On some forms of logical connections between theories

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Alter-theories

A sentence A is called independent of T iff when both T + A and T + ~ A are consistent.

If T is a consistent theory with non-logical axioms A1, A2, ..., then it is always possible to choose a subset of axioms Ak1, Ak2, ..., such that these axioms are independent of the others and their set of logical consequences is identical to T.

However, we do not know whether the axioms of every {~Tki} theory i.e. a theory in which one of the axioms, say Aki, is replaced with its negation, i.e. ~Aki, are still independent.

If they are still independent, then we can construct another consistent theory, e.g. ~Ak1, ~Ak2, Ak3, ...

In particular, under certain conditions, it is possible to create a consistent theory, the **so-called alter-theory of T, AT,** whose axioms are the negations of all specific axioms of the initial theory T.

Alter-theories

Example. T- consistent theory, non-logical axioms

A1: $\exists x.\phi(x)$

A2: $\exists x.\sim\phi(x)$.

If these sentences are consistent, then they are also independent, since the theories {~A1, A2}, {A1, ~A2} are consistent. However, their axioms are no longer independent, as they are ”negatively dependent”: $\llcorner \sim A1 \rightarrow A2 \lrcorner$, which follows from the fact that $\sim A1 \equiv \sim[\exists x.\phi(x)] \equiv [\forall x.\sim\phi(x)] \rightarrow \exists x.\sim\phi(x) \equiv A2$. The same applies to $\llcorner \sim A2 \rightarrow A1 \lrcorner$.

Alter-theories cont.

It is always possible to construct a consistent AT for a given consistent T if the specific axioms of T are **negatively independent (negative-independent)**, i.e., if none of the remaining (non-negated) axioms of T can be derived from the conjunction of any finite number of negations of the non-logical axioms of T.

Similarly, we say that the axioms of a consistent theory T are **partially negatively independent** if, from any finite number of negations of these axioms together with a finite number of non-negated axioms, it is impossible to derive any of the remaining axioms of T.

The axioms of a consistent theory T are **absolutely independent** if they are partially negatively independent, and negatively independent.

Types of Alter-theories

The concepts introduced above allow us to distinguish between different types of consistent alter-theories:

AT – alter-theories, i.e., theories that arise from a given consistent theory T **by negating all of its non-logical axioms.**

P-AT – partial alter-theories, i.e., theories that arise **by negating only some of the axioms** of the initial theory T, while the remaining axioms are identical to the axioms of T.

Other types can also be distinguished, e.g., **M-AT** (mixed alter-theories), **M-P-AT** (partial mixed alter-theories), **LT** – (lower theories), **UT** – (upper theories), etc.

Types of Alter-theories

Usually, in mathematics and logic, only P-AT and M-P-AT are considered, and this is usually limited to the negation of only one axiom, after demonstrating, of course, that it is independent of the others.

Examples. ZF(C)+CH, ZF(C)+~CH, or commutative and non-commutative groups – these are P-AT.

Euclidean geometry, Non-euclidean geometries or Non-well founded set theories – these are typical M-P-ATs, i.e., they do not use a "complete" negation of an axiom (e.g., Euclid's fifth postulate, the axiom of extensionality), but only a "partial" negation of it, i.e., they accept as an axiom a statement or property that results from the "complete" negation of a given axiom.

Properties of Alter-theories

Consistent Alter-theories have very interesting properties.

For example, if A is a non-logical thesis of AT, then it cannot be a thesis of T and vice versa, i.e., T and AT do not have any specific (non-logical) theses in common.
This allows (under certain assumptions) for purely syntactic inquiry of sentences independent of a given initial theory T.
Similarly, if T and its P-AT (provided they are consistent) have common non-logical consequences, then these consequences are exclusively the consequences of their common (non-negated) axioms.

Properties of Alter-theories

A given theory T may have a consistent AT but may not have consistent P-ATs (or only some of the possible P-ATs are consistent) and *vice versa*.

ATs are a certain idealization, but it turns out that the axioms of a given consistent theory T, even if they are dependent on each other, if they are negatively independent, then there exists a consistent AT (P-ATs must be inconsistent, at least some of them).

The above conclusion allows us to construct AT without the special and laborious task of determining which axioms are dependent on others.

Properties of Alter-theories

If we limit ourselves to theories (not necessarily only first-order) based on functional calculus (logic with functions) with identity (=) and with modus ponens as the only rule of inference, then – as is well known – the specific axioms of a given theory, e.g., group theory or Peano arithmetic of natural numbers, PAr, are most often dependent on the axioms of identity and/or vice versa.

Therefore, when constructing an alter-theory of such a theory, the axioms of the theory of identity ID= must also be negated.

This means that - if there is a consistent AT of a certain theory with identity, then it is (often) not itself a theory with identity.

Theory of Identity

ID.1. $\forall x.\ x=x$;
ID.2. $\forall x,y.\ x=y \rightarrow y=x$;
ID.3. $\forall x,y.\ x=y \wedge y=z \rightarrow x=z$;
ID.4. (P). $\forall \ldots \forall x,y.\ (x=y \wedge P(x1, \ldots, x, \ldots.)) \rightarrow P(x1, \ldots, y, \ldots.)$, for each primitive relation P, where $P(x1, \ldots, y, \ldots.)$ is a formula obtained from the formula $P(x1, \ldots, x, \ldots.)$, in which all occurrences of the free variable x have been replaced by the variable y (provided that y is free for x in $P(x1, \ldots, x, \ldots.)$);
ID.5. (F). $\forall \ldots \forall x,y.\ x=y \rightarrow (F(x1, \ldots, x, \ldots.) = F(x1, \ldots, y, \ldots.))$, for each primitive function F, where $F(x1, \ldots, y, \ldots.)$ is a term obtained from the term $F(x1, \ldots, x, \ldots.)$, in which all occurrences of the free variable x have been replaced by the variable y.

Properties of Alter-theories

AID= has the following axioms:

AID.1. $\exists x.\ \sim I(x,x)$;

AID.2. $\exists x,y.\ I(x,y) \wedge \sim I(y,x)$;

AID.3. $\exists x,y.\ I(x,y) \wedge I(y,z) \wedge \sim I(x,z)$;

AID.4. $(F)\ \exists x,y.\ I(x,y) \wedge \sim(F(x1, \ldots, x, \ldots.) = F(x1, \ldots, y, \ldots.))$;

AID.5. $(R)\ \exists x,y.\ I(x,y) \wedge R(x1, \ldots, x, \ldots.) \wedge \sim R(x1, \ldots, y, \ldots.))$.

AID= axioms are not all independent, because, for example, in ID=, not-ID.5. is inconsistent with ID.1.; (cf. P(x,y)/x=y). However, these axioms are negatively independent, because the alter-theory **AID=** is consistent, which can be shown by constructing a model of this theory, e.g., a non-transitive model, where we interpret the relation I(x,y) as "ϵ" ("x belongs to y").

PAr

PA.a⁼. $\forall x.\ x=x;$

PA.b⁼. $\forall x,y.\ x=y \rightarrow y=x;$

PA.c⁼. $\forall x,y,z.\ x=y \wedge y=z \rightarrow x=z.$

PA.d⁼. $(S,+,\bullet)\ \forall\ldots\forall x,y.\ x=y \rightarrow (F(x1, \ldots, x, \ldots.) = F(x1, \ldots, y, \ldots.))$

PA.e⁼.. $(=)\ \forall\ldots\forall x,y.\ (x=y \wedge P(x1, \ldots, x, \ldots.)) \rightarrow P(x1, \ldots, y, \ldots.)$

Specific (non-logical) axioms:

(PA1) $\forall x.\ 0 \neq S(x);$

(PA2) $\forall x,y.\ S(x)=S(y) \rightarrow x=y;$

(PA3) $\forall x.\ x+0 = x;$

(PA4) $\forall x,y.\ x+S(y) = S(x+y);$

(PA5) $\forall x.\ x\bullet 0 = 0;$

(PA6) $\forall x,y.\ x\bullet S(y) = (x\bullet y) + x;$

and the induction axiom schema:

