Fold Maps, Positive Topological Field Theories, and Exotic Spheres

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TFT & Gluing

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gluing axiom: $(M^n, N^n, P^n) \rightsquigarrow$ contraction product:

$$\langle \cdot, \cdot \rangle \colon Z(M \sqcup N) \otimes Z(N \sqcup P) \quad \longrightarrow \quad Z(M \sqcup P),$$

s.t. $Z_{\boldsymbol{W}} = \langle Z_{\boldsymbol{W'}}, Z_{\boldsymbol{W''}} \rangle$ whenever $\boldsymbol{W} \colon M \xrightarrow{\boldsymbol{W'}} N \xrightarrow{\boldsymbol{W''}} P$

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s.t. $Z_{W} = \langle Z_{W'}, Z_{W''}\rangle$ whenever $W\colon M \xrightarrow{W'} N \xrightarrow{W''} P$

further axioms: $Z(M \sqcup N) \cong Z(M) \otimes Z(N)$, $Z_{W \sqcup V} \cong Z_W \otimes Z_V$, $Z(-M) = Z(M)^*$ (unitary theory), $Z_{M\times[0,1]} = \mathrm{id}_{Z(M)}$

Examples (Gluing)

$W^{n+1}\colon M^n \xrightarrow{W'} N^n \xrightarrow{W''} P^n$

- Euler characteristic (n odd):

$$\chi(W) = \chi(W') + \chi(W'')$$

- Novikov additivity (compatibly oriented cobordisms):

$$\sigma(W) = \sigma(W') + \sigma(W'')$$

- Pontrjagin numbers ($n = 7$, compatibly oriented cobordisms, $M = P = \emptyset$, $\boldsymbol{H^3(N^7) = H^4(N^7) = 0}$):

$$p_1^2[W] = p_1^2[W'] + p_1^2[W'']$$

$\rightsquigarrow$ **Milnor's invariant $\boldsymbol{\lambda(N^7)}$**

A convenient setting for topological invariants

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- accept certain deviations from Atiyah's axioms
- obtain **positive TFT** & construct high-dimensional invariants!

Semirings

Definition

A **semiring** is a tuple $S = (S, +, \cdot, 0, 1)$, where

- $(S, +, 0)$ comm. monoid
- $(S, \cdot, 1)$ monoid

satisfying distributivity: $a(b + c) = ab + ac$, $(a + b)c = ac + bc$, and such that 0 is absorbing: $0 \cdot a = a \cdot 0 = 0$.

Example

- Boolean semiring $\mathbb{B} = \{0, 1\}$, require $1 + 1 = 1$
- semiring of formal power series $\mathbb{B}[\![q]\!]$
- tropical semiring $(\mathbb{R} \cup \{\infty\}, \min, +, \infty, 0)$
- ...

Eilenberg-Completeness

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Definition

1. A comm. monoid $(C, +, 0)$ is called **complete** if "$+$" extends to

$$\sum : \quad \{c_i\}_{i \in I} \longmapsto \sum_{i \in I} c_i \in C$$

   satisfying *Fubini's law*: $I = \dot{\bigcup}_{j \in J} I_j \Rightarrow \sum_{i \in I} c_i = \sum_{j \in J} \sum_{i \in I_j} c_i$.

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Eilenberg swindle: If S is an Eilenberg-complete *ring*, then

$$s := 1 + 1 + \cdots = 1 + (1 + \dots) = 1 + s \Rightarrow 0 = 1 \Rightarrow S = 0.$$

Banagl's Abstract Framework of Positive TFT

$\boldsymbol{C}$: small strict monoidal category
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- sets of **fields** $\mathcal{F}(W^{n+1})$, $\mathcal{F}(M^n)$,
  and compatible restriction maps (for codim. 0, 1):
  $\mathcal{F}(W) \to \mathcal{F}(W')$, $\quad \mathcal{F}(W) \to \mathcal{F}(M)$, $\quad \mathcal{F}(M) \to \mathcal{F}(M')$
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  in particular: $\mathcal{F}(W) \to \mathcal{F}(\partial W)$, $F \mapsto F|_{\partial W}$
- **action functional** $\mathbb{T}_W : \mathcal{F}(W) \to \mathrm{Mor}(\boldsymbol{C})$,
  $\mathbb{T}_{W' \sqcup W''}(F) = \mathbb{T}_{W'}(F|_{W'}) \otimes \mathbb{T}_{W''}(F|_{W''})$
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Step 1: Time-Interacting Fields

