Hybrid-Dynamic Ehrenfeucht-Fraïssé Games

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Introduction

- Idea: take a hodgepodge of existing results and make a systematic sense of them.
- Ingredients:
  - Some results on bisimulations in hybrid and dynamic logics, for example [Areces et al., 2001, Hodkinson and Tahiri, 2010].
  - Classical first-order stuff: Ehrenfeucht-Fraïssé games, partial isomorphisms, back-and-forth.
  - Institution theory framework.
- Recipe:
  - Mix, shake well, simmer in a pot (see [Badia et al., 2025]) long enough for Daniel to lose patience.
  - Rewrite everything in institution theory framework, get some new results on the way.

Signatures

- All signatures will be of the form $(\Sigma, \texttt{Prop})$, where $\Sigma = (F, P)$ is a single-sorted first-order signature consisting of
  - a set of constants F (nominals), and
  - a set of binary relation symbols P (accessibility),

  and $\texttt{Prop}$ is a set of propositional symbols.
- We let Δ range over signatures of the form $(\Sigma, \texttt{Prop})$. When necessary, we will use indexing in a natural way.
- A signature morphism $\chi : \Delta_1 \to \Delta_2$ consists of a first-order signature morphism $\chi : \Sigma_1 \to \Sigma_2$ and a function $\chi : \texttt{Prop}_1 \to \texttt{Prop}_2$.
- The extension of Δ by a fresh nominal k is denoted $\Delta[k]$ yielding the inclusion $\Delta \hookrightarrow \Delta[k]$.
- We denote by $\mathsf{Sig}^{\mathsf{HDPL}}$ the category of HDPL signatures.

Models

- Models over a signature Δ are standard Kripke structures $\mathfrak{M} = (W, M)$, where:
  - W is a first-order structure over Σ, that is, $W = \big(|\mathfrak{M}|, (\lambda^W)_{\lambda \in P}\big)$. So $|\mathfrak{M}|$ is the set of worlds, and λ^W are accessibility relations.
  - M is a propositional valuation, that is, a map $M\colon |\mathfrak{M}| \to |\mathsf{Mod}^{\mathsf{PL}}(\mathtt{Prop})|$, where $\mathsf{Mod}^{\mathsf{PL}}$ stands for power-set.
- Convention: $\mathfrak{M}$ and $\mathfrak{N}$ range over Kripke structures of the form (W, M) and (V, N).
- Δ-homomorphism $h : \mathfrak{M} \to \mathfrak{N}$ between two Kripke structures $\mathfrak{M}$ and $\mathfrak{N}$ is a first-order homomorphism $h : W \to V$ such that $h(M(w)) \subseteq N(h(w))$ for all states $w \in |\mathfrak{M}|$, that is, h preserves truth of propositions.
- Δ-homomorphisms form a category $\mathsf{Mod}^{\mathsf{HDPL}}(\Delta)$.

Actions / Programs

The set of actions $\mathcal{A}^{\mathsf{HDPL}}(\Delta)$ over a signature Δ is defined by the following grammar:

$$\mathfrak{a} ::= \lambda \mid \mathfrak{a} \cup \mathfrak{a} \mid \mathfrak{a} \mathbin{;} \mathfrak{a} \mid \mathfrak{a}^* \ ,$$

where λ is a binary relation on nominals. Actions are interpreted in Kripke structures (W, M) as accessibility relations, as follows:

- $\lambda^{\mathfrak{M}} = \lambda^{W}$ for all binary relations λ in Δ,
- $(\mathfrak{a}_1 \cup \mathfrak{a}_2)^{\mathfrak{M}} = \mathfrak{a}_1^{\mathfrak{M}} \cup \mathfrak{a}_2^{\mathfrak{M}}$ (union),
- $(\mathfrak{a}_1 \mathbin{;} \mathfrak{a}_2)^{\mathfrak{M}} = \mathfrak{a}_1^{\mathfrak{M}} \mathbin{;} \mathfrak{a}_2^{\mathfrak{M}}$ (composition),
- $(\mathfrak{a}^*)^{\mathfrak{M}} = (\mathfrak{a}^{\mathfrak{M}})^*$ (reflexive & transitive closure).