(Ind). $(\phi)\ \forall u_1, u_2,\ldots, u_n\{[\phi(0) \wedge \forall x(\phi(x) \rightarrow \phi(S(x))] \rightarrow \forall x.\phi(x)\}.$

A-PAr

Note that when constructing A-PAr, we cannot create a consistent negation of the induction axiom schema because, for example, for formulas ϕ without parameters, negating every axiom falling under the **Ind** schema will immediately give us a pair of contradictory statements ϕ and $\sim\phi$. This means that some axioms falling under the **Ind** schema are negatively dependent.

We can therefore only create partial alter-theories of PAr.

So, the theory

(A-Ind). $(\phi)\ \exists u_1, u_2, \ldots, u_n\{[\phi(0) \wedge \forall x(\phi(x) \rightarrow \phi(S(x))] \rightarrow \exists x.\sim\phi(x)\}$

is inconsistent.

A-PAr

Ind is not an a priori, unchanging construct that is attached to a theory in an unaltered form but depending on the other axioms of a given theory, we determine which functions, constants, and relations can form the relevant formulas ϕ. **Ind** is also not a concept that is consistent only with PAr or theories in which PAr is constructable, such as ZF(C).

For example, let us consider the theory of the axioms of identity theory, AI and Ind, i.e.

ID=

A.1a. $\forall x.\ S(x)=0$

(Ind). $(\phi)\ \forall u_1, u_2,\ldots, u_n\{[\phi(0) \wedge \forall x(\phi(x) \rightarrow \phi(S(x))] \rightarrow \forall x.\phi(x)\}$.

In this theory, using Ind, we can obtain the thesis $\forall x.\ S(x)=x$.

Let us first examine the alter-theory of PAr-(Ind),i.e. the alter-theory of PAr without the **Ind** schema. This theory, let us call it **1P-A-PAr**, has the following axioms:

AID.1. ∃x. ~I(x,x);

AID.2. ∃x,y. I(x,y) ∧ ~I(y,x);

AID.3. ∃x,y. I(x,y) ∧ I(y,z) ∧ ~I(x,z);

AID.4. (S.+, ×) ∃x,y. I(x,y) ∧ ~(F(x1, …, x, ….) = F(x1, …, y, ….));

AID.5. (I) ∃x,y. I(x,y) ∧ R(x1, …, x, ….) ∧ ~R(x1, …, y, ….)).

(A-PAr1) ∃x. 0 = S(x); [i.e. ∃x. I(0,S(x))]

(A-PAr2) ∃x,y. S(x)=S(y) ∧ ~(x=y); [i.e. ∃x,y. I(S(x),S(y)) ∧ ~I(x,y)]

(A-PAr3) ∃x. ~(x+0 = x); [as above]

(A-PAr4) ∃x,y. ~(x+S(y) = S(x+y)); [as above]

(A-PAr5) ∃x. ~(x×0 = 0); [as above]

(A-PAr6) ∃x,y. ~(x×S(y) = (x×y) + x); [as above]

It turns out that it is possible to construct a model of this theory (again, it suffices to interpret I(x,y) as a relation “$\in$” (“x belongs to y”) and select appropriate elements so that the axioms of our theory are satisfied.

This means that 1P-A-PAr is consistent, and therefore that its axioms are negatively independent.

This also means that we have a proof of the consistency of PAr-(Ind), since AT is consistent iff T is consistent.

Conclusions:

1. The AT construction method revives hopes for the possibility of saving Hilbert's original program for a broader class of theories (/sentences) than previously thought.
2. The AT method allows the independence of (some) statements to be examined in a purely syntactic manner.
3. There is a class of problems concerning the stoping of a Turing machine that are undecidable for a single Turing machine but become decidable for two Turing machines operating “in parallel,” i.e., one examines the problem for formula A in T and the other for non-A in the corresponding alter-theory.

Thank you for your attention!

Ideal K-calculus:

Axiom 1 (extensionality): Two sets are identical if and only if they have the same elements.

Axiom (schema) 2 (comprehension): For every property P, there exists a set consisting of those and only those elements that have this property.

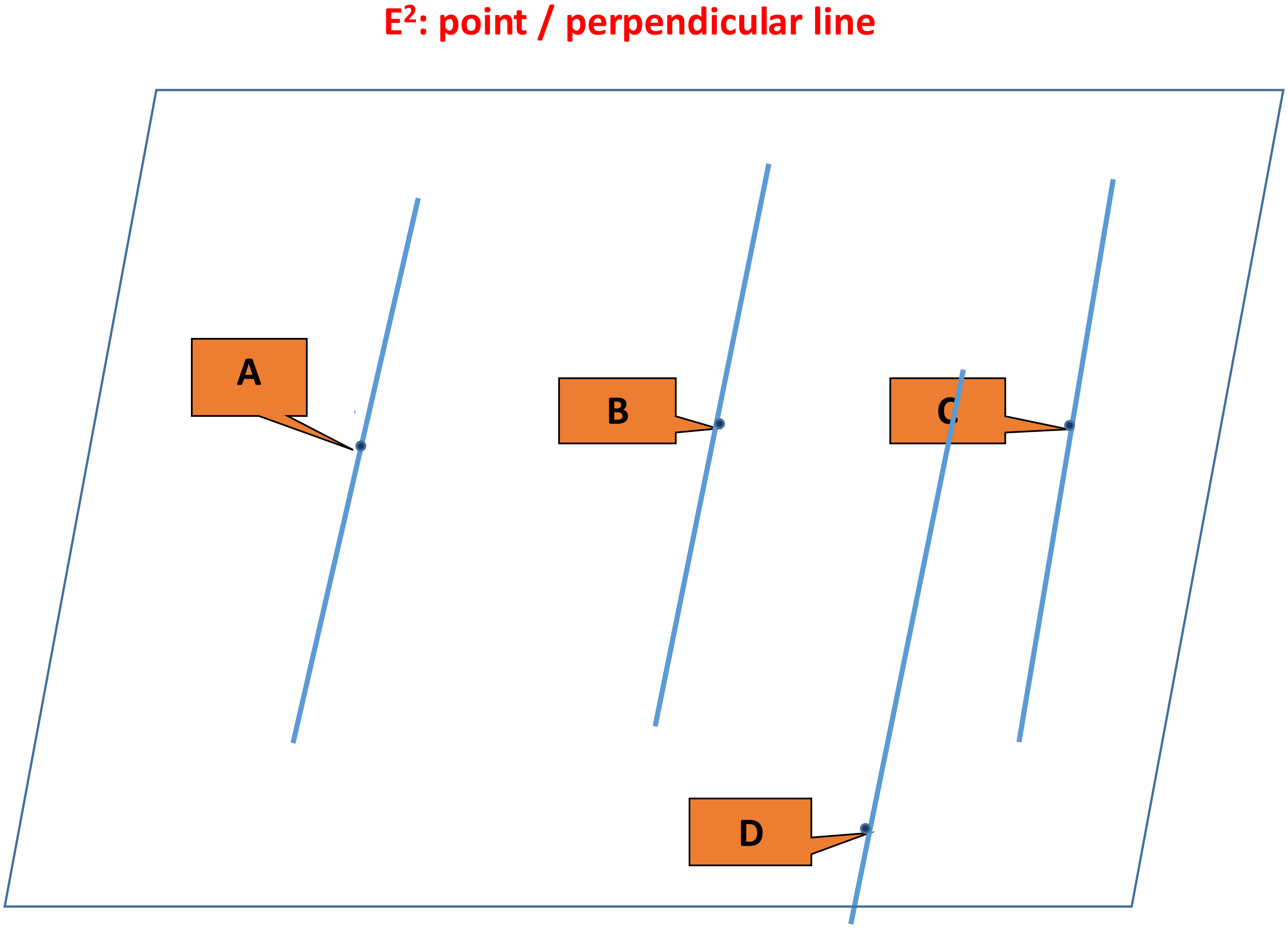
E.1. $\forall x\,[(x \in y \equiv x \in z) \rightarrow y = z]$