$F\colon W^{n+1} \to \mathbb{R}^2$ is called **fold map** if F looks at every singular point $c \in \boldsymbol{S(F)}$ in suitable coordinates centered at c and $F(c)$ like

$$(t, x) \mapsto (t, -x_1^2 - \cdots - x_i^2 + x_{i+1}^2 + \cdots + x_n^2).$$

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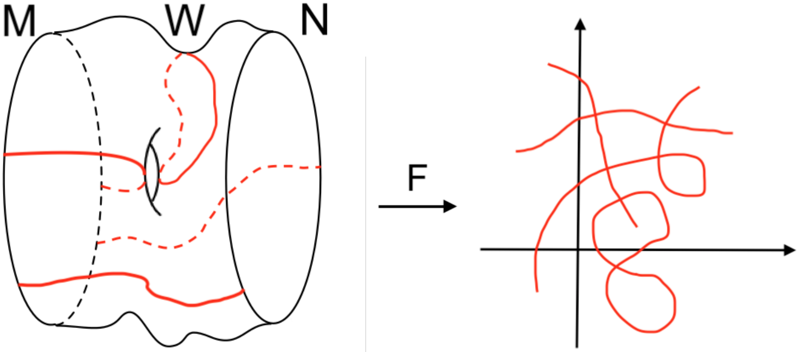

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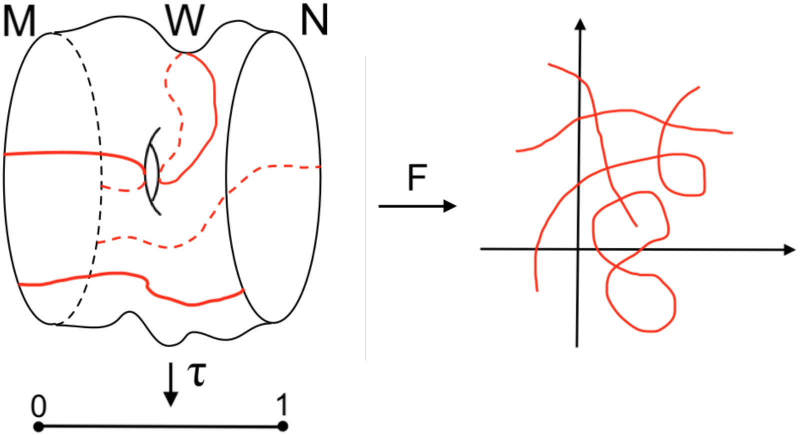

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$$\mathcal{F}(W) = \{F \text{ fold map} \mid \exists \textbf{ residual} \text{ subset } 0,1 \in A \subset [0,1]\ \forall t \in A: \\ t \in \mathrm{Reg}(\tau),\ \boldsymbol{S(F)} \pitchfork W_t,\ \mathrm{Im} \circ F \text{ is injective on } \boldsymbol{S(F)} \cap W_t\}$$

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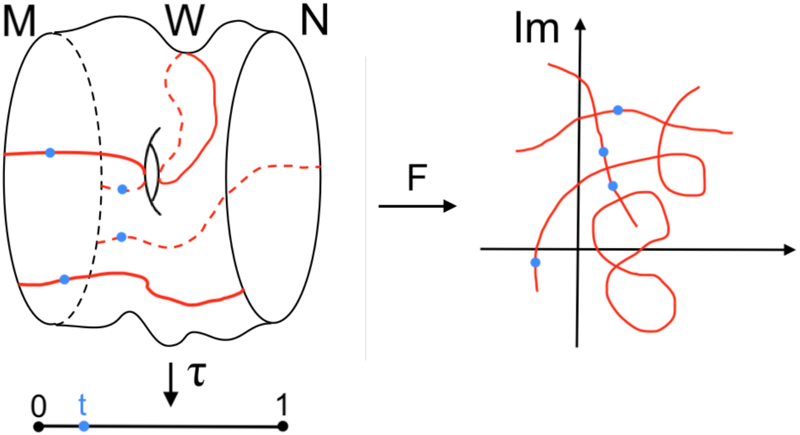
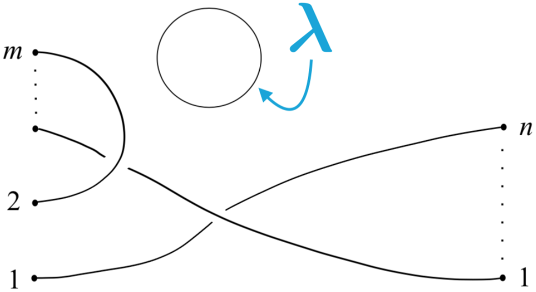


Step 2: Action Functional

The **Brauer category** $\boldsymbol{C} = (\boldsymbol{Br}, \otimes, [0], b)$ is the categorification of the Brauer algebras D_m arising in representation theory of $O(n)$:

- Ob $\boldsymbol{Br}$: $[0] = \emptyset, \ [1] = \{1\}, \ [2] = \{1, 2\}, \ \ldots$
- $\mathrm{Mor}_{\boldsymbol{Br}}([m], [n])$:



- $[m] \otimes [n] = [m + n]$; $\otimes$ of morphisms: vertical stacking
- braiding $b =$  $\in \mathrm{Mor}_{\boldsymbol{Br}}([2], [2])$

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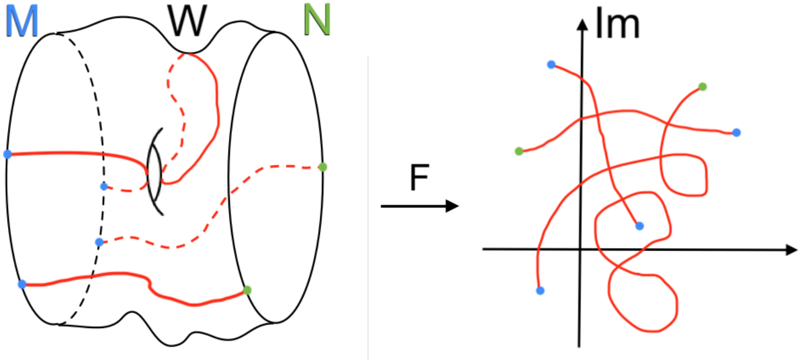

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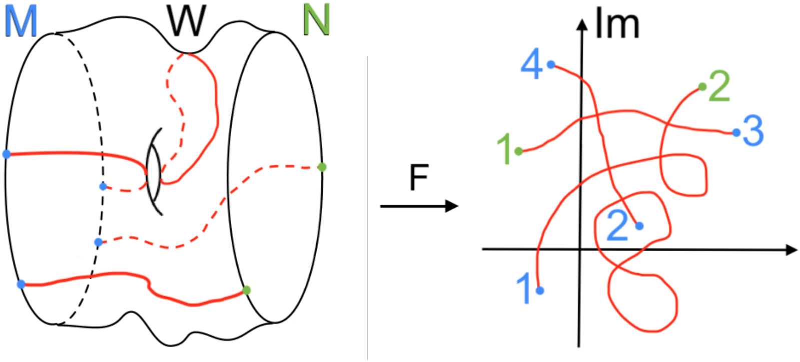

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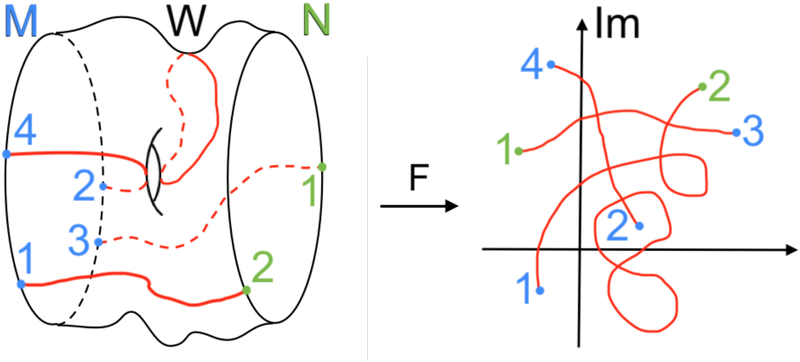

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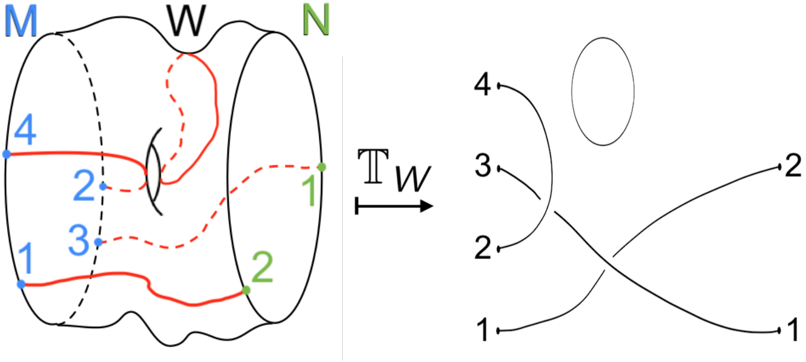

Which fold patterns are excluded?

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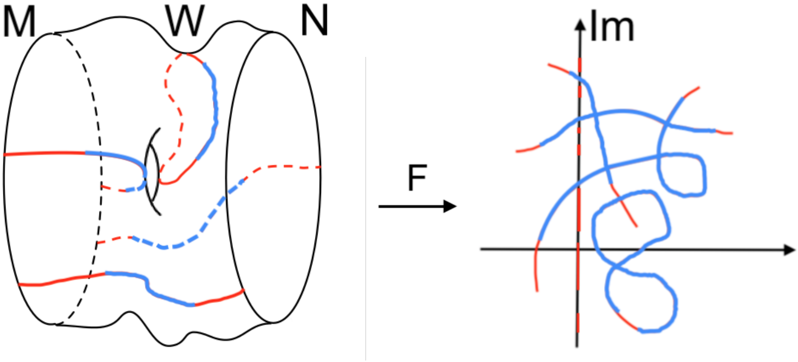
Theorem (W.)

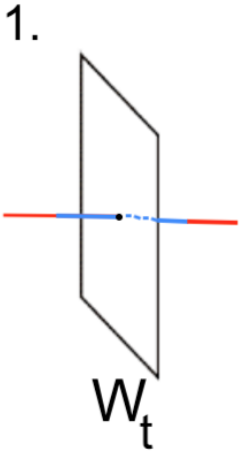
Every fold map $F: W^{n+1} \to \mathbb{R}^2$ satisfying $F|_{[0,\varepsilon]\times M} \in \mathcal{F}(M)$ and $F|_{[1-\varepsilon,1]\times N} \in \mathcal{F}(N)$ is homotopic rel $[0,\varepsilon] \times M \sqcup [1-\varepsilon, 1] \times N$ to a field $G \in \mathcal{F}(W)$ such that $\mathbb{T}_W(F) = \mathbb{T}_W(G)$.

Sketch of Proof



$\operatorname{Im} \circ F$ is injective on $S(F) \setminus X$

Sketch of Proof (continued)



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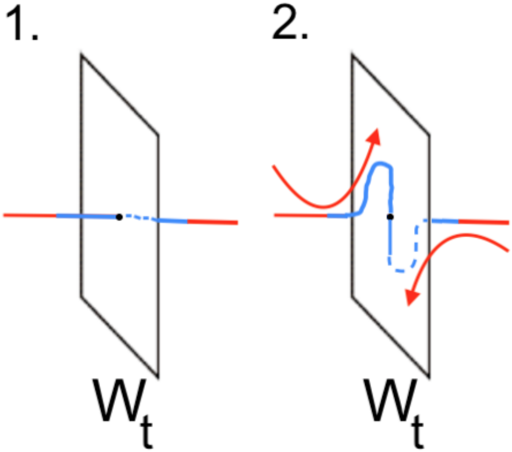