Sentences

The set of sentences $\mathsf{Sen}^{\mathsf{HDPL}}(\Delta)$ over a signature Δ is defined by:

$$\phi ::= p \mid k \mid \bigwedge \Phi \mid \neg\phi \mid \langle\mathfrak{a}\rangle\phi \mid @_k\,\phi \mid {\downarrow} x \cdot \phi_x \mid \exists x \cdot \phi_x \ ,$$

where (i) p is a propositional symbol, (ii) k is a nominal, (iii) Φ is a finite set of sentences over Δ, (iv) x is a variable for Δ, (v) $\mathfrak{a}$ is an action over Δ, (vi) $\phi_x \in \mathsf{Sen}^{\mathsf{HDPL}}(\Delta[x])$.

- $\langle\mathfrak{a}\rangle\phi$ is read ϕ holds after a run of $\mathfrak{a}$ (possibility),
- $@_k\,\phi$ is read ϕ holds at state k (retrieve),
- ${\downarrow} x \cdot \phi_x$ is read ϕ_x holds with the current state set to x (store).

Usual abbreviations are in force: $\bigvee \Phi$ for $\neg(\bigwedge_{\phi\in\Phi} \neg\phi)$, $[\mathfrak{a}]\phi$ for $\neg\langle\mathfrak{a}\rangle\neg\phi$, and $\forall x \cdot \phi_x$ for $\neg\exists x \cdot \neg\phi_x$.

Local satisfaction relation

Satisfaction of a sentence ϕ at a world $w \in |\mathfrak{M}|$ in a model $\mathfrak{M} = (W, M)$ over a signature Δ, is defined by induction:

- $(\mathfrak{M}, w) \models p$ if $p \in M(w)$;
- $(\mathfrak{M}, w) \models k$ if $w = k^{\mathfrak{M}}$
- $(\mathfrak{M}, w) \models \bigwedge \Phi$ if $(\mathfrak{M}, w) \models \phi$ for all $\phi \in \Phi$;
- $(\mathfrak{M}, w) \models \neg\phi$ if $(\mathfrak{M}, w) \not\models \phi$;
- $(\mathfrak{M}, w) \models @_k\, \phi$ if $(\mathfrak{M}, k^{\mathfrak{M}}) \models \phi$;
- $(\mathfrak{M}, w) \models \langle \mathfrak{a} \rangle \phi$ if $(\mathfrak{M}, v) \models \phi$ for some $v \in \mathfrak{a}^{\mathfrak{M}}(w) := \{w' \in |\mathfrak{M}| \mid (w, w') \in \mathfrak{a}^{\mathfrak{M}}\}$;
- $(\mathfrak{M}, w) \models {\downarrow} x \cdot \phi_x$ if $(\mathfrak{M}^{x \leftarrow w}, w) \models \phi_x$, where $\mathfrak{M}^{x \leftarrow w}$ is the expansion of $\mathfrak{M}$ to $\Delta[x]$ interpreting x as w;
- $(\mathfrak{M}, w) \models \exists x \cdot \phi_x$ if $(\mathfrak{M}^{x \leftarrow v}, w) \models \phi_x$ for some $v \in |\mathfrak{M}|$.

Local satisfaction condition

We call the pair $(\mathfrak{M}, w)$ a pointed model, and w the current state.

Theorem (Local satisfaction condition)

For all signature morphisms $\chi : \Delta_1 \to \Delta_2$, all Δ_2-models $\mathfrak{M}$, all states $w \in |\mathfrak{M}|$, all Δ_1-sentences ϕ, we have

$$(\mathfrak{M}, w) \models \chi(\phi) \textit{ iff } (\mathfrak{M}|\chi, w) \models \phi \ .$$

where $\mathfrak{M}|\chi$ is the χ-reduct of $\mathfrak{M}$.

- Technically, this shows that HDPL is a stratified institution (see [Diaconescu, 2017, Găină, 2020]).
- In practice, it means that signature morphisms are very decent translations.
- Folklore in institution theory (see [Diaconescu, 2016]).