K.1. $(\varphi)\ \forall x \exists z\,[x \in z \equiv \varphi(x)]$

The inconsistency of K.1 formulated in L_c is obvious for intuitive reasons. In L_c, we can formulate the intentional property "x is not an element of any set," i.e., $\varphi^p(x) \equiv \sim(\exists y.\ x \in y)$. No set can correspond to this property, in particular the set postulated by K.1. For if $x \in z$ and at the same time $\sim(\exists y.\ x \in y)$, we obtain a contradiction. Russell's property is a special case of $\varphi^p(x)$ and follows from it. and we obtain a contradiction.

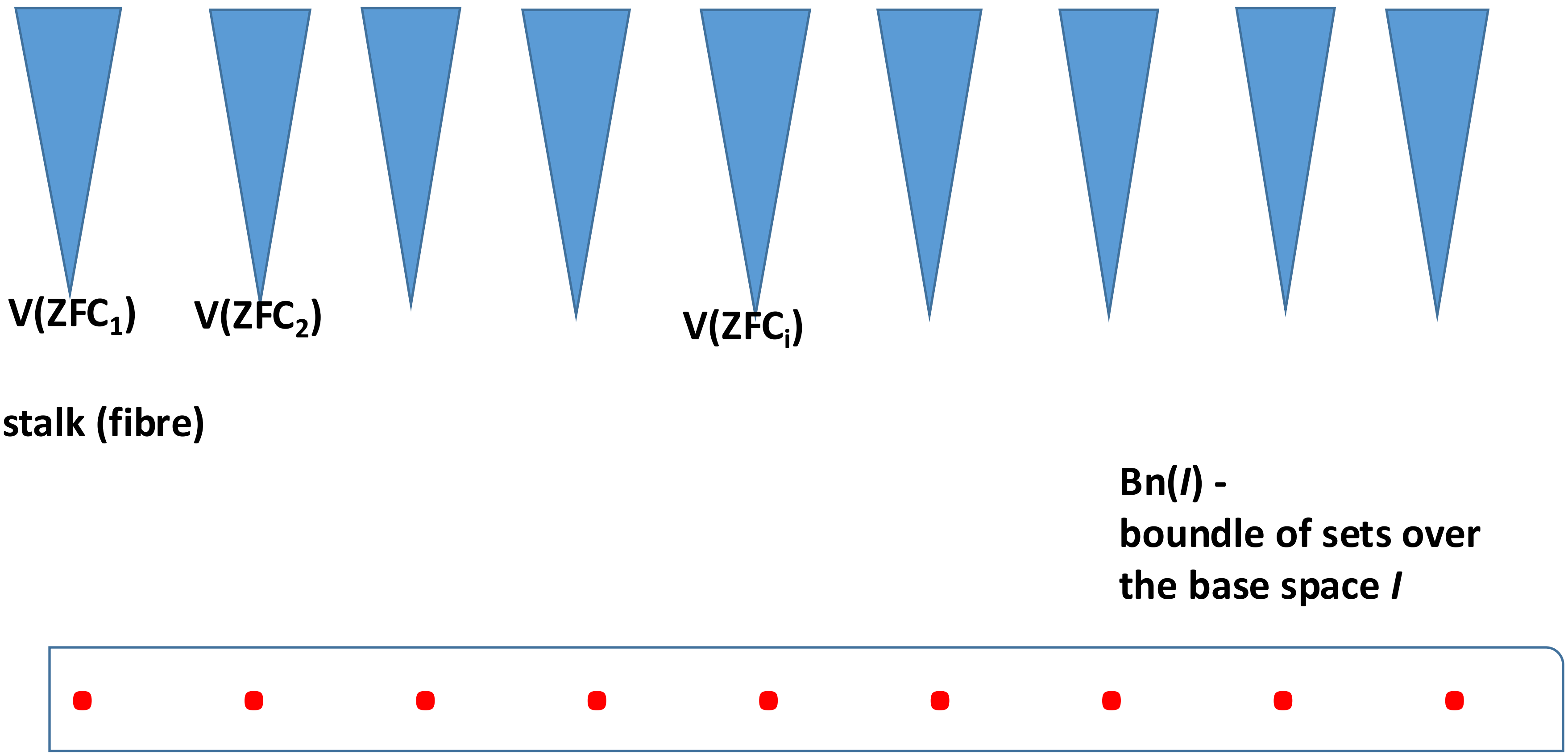
In first-order logic, no two-argument relation (and therefore also the relation "$\in$") can be defined by a certain formula $\varphi(x)$ in this theory. K.1. is false in a homogeneous environment (i.e., with one type of belonging relation) for purely intuitive (intentional) reasons.

An example of substitution



General description of global substitutions

The stalk space

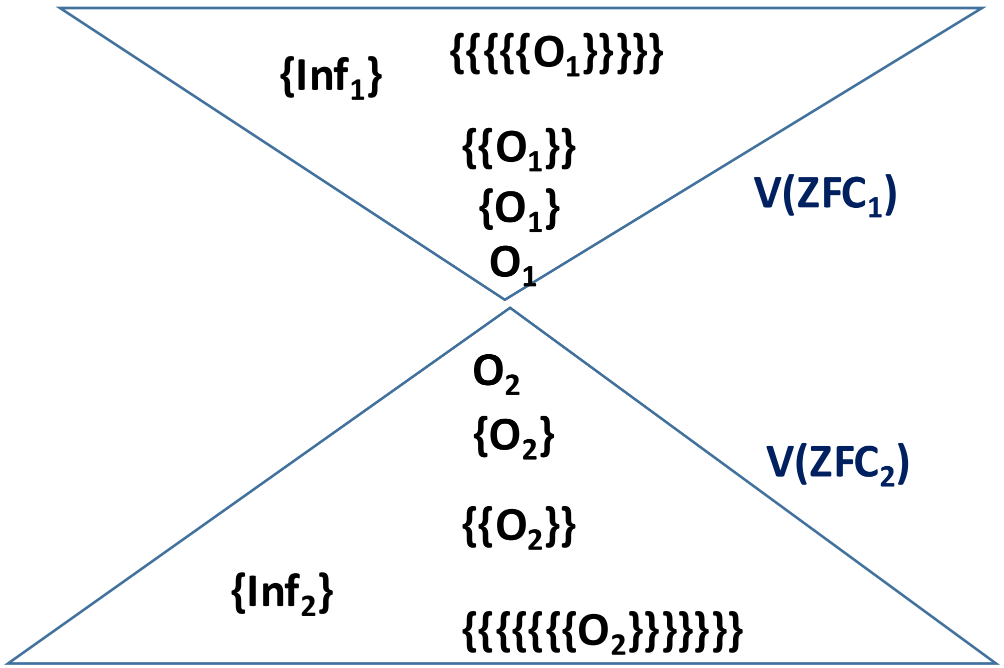


I **– the set of indexes (R, N, E^2, …)**
= the base space

Bn(*I*)

Example: local substitution of ZFC in ZFC from the point of view of models and a theory

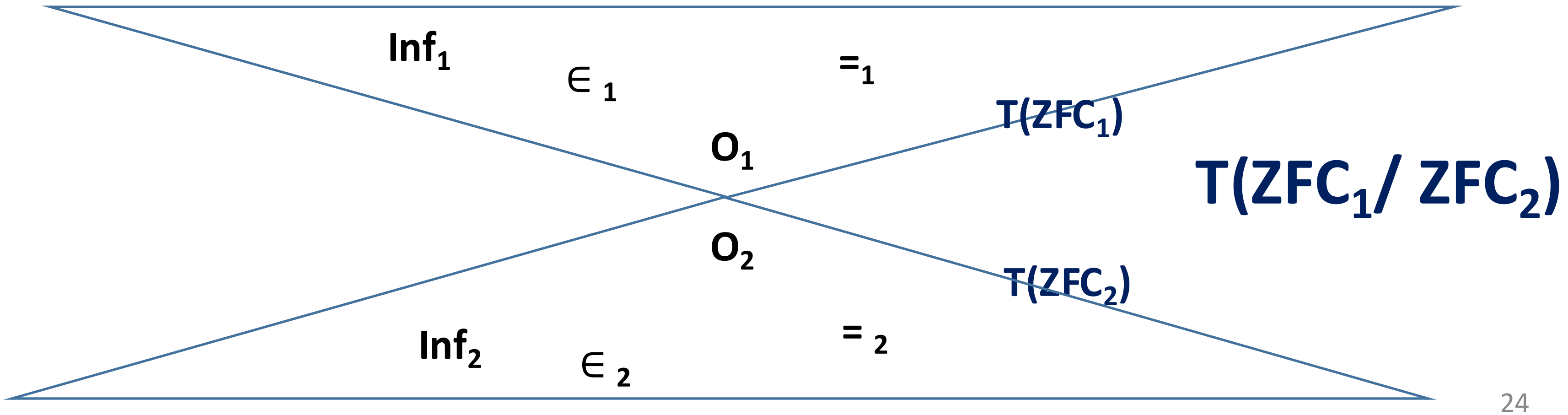
- A substitution of ZFC_2 in ZFC_1: substitute a transitive model of ZFC_2 for the empty set **0** from ZFC_1; "local singular self-substitution".
- The two "copies" of ZFC: ZFC_1 and ZFC_2, must be written down in such a way that they have no relational and constant symbols in common: e.g. the symbols "$\in_1$" and "$=_1$" in ZFC_1 correspond to "$\in_2$" and "$=_2$" in ZFC_2. The same concerns the symbols designating constants: "$\mathbf{0_1}$" and "$\mathbf{0_2}$", infinite set, etc.
- It is possible to create the third theory, $T(ZFC_1/ ZFC_2)$, in the following way. Write down every axiom of ZFC_1, and add to them the set of renamed and supplemented axioms of ZFC_2



$V(ZFC_1/ ZFC_2)$

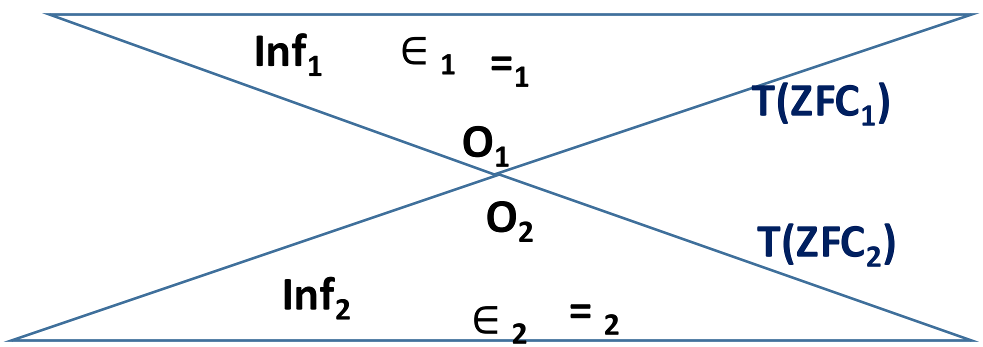
Example: local substitutions of ZFC in ZFC - a formal description

- Ax. of extensionality of ZFC_2: $\forall \mathbf{x}, \mathbf{y}, \mathbf{z}. [(\mathbf{x} \in_2 \mathbf{0_1} \wedge \mathbf{y} \in_2 \mathbf{0_1} \wedge \mathbf{z} \in_2 \mathbf{0_1}) \rightarrow (\mathbf{x} \in_2 \boldsymbol{y} \equiv \mathbf{x} \in_2 \mathbf{z} \rightarrow \mathbf{y} =_2 \mathbf{z})]$.
- Axiom of pairing: $\forall \mathbf{x} \in_2 \mathbf{0_1}, \forall \mathbf{y} \in_2 \mathbf{0_1} \exists \mathbf{z} \in_2 \mathbf{0_1} \forall \mathbf{u} \in_2 \mathbf{z}. (\mathbf{x} \in_2 \mathbf{z} \equiv \mathbf{u} =_2 \mathbf{x} \vee \mathbf{u} =_2 \mathbf{y})$.
- Axiom of infinity: $\exists \mathbf{z} \in_2 \mathbf{0_1}. [(\mathbf{0_1} \in_2 \mathbf{z}) \wedge (\forall \mathbf{x}. \mathbf{x} \in_2 \mathbf{z} \rightarrow \mathbf{x} \cup \{x\} \in_2 \mathbf{z})]$.
- In a similar way the remaining axioms of ZFC_2 should be adapted.
- One new axiom, the axiom of transitivity: $\forall \mathbf{x}, \mathbf{y}. (\mathbf{x} \in_2 \mathbf{y} \wedge \mathbf{y} \in_2 \mathbf{0_1}) \rightarrow (\mathbf{x} \in_2 \mathbf{0_1})$, in order to have every set from ZFC_2 “inside” $\mathbf{0_1}$.
- The usual rules concerning the substitutions of identical objects are restricted, e.g. if $(\mathbf{x} =_1 \mathbf{y})$, one can substitute **x/y** only in the formulae built with the use of “$=_1$” or “$\in_1$”.
- We refer to the theory $T(ZFC_1/ZFC_2)$ as *substitutional theory* which corresponds to a *substitutional model*.



Example: local substitutions of ZFC in ZFC

- One has to decide, for instance, if $\mathbf{0}_1 =_1 \mathbf{0}_2$, or $\forall \mathbf{x}. [(\mathbf{x} \in_2 \mathbf{0}_1) \to (\mathbf{x} =_1 \mathbf{O}_1)]$ or $\forall \mathbf{x}. [\sim (\mathbf{x} \in_2 \mathbf{0}_1) \to (\mathbf{x} =_1 \mathbf{O}_2)]$, or introduce *global relation of identity*: $\forall \mathbf{x}, \mathbf{y}. (\mathbf{x} = \mathbf{y}) \equiv (\mathbf{x} =_1 \mathbf{y} \wedge \mathbf{x} =_2 \mathbf{y})$.
- In $T(ZFC_1/ZFC_2)$ unrestricted quantifiers in the part corresponding to ZFC_1 are used. The resulting possible mixture of sets belonging to the universes of the component ZFC systems causes that the resulting $V(ZFC_1/ZFC_2)$ is not "a pure" substitution without some additional axioms.
- In the case when $\sim(\mathbf{0}_1 =_1 \mathbf{0}_2)$, the quantifiers act over the objects of both initial systems of ZFC, however, **the quantifiers cannot detect all the relevant ontological commitment of $T(ZFC_1/ZFC_2)$. The objects of ZFC_1/ZFC_2 "change their nature" and this fact is "unseen" by the quantifiers.**



$T(ZFC_1/ ZFC_2)$

- It is now straightforward to define in $T(ZFC_1/ZFC_2)$ the relations "$\in_2$" and "$=_2$", as well as the constant $\mathbf{0_1}$. from ZFC_2. For instance, $\forall \mathbf{x}, \mathbf{y}. (\mathbf{x} \in_2 \mathbf{y})_{ZFC2} \equiv \forall \mathbf{x}, \mathbf{y}. (\mathbf{x} \in_2 \mathbf{y})_{ZFC1/ZFC2}$ or $\mathbf{0_1}_{ZFC2} =_2 \mathbf{0_1}$. Therefore ZFC_2 (and ZFC_1) is interpretable in $T(ZFC_1/ZFC_2)$.
- Mutual interpretability of theories, especially of ZFC, is inquired in M. Friedman's *concept calculi*. The obtained results demonstrates the fallibility of **QCE** and **M-QCE**. Moreover, such examples as ZFC_1/ZFC_2, also indicate the inadequacy of the general idea of Quine's ontological commitments of theories because **extensional theories are intensionally undetermined.**
- ZFC can speak about sets or be interpreted as, say, a recipe.
- **Ontology appears independent from quantification; from the point of view of extensional theories which are intensionally undetermined, ontological properties belong to unessential "intensional decoration".**