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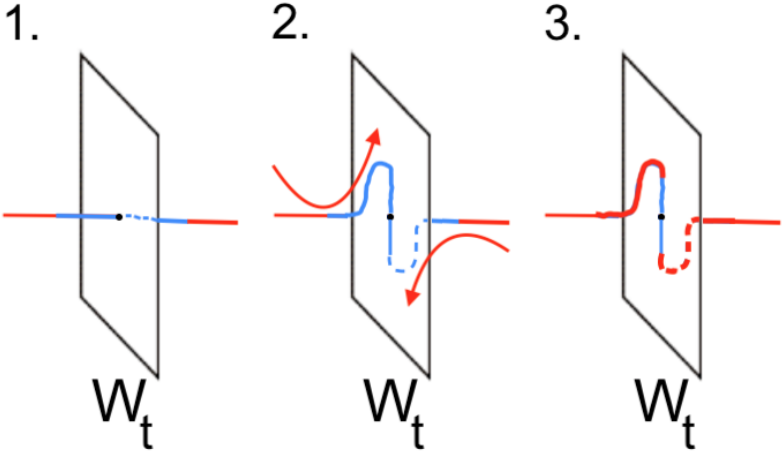

Step 3: Quantization

- **profinite idempotent completion:**

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Theorem (Banagl, 2015)

Z is a **positive TFT**. *In particular, time-consistent diffeomorphism invariance and the gluing axiom hold.*

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Theorem (W.)

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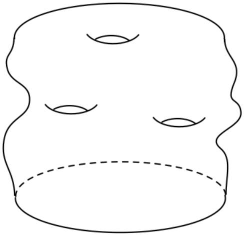

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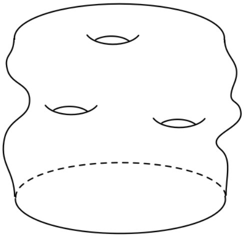

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Definition (aggregate invariant)

$$\mathfrak{A}(M^n) := \sum_{W^{n+1}} Z_W \quad \in Z(M).$$

Application: Exotic Smooth Spheres

Let $n \geq 5$.

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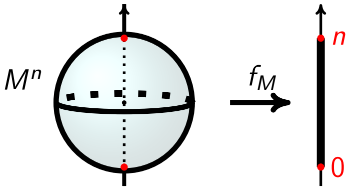
FACT. $M^n = S^n$ and $M^n = \Sigma^n$ have Morse number 2:



Application: Exotic Smooth Spheres

Let $n \geq 5$.

Definition

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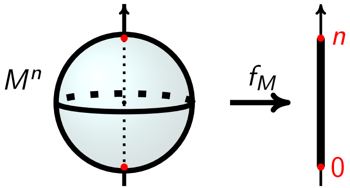

Theorem (Banagl, 2015)

$$M^n \cong S^n \iff \mathfrak{A}(M^n)(\bar{f}_M) \notin q \cdot Q.$$

Detecting Exotic Kervaire Spheres

Let $n = 4k + 1$, $k \geq 1$.

- Σ_K^n: unique **Kervaire sphere** of dimension n
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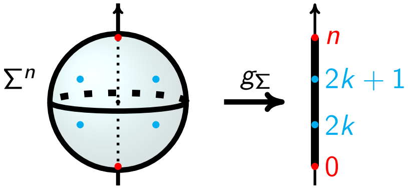

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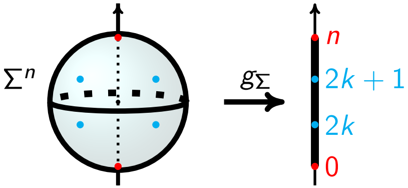

Theorem (W.)

Let $n \geq 237$ *and* $n \equiv 13 \pmod{16}$. *Then,*

$$\Sigma^n \cong \Sigma^n_K \iff \mathfrak{A}(\Sigma^n)(\overline{g}_\Sigma) \notin q \cdot Q.$$

Thank you for your attention!