

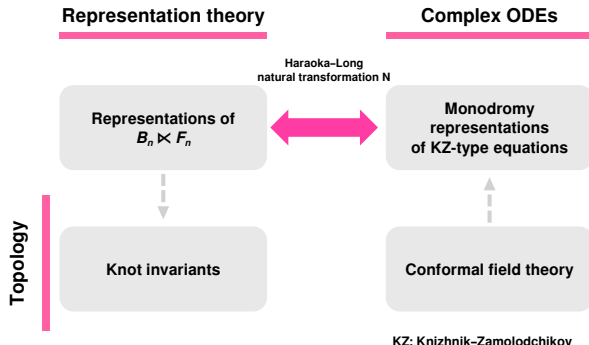
Foundations of Quantum Programming Contributed talk

From hypergeometric functions to quantum computing

Haru Negami

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We use algebraic and analytic methods to study braid group representations and middle convolution for KZ-type equations.

Goal and contributions

We construct a family of unitary Katz–Long–Moody (KLM) braid representations from character-twisted Burau inputs

Two questions about the resulting gates

1. What do their spectra and invariant subspaces tell us about non-Clifford gates and entanglement?
2. How can we compile them into elementary quantum circuits?

We explicitly describe the spectra and invariant summands and prove an $O(M)$ bound on circuit length, where M is the output dimension.

The construction, invariant forms, and signature criterion are prior results.

Representations and braid gates

A representation assigns a matrix to each group element:

$$\rho : G \longrightarrow \mathrm{GL}(V), \quad \rho(gh) = \rho(g)\rho(h).$$

For a unitary braid representation,

$$\rho(\sigma_i) = U_i \in U(V), \quad \rho(\sigma_{i_1} \cdots \sigma_{i_s}) = U_{i_1} \cdots U_{i_s}.$$

We construct the matrices and study the gates they define.

This is an algebraic construction; the talk does not assume a physical anyon implementation.

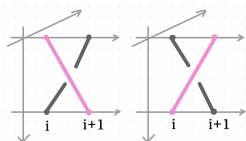
Braid groups: a geometric viewpoint

Definition: Artin's braid group B_n ¹

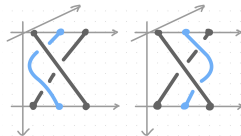
For $n > 1$, the group with generators $\sigma_1, \dots, \sigma_{n-1}$ and braid relations

1. $\sigma_i \sigma_j = \sigma_j \sigma_i$ ($|j - i| > 1$)
2. $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ ($i = 1, \dots, n - 2$)

Geometrically, B_n is the group of braids formed by n intertwined strands.



(a) σ_i and σ_i^{-1}



(b) Reidemeister moves

Figure 1: Geometric realizations of braids

¹[Artin, 1925]

Background: monodromy representations

Monodromy of complex linear differential equations

- Group: the fundamental group of the domain
- Space: the vector space of solutions
- Action: analytic continuation around a singularity induces a linear transformation of the solution space.



Along a fundamental-group generator x_i , analytic continuation maps the solution space $\mathcal{V}$ linearly to $M_i \mathcal{V}$.

KZ-type equations

Definition: KZ-type equations [Haraoka, 2020]

An integrable system with constant $N \times N$ matrices A_{ij} :

$$\frac{\partial u}{\partial z_i} = \sum_{j \neq i} \frac{A_{ij}}{z_i - z_j} u, \quad A_{ij} = A_{ji}, \quad i = 1, \dots, n$$

$$\begin{array}{ccccc}
 \rho^{\text{anti}} & \xrightarrow{\text{monodromy representation}} & u(z_1, \dots, z_n) & \xrightarrow{\text{solution}} & u_{z_i} = \sum \left(\frac{A_{ij}}{z_i - z_j} \right) u \\
 \downarrow \text{convolution} & & \downarrow \text{Euler transform} & & \downarrow \\
 (\rho^{\text{anti}})_\lambda^{\text{H}} & \xrightarrow{\quad} & \hat{u}(z_1, \dots, z_n) = \int u(t, z_2, \dots, z_n) (z_1 - t)^\lambda dt & - & \hat{u}_{z_i} = \sum \left(\frac{\widehat{A_{ij}}}{z_i - z_j} \right) \hat{u}
 \end{array}$$