Framework for fragments

HDPL is too powerful, so we want to consider fragments. Let $\mathcal{L} = (\mathsf{Sig}^{\mathsf{HDPL}}, \mathsf{Sen}^{\mathcal{L}}, \mathsf{Mod}^{\mathsf{HDPL}}, \models)$ of HDPL be an arbitrary fragment of HDPL closed under Boolean connectives. Boolean closure in not needed for the games, but it is for game sentences.

In practice, any $\mathcal{L}$ is obtained from HDPL by discarding:

- some constructors for actions from the grammar which defines actions in HDPL, and/or
- some constructors for sentences from the grammar which defines sentences in HDPL.

Two extreme cases:

- drop the action part, get hybrid propositional logic (HPL),
- drop the hybrid part, get propositional dynamic logic (PDL).

Notational conventions

- Let $\mathcal{O} \subseteq \{\Diamond, @, \downarrow, \exists\}$ be the subset of the sentence constructors which belong to $\mathcal{L}$.
  - $\Diamond \in \mathcal{O}$ means that $\mathcal{L}$ is closed under possibility over actions, but no assumption is made concerning the existence of action constructors. One or more constructors for actions can be discarded from $\mathcal{L}$.
- Let $\mathsf{Mod}_p(\Delta) = \{(\mathfrak{M}, w) \mid \mathfrak{M} \in |\mathsf{Mod}(\Delta)| \text{ and } w \in |\mathfrak{M}|\}$ be the class of pointed Δ-models.
- Let $\mathsf{Mod}_p(\Delta, \phi) = \{(\mathfrak{M}, w) \in \mathsf{Mod}_p(\Delta) \mid (\mathfrak{M}, w) \models \phi\}$ be the class of pointed Δ-models which satisfy a sentence ϕ.
- Let $\mathsf{Sen}_b : \mathsf{Sig} \to \mathbb{S}\mathsf{et}$ be the subfunctor of $\mathsf{Sen} : \mathsf{Sig} \to \mathbb{S}\mathsf{et}$ which maps each signature $\Delta = (\Sigma, \mathtt{Prop})$ to the set of basic (or atomic) sentences $F \cup \mathtt{Prop}$.

Elementary equivalence

Definition

Let Δ be a signature.

- Two pointed Δ-models $(\mathfrak{M}, w)$ and $(\mathfrak{N}, v)$ are $\mathcal{L}$-elementarily equivalent, in symbols, $(\mathfrak{M}, w) \equiv (\mathfrak{N}, v)$, when $(\mathfrak{M}, w) \models \phi$ iff $(\mathfrak{N}, v) \models \phi$, for all Δ-sentences $\phi \in \mathsf{Sen}^{\mathcal{L}}(\Delta)$.
- Two Δ-sentences ϕ_1 and ϕ_2 are semantically equivalent if they are satisfied by the same pointed models, that is, $\mathsf{Mod}_p(\Delta, \phi_1) = \mathsf{Mod}_p(\Delta, \phi_2)$.

- $\mathcal{L}$-elementary equivalence is an equivalence relation for any $\mathcal{L}$.
- Strength of $\equiv$ depends on $\mathcal{L}$.

Ehrenfeucht-Fraïssé games in our framework

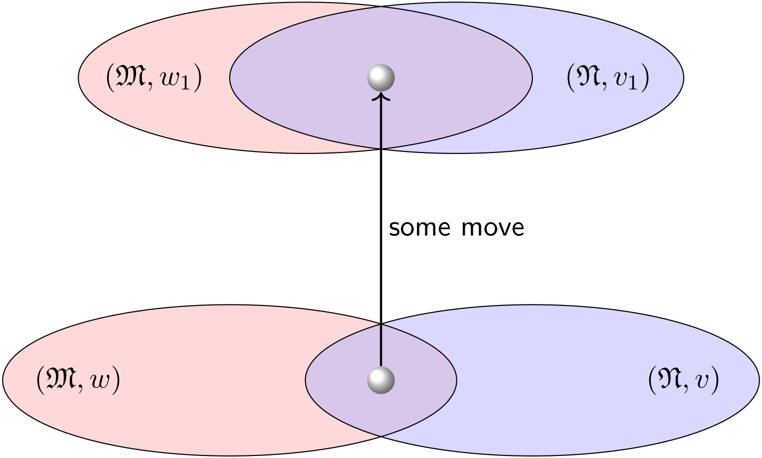
Idea based on [Găină and Kowalski, 2020].

- A game is a collection of all its plays over a set of gameboard trees.
- A play of the game starts with two pointed models $(\mathfrak{M}, w)$ and $(\mathfrak{N}, v)$ defined over the same signature Δ, and a gameboard tree tr with $root(tr) = \Delta$.
- ∃loise loses if the following game property is not satisfied:

$$(\mathfrak{M}, w) \models \phi \text{ iff } (\mathfrak{N}, v) \models \phi \quad \text{for all } \phi \in \mathsf{Sen}_b(\Delta).$$

- Otherwise, the play can continue and ∀belard can move one of the pointed models along an edge of the gameboard tree. Wlog, ∀belard picks up the first pointed model $(\mathfrak{M}, w)$.
- A winning strategy for a player is a sequence of moves that ensure that the player wins any play of the game.

Idea illustrated

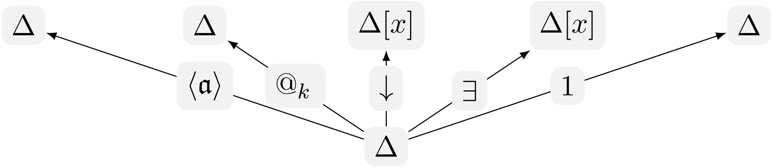


Gameboard trees

A gameboard tree describes the possible moves in the play of a game. The edges are labelled to account for different kinds of "quantifiers": $\langle\mathfrak{a}\rangle$, $@_k$, $\downarrow$, $\exists$.

- Nodes are labelled by signatures
- Edges are labelled by sentence operators; they are uniquely identified by source and label.

Here are all possible labels for HDPL. For a particular $\mathcal{L}$ there may be fewer possible labels. But the idle label 1 always occurs for technical reasons.



Labels

- $\Diamond \in \mathcal{O}$: A pair consisting of the identity signature morphism $1_\Delta : \Delta \to \Delta$ and the modal operator $\langle \mathfrak{a} \rangle$.
- $@ \in \mathcal{O}$: A pair consisting of the identity signature morphism $1_\Delta : \Delta \to \Delta$ and the operator retrieve $@_k$.
- $\downarrow \in \mathcal{O}$: A pair consisting of a signature inclusion $\Delta \longrightarrow \Delta[x]$ and the operator store $\downarrow x$.
- $\exists \in \mathcal{O}$: A pair consisting of a signature inclusion $\Delta \longrightarrow \Delta[x]$ and the first-order quantifier $\exists x$.
- idle: The identity signature morphism $1_\Delta : \Delta \to \Delta$. The idle edge serves to construct complex gameboard trees from simpler components and to define game sentences that represent conjunctions.

Moves

$\Diamond \in \mathcal{O}$:

A move along $\Delta \xrightarrow{\langle \mathfrak{a} \rangle} \Delta$. The next state w_1 chosen by $\forall$belard is accessible from w via $\mathfrak{a}^{\mathfrak{M}}$. $\exists$loise needs to find a state v_1 accessible from v via $\mathfrak{a}^{\mathfrak{N}}$ such that for her new pointed model $(\mathfrak{N}, v_1)$ the game property still holds.

$$\begin{tikzcd}
(\mathfrak{M}, w) & & (\mathfrak{M}, w_1) \\
\Delta \arrow[rr, "\langle \mathfrak{a} \rangle"] & & \Delta \\
(\mathfrak{N}, v) & & (\mathfrak{N}, v_1)
\end{tikzcd}$$

$@ \in \mathcal{O}$:

A move along an edge $\Delta \xrightarrow{@_k} \Delta$. $\forall$belard changes the current state w to $k^{\mathfrak{M}}$. The only possible choice for $\exists$loise is $(\mathfrak{N}, k^{\mathfrak{N}})$. If the game property still holds, the play continues with $(\mathfrak{M}, k^{\mathfrak{M}})$ and $(\mathfrak{N}, k^{\mathfrak{N}})$.

$$\begin{tikzcd}
(\mathfrak{M}, w) & & (\mathfrak{M}, k^{\mathfrak{M}}) \\
\Delta \arrow[rr, "@_k"] & & \Delta \\
(\mathfrak{N}, v) & & (\mathfrak{N}, k^{\mathfrak{N}})
\end{tikzcd}$$

$\downarrow \in \mathcal{O}$:

A move along $\Delta \xrightarrow{\downarrow} \Delta[x]$. $\forall$belard gives name x to his current state. $\exists$loise can only respond by giving name x to her current state.

$$\begin{tikzcd}
(\mathfrak{M}, w) & & (\mathfrak{M}^{x \leftarrow w}, w) \\
\Delta \arrow[rr, "\downarrow"] & & \Delta[x] \\
(\mathfrak{N}, v) & & (\mathfrak{N}^{x \leftarrow v}, v)
\end{tikzcd}$$

$\exists \in \mathcal{O}$:

A move along $\Delta \xrightarrow{\exists} \Delta[x]$. $\forall$belard gives name x to a new arbitrary state $w_1 \in |\mathfrak{M}|$ without changing his current state. $\exists$loise needs to match $\forall$belard's choice by giving name x to a state $v_1 \in |\mathfrak{N}|$.

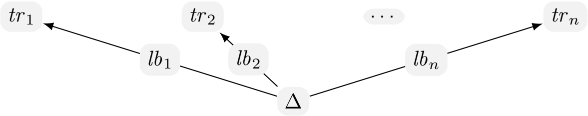
$$\begin{tikzcd}
(\mathfrak{M}, w) & & (\mathfrak{M}^{x \leftarrow w_1}, w) \\
\Delta \arrow[rr, "\exists"] & & \Delta[x] \\
(\mathfrak{N}, v) & & (\mathfrak{N}^{x \leftarrow v_1}, v)
\end{tikzcd}$$

idle: $\Delta \xrightarrow{1} \Delta$, no change in models.

Game sentences

The set of game sentences Θ_{tr} over a gameboard tree tr with root labelled by a finite signature Δ, is defined by structural induction on gameboard trees:

- Base case ($tr = \Delta$). Put $\Theta_\Delta = \{\bigwedge_{\rho \in \mathsf{Sen}_b(\Delta)} \rho^{f(\rho)} \mid f : \mathsf{Sen}_b(\Delta) \to \{0,1\}\}$, where $\rho^0 = \rho$ and $\rho^1 = \neg\rho$. (All atomic "state descriptions")
- Inductive step ($tr = \Delta(\xrightarrow{lb_1} tr_1 \dots \xrightarrow{lb_n} tr_n)$), as below:



- Fix an index $i \in \{1, \dots, n\}$. Then define
  1. a subset $S_i \subseteq \mathcal{P}(\Theta_{tr_i})$ of the powerset of Θ_{tr_i}, and
  2. a Δ-sentence φ_Γ for each set of game sentences $\Gamma \in S_i$.

  Depending on the label lb_i, there are five cases:

Game sentences

which we will not go into. Except two:

- Possibility move ($\Delta \xrightarrow{\langle\mathfrak{a}\rangle} \Delta$): This case assumes $\Diamond \in \mathcal{O}$.
  1. $S_i \coloneqq \mathcal{P}(\Theta_{tr_i})$, and
  2. $\varphi_\Gamma \coloneqq (\bigwedge_{\gamma\in\Gamma}\langle\mathfrak{a}\rangle\gamma) \wedge ([\mathfrak{a}]\bigvee\Gamma)$ for all $\Gamma \in S_i$.
     (For any $\gamma \in \Gamma$ one can go somewhere where γ holds, and wherever one goes some γ will hold.)
- Existential q. move: ($\Delta \xrightarrow{\exists} \Delta[x]$): This case assumes $\exists \in \mathcal{O}$.
  1. $S_i \coloneqq \mathcal{P}(\Theta_{tr_i})$, and
  2. $\varphi_\Gamma \coloneqq (\bigwedge_{\gamma\in\Gamma}\exists x\cdot\gamma) \wedge (\forall x\cdot\bