s} = -\eta_s$ for all $s \in S^k$, and such that η restricts for every $s \in S^k$ to a **special generic function** $\eta_s \colon M^n \to \mathbb{R}, \eta_s(x) = \eta(s, x)$.

Theorem. *(M. Weiss, 1993)* If Σ^n is a homotopy sphere of dimension $n \geq 7$, then

$$\text{Gromoll filtration of } \Sigma^n \quad \leq \quad (\text{Morse perfection of } \Sigma^n) + 1.$$

Standard Special Generic Maps & Fold Perfection

The *Stein factorization* of a map $f \colon X \to Y$ is . . .

. . . the quotient $W_f := X/\sim_f$ of X by the following equivalence relation:
$x_1 \sim_f x_2 \quad :\Leftrightarrow \quad y := f(x_1) = f(x_2)$, and x_1, x_2 lie in the same component of $f^{-1}(y)$.
The quotient map $q_f \colon X \to W_f$ and the structure map $\overline{f} \colon W_f \to Y$ satisfy $f = \overline{f} \circ q_f$.

Proposition. If $f \colon M^n \to \mathbb{R}^p$ ($p < n$) is a special generic map, then W_f is a smooth p-manifold with boundary such that q_f is smooth, the singular locus is $S(f) = q_f^{-1}(\partial W_f)$, and $\overline{f}$ is an immersion. Moreover, q_f restricts to a diffeomorphism $S(f) \cong \partial W_f$.

Let Σ^n denote a homotopy sphere of dimension $n \geq 7$.

Proposition. *(O. Saeki, 1993)* If $f \colon \Sigma^n \to \mathbb{R}^p$ ($p < n$) is a special generic map, then W_f is a compact contractible smooth manifold whose boundary is a homology sphere.

A *standard special generic map* on Σ^n is . . .

. . . a special generic map $f \colon \Sigma^n \to \mathbb{R}^p$ ($p < n$) such that $W_f \cong D^p$.

Proposition. Any special generic map $f \colon \Sigma^n \to \mathbb{R}^p$ with $p \in \{1, 2, 3\}$ is standard.

The *fold perfection* of Σ^n is . . .

. . . the greatest integer $p \geq 1$ for which Σ^n admits a s.s.g.m. $\Sigma^n \to \mathbb{R}^p$.

The *standard filtration* of Θ^n is . . .

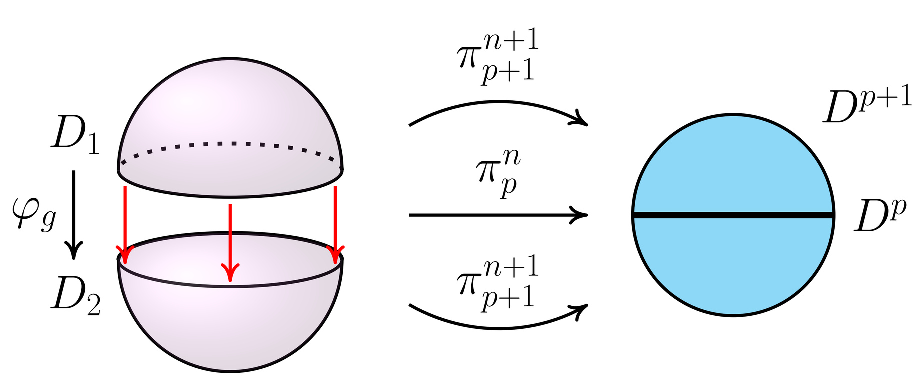
. . . a subgroup filtration of the form $F^n_{n-1} \subset \dots \subset F^n_1 \subset \Theta_n$, where

$$[\Sigma^n] \in F^n_p \quad :\Leftrightarrow \quad \Sigma^n \text{ has fold perfection} \geq p.$$

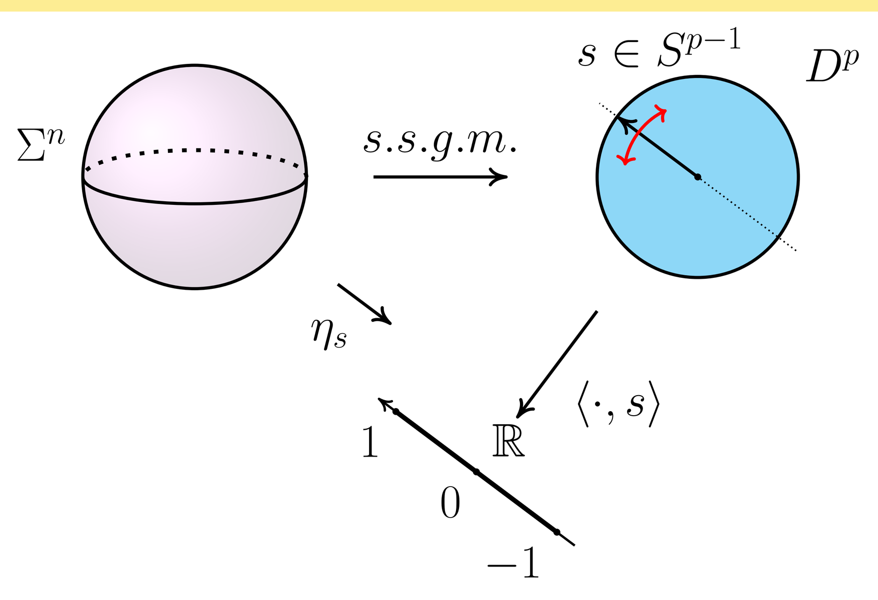
Main Result

Theorem. *(W., 2017)* Let Σ^n denote a homotopy sphere of dimension $n \geq 7$. Then,

$$\text{Gromoll filtration of } \Sigma^n \quad \leq \quad \text{fold perfection of } \Sigma^n \quad \leq \quad (\text{Morse perfection of } \Sigma^n) + 1.$$



$$\pi^{n-1}_p \circ g = \pi^{n-1}_p \quad \Rightarrow \quad \pi^n_p \circ \varphi_g = \pi^n_p$$



Application: Milnor Spheres

Suppose $n = 4k - 1$ for some integer $k \geq 2$.

The *Milnor n-sphere* Σ^n_M is . . .

. . . the unique homotopy sphere in Γ^n satisfying $\Sigma^n_M = \partial W^{n+1}$ for some parallelizable cobordism W^{n+1} of signature $\sigma(W) = 8$.

Theorem. *(M. Weiss, 1993)*

$$\text{Gromoll filtration of } \Sigma^n_M = 2 = (\text{Morse perfection of } \Sigma^n_M) + 1.$$

Corollary. *(W., 2017)* The fold perfection of Σ^n_M is 2, and

$$\mathcal{S}(\Sigma^7_M) = \{1, 2, 7\}.$$

In fact, this holds for all exotic 7-spheres in $\Theta_7 \setminus F^7_3$ (at least 14).

References

- O. Saeki, *Topology of special generic maps into* $\mathbb{R}^3$, Workshop on Real and Complex Singularities (São Carlos, 1992), Mat. Contemp. **5** (1993) 161–186.
- M. Weiss, *Pinching and concordance theory*, J. Differential Geometry **38** (1993) 387–416.
- D.J. Wrazidlo, *Standard special generic maps of homotopy spheres into Euclidean spaces*, submitted preprint (2017), arXiv: `https://arxiv.org/abs/1707.08646`