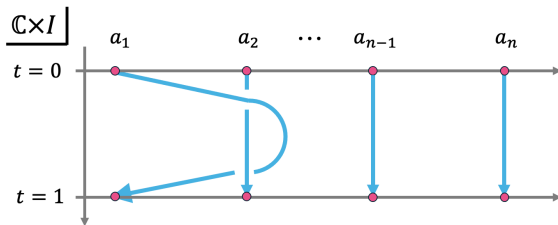
Fixing all but one variable in a KZ-type equation yields a Fuchsian system in one variable, with hypergeometric equations arising as special cases.

KZ-type equations and pure braids

The domain of a KZ-type equation is the ordered configuration space

$$\text{Conf}(\mathbb{C}^n) := \{(z_1, \dots, z_n) \mid z_i \neq z_j (i \neq j)\}$$

Its fundamental group is P_n , generated by the corresponding braid paths.



$$\gamma_{1,2}(t) = (\gamma_1(t), \dots, \gamma_n(t)) \in \{(z_1, \dots, z_n) \mid z_i \neq z_j \text{ for any } i \neq j\}$$

The action of B_n on F_n (Artin representation)

Let F_n be the free group of rank n ,
with generators $(x_j)_{j=1,\dots,n}$.

The action of B_n on F_n (Artin representation)^a

$$\Phi: B_n \ni \sigma_i \longmapsto \overline{\sigma}_i \in \text{Aut}(F_n)$$

is defined by

$$\overline{\sigma}_i(x_j) := \begin{cases} x_{i+1} & j = i \\ x_{i+1}^{-1} x_i x_{i+1} & j = i + 1 \\ x_j & \text{otherwise} \end{cases}$$

^a[Artin, 1925]

Together these actions form $F_n \rtimes B_n$. A representation of this group is the compatible input for KLM.

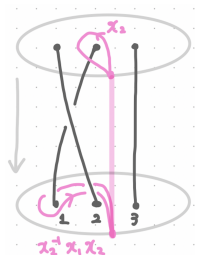


Figure 2: The action of σ_1 on F_3

Example: the Burau representation

The following representation of B_n is the Burau representation.

$$\begin{array}{ccc} \rho: & B_n & \longrightarrow & \mathrm{GL}(\mathbb{Z}[t^{-1}, t]^{\oplus n}) \\ & \Psi & & \Psi \end{array}$$
$$\sigma_i \longmapsto \begin{pmatrix} I_{i-1} & & \\ & \begin{pmatrix} 0 & t \\ 1 & 1-t \end{pmatrix} & \\ & & I_{n-i-1} \end{pmatrix}$$

For our seed, use $n + 1$ strands and set $t = q$. The character twist multiplies each free-group input operator by τ .²

²Burau representation: [Burau, 1935].

The Long-Moody construction

Definition: the Long-Moody construction [Long, 1994]

For an input $\rho : F_n \rtimes B_n \rightarrow \mathrm{GL}(V)$, with $N = \dim V$, the LM construction gives

$$\rho^{\mathrm{LM}} : B_n \longrightarrow \mathrm{GL}(V^{\oplus n})$$

On $V^{\oplus n}$, each braid generator acts by

$$\rho^{\mathrm{LM}}(\sigma_i) := \begin{pmatrix} \rho(\sigma_i) \otimes I_{i-1} & & & \\ & \begin{matrix} 0 & \rho(\sigma_i x_i) \\ \rho(\sigma_i) & \rho(\sigma_i)(1 - \rho(x_{i+1})) \end{matrix} & & \\ & & & \rho(\sigma_i) \otimes I_{n-i-1} \end{pmatrix}$$

The output dimension is nN .

Dettweiler–Reiter convolution

Definition [Dettweiler and Reiter, 2000]

Let V be an N -dimensional k -vector space, $\lambda \in k^\times$, and (x_j) a generating system of F_n .

Given $\rho: F_n \longrightarrow \mathrm{GL}(V)$, define $\rho_\lambda^{\mathrm{DR}}: F_n \longrightarrow \mathrm{GL}(V^{\oplus n})$ by

$$\rho_\lambda^{\mathrm{DR}}(x_j) := \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ \lambda(\rho(x_1) - 1) & \cdots & \lambda(\rho(x_{j-1}) - 1) & \lambda\rho(x_j) & \rho(x_{j+1}) - 1 & \cdots & \rho(x_n) - 1 \\ & & & 1 & & & \\ & & & & \ddots & & \\ & & & & & 1 & \end{pmatrix}$$

The twisted Long–Moody construction

The extension combines the Long–Moody braid action with the Dettweiler–Reiter convolution of the free-group action.

$$\rho : F_n \rtimes B_n \longrightarrow \mathrm{GL}(V) \quad \rightsquigarrow \quad \rho_\lambda^{\mathrm{LM}} : F_n \rtimes B_n \longrightarrow \mathrm{GL}(V^{\oplus n}).$$

$$\rho_\lambda^{\mathrm{LM}}(\sigma_i) = \rho^{\mathrm{LM}}(\sigma_i),$$

$$\rho_\lambda^{\mathrm{LM}}(x_j) = \rho_\lambda^{\mathrm{DR}}(x_j).$$

The two actions remain compatible.

$\lambda \neq 0$. The braid matrices themselves are independent of λ .
[Hiroe and Negami, 2023, Negami, 2025]

The Katz–Long–Moody construction

Let $g_j := \rho(x_j)$ and let the tLM representation ρ_λ^{LM} act on $V^{\oplus n}$.

$$K := \left\{ (w_1, \dots, w_n)^T; w_j \in \text{Ker}(g_j - I) \right\},$$

$$L := \bigcap_{j=1}^n \text{Ker} \left(\rho_\lambda^{\text{LM}}(x_j) - I_{Nn} \right).$$

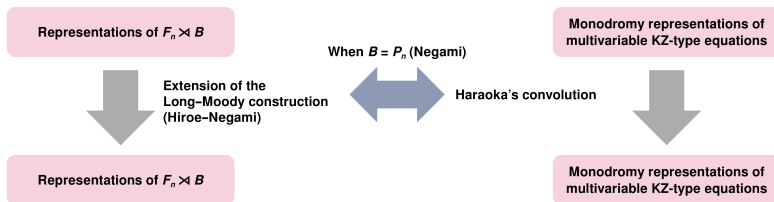
The Katz–Long–Moody construction [Hiroe and Negami, 2023, Negami, 2025]

$K + L$ is ρ_λ^{LM} -invariant. The KLM construction $\rho_\lambda^{\text{KLM}}$ is the induced representation on the quotient

$$V^{\oplus n} / (K + L)$$

KLM and middle convolution

The algebraic construction corresponds to the integral transform for KZ-type equations, after the appropriate quotient.



This correspondence transfers questions about differential equations to explicit braid-representation matrices.

[Hiroe and Negami, 2023, Haraoka, 2020, Negami, 2025]

Invariant forms and positive inner products

Earlier work constructs a canonical invariant Hermitian form H_λ on the KLM quotient:

$$A_i^\dagger H_\lambda A_i = H_\lambda.$$

A_i : the matrix representing the braid generator σ_i on the KLM quotient.

Its signature counts positive and negative eigenvalues:

$$\text{sig}(H_\lambda) = (n_+, n_-).$$

- A nondegenerate invariant form can be indefinite.
- Quantum gates require a positive inner product.

When does the canonical form have $n_- = 0$?

Invariant forms: [Negami, 2025]. Signature formula: [Negami, 2026].

A signature formula from input eigenangles

Assume a positive-definite unitary input, with $N = \dim V$. Put $g_j = \rho(x_j)$, $P_n = g_1 \cdots g_n$, and $\lambda = e^{2\pi i l}$, $0 < l < 1$.

- $a_j^{(k)}$: eigenangles of g_j in $[0, 1)$.
- $\beta_k(l)$: eigenangles of λP_n in $[0, 1)$.

$$R = \sum_j \text{rank}(g_j - I), \quad m(l) = Nl + \sum_{j,k} a_j^{(k)} - \sum_k \beta_k(l).$$

Away from the walls $1 \in \text{spec}(\lambda P_n)$,

$$\boxed{\text{sig}(H_\lambda) = (R - m(l), m(l)).}$$

$m(l) = 0$ exactly when this quotient form is positive definite.

$\lambda^{1/2} = e^{\pi i l}$. [Negami, 2026, Theorem 1.1 and Corollary 1.2]

Example: the Burau representation

Recall the definition of the Burau representation of B_n :

$$\begin{array}{ccc} \rho: & B_n & \longrightarrow & \mathrm{GL}(\mathbb{Z}[t^{-1}, t]^{\oplus n}) \\ & \Downarrow & & \Downarrow \\ & \sigma_i & \longmapsto & \begin{pmatrix} I_{i-1} & & \\ & \boxed{\begin{pmatrix} 0 & t \\ 1 & 1-t \end{pmatrix}} & \\ & & & I_{n-i-1} \end{pmatrix} \end{array}$$

For our seed, use $n + 1$ strands and set $t = q$. The character twist multiplies each free-group input operator by τ .³

³Burau representation: [Burau, 1935].

Character-twisted Burau input

Let $\rho_q : B_{n+1} \rightarrow \mathrm{GL}(V)$ be the unreduced Burau representation, with $V = \mathbb{C}^{n+1}$ and $q \in \mathbb{C}^\times$.

Use the subgroup $F_n \rtimes B_n \subset B_{n+1}$ fixing strand 0:

$$x_1 = \sigma_0^2, \quad x_{j+1} = \sigma_j x_j \sigma_j^{-1} \quad (1 \leq j < n).$$

Definition: the character twist

For $\tau \in \mathbb{C}^\times$, define $\chi_\tau(x_j) = \tau$ and $\chi_\tau(\sigma_i) = 1$. The input representation is

$$\rho_{q,\tau} := \chi_\tau \otimes \rho_q|_{F_n \rtimes B_n},$$

$$g_j = \tau \rho_q(x_j), \quad s_i = \rho_q(\sigma_i).$$

q : Burau parameter, τ : character parameter, λ : convolution parameter.

The positive window for the Bureau family

Take $n \geq 2$, $N = n + 1$, and

$$q = e^{2\pi ia}, \quad \tau = e^{2\pi ic}, \quad a, c > 0, \quad nc + (n + 1)a < 1.$$

The boundary product $P_n = g_1 \cdots g_n$ has eigenangles

$$\gamma_k : \quad nc, \quad \underbrace{nc + a, \dots, nc + a}_{n-1 \text{ times}}, \quad nc + (n + 1)a.$$

Here $R = M = n(n + 1)$ and, away from the walls,

$$m(l) = \#\{k : l + \gamma_k > 1\}.$$

Before the first eigenangle crosses 1,

$$0 < l < 1 - nc - (n + 1)a \implies \text{sig}(H_\lambda) = (M, 0).$$

An application of [Negami, 2026, Corollary 6.10] to the boundary spectrum.

A family of unitary braid gates

Twisted Burau input of dimension $n + 1$, with $n \geq 2$.

$$M = n(n + 1), \quad U_i = CA_iC^{-1} \in U(M).$$

Earlier form and signature results supply $H = C^\dagger C > 0$.

One common change of basis for all braid generators.

$$q = e^{2\pi ia}, \tau = e^{2\pi ic}, \text{ with } a, c > 0 \text{ and } nc + (n + 1)a < 1.$$

Unreduced and reduced Burau representations

Recall $\rho_t : B_n \rightarrow \mathrm{GL}(\mathbb{C}^n)$, with $t \in \mathbb{C}^\times$:

$$\rho_t(\sigma_i) = I_{i-1} \oplus \begin{pmatrix} 0 & t \\ 1 & 1-t \end{pmatrix} \oplus I_{n-i-1}.$$

In this convention, the coordinate sum is invariant:

$$\ell(v_1, \dots, v_n) = v_1 + \dots + v_n, \quad \ell \circ \rho_t(\beta) = \ell \quad (\beta \in B_n).$$

The reduced Burau representation

Restrict to the invariant hyperplane $V_{\mathrm{red}} = \ker \ell$:

$$\mathrm{Bur}_n^{\mathrm{red}}(t) = \rho_t|_{V_{\mathrm{red}}}, \quad \dim V_{\mathrm{red}} = n - 1.$$

In our parameter region, $\rho_t \cong \mathbf{1} \oplus \mathrm{Bur}_n^{\mathrm{red}}(t)$ for $t = q, \tau$.

The seed uses $n + 1$ strands; the reduced summands below use n strands.

Eigenvalues: $1, -q, -\tau, \pm q^{3/2}\tau$.

The eigenvalue multiset recovers (q, τ) .

$$\mathbb{C}^M \cong \mathbf{1}^{\oplus 2} \oplus \text{Bur}_n^{\text{red}}(\tau) \oplus \text{Bur}_n^{\text{red}}(q) \oplus \mathcal{R}_n.$$

- $\dim \mathcal{R}_n = n(n-1)$, generically irreducible for $n = 3, 4$.
- Full braid image: **not projectively dense**.

$\text{Bur}_n^{\text{red}}$ has dimension $n-1$. $\mathcal{R}_n$ is Long's reduced construction.

Non-Clifford and entangling gates

If a or c is irrational:

- Every generator is non-Clifford in every Weyl–Heisenberg frame.

If $1, a, c$ are linearly independent over $\mathbb{Q}$:

- Every generator is entangling for every nontrivial bipartition.

For each bipartition, some product input becomes entangled.

These statements concern $\mathbb{C}^M$ in the positive parameter region defined above.

One basis, different block partitions

For $n = 3$, the output has dimension $M = 12$. In the positive window, **one orthonormal basis** $b_1, \dots, b_{12}$ gives

$$U_1 = \text{diag} \left(\underbrace{C_1}_{8 \times 8}, 1, \underbrace{D_1}_{2 \times 2}, 1 \right),$$

$$U_2 = \text{diag} \left(I_2, \underbrace{D_2}_{2 \times 2}, \underbrace{C_2}_{8 \times 8} \right).$$

Here $C_i \in U(8)$ and $D_i \in U(2)$ denote the indicated diagonal blocks of U_i .

- For U_1 , the large block acts on $\text{span}(b_1, \dots, b_8)$.
- For U_2 , the large block acts on $\text{span}(b_5, \dots, b_{12})$.

The basis is fixed; the block partitions depend on the generator.

A block for one generator need not be invariant under another.

This structure lets us synthesize every generator in one fixed encoding.

Exact circuits of size $O(M)$

A common orthonormal basis gives small blocks for every generator.

Exact gate count $O(M)$.

- Arbitrary one-qubit gates and all-to-all CNOTs.
- No ancillas. Direction preparation and its inverse are included.

Linear in dimension, exponential in encoded qubit count.
Classical preprocessing is excluded.

CNOT counts: our construction versus QSD

Quantum Shannon decomposition (QSD) is a general method for converting a unitary matrix into a circuit of one-qubit gates and CNOTs.

We compare two methods on **the same braid-generator matrices**, using the same encoding and number of qubits.

n	Qubits	Our construction	Reference QSD
3	4	198–202	144–164
4	5	554–576	718–720
8	7	3178–3218	12096

Both methods use arbitrary one-qubit gates and all-to-all CNOTs, with **no ancillary qubits**.

Our construction uses fewer CNOTs in the larger examples shown.

The reference QSD implementation omits the optimal two-qubit base case and additional optimizations.

Conclusions and questions

- The signature formula identifies positive inner products for a concrete KLM braid-representation family.
- Spectral structure gives non-Clifford and entangling gates under explicit parameter conditions.
- A common basis gives circuits of size $O(M)$, with a matched Quantum Shannon reference comparison.

Which quantum-computing tasks can exploit this structure?

Quantum gates from the middle convolution of twisted Burau representations
Haru Negami, preprint, September 25, 2026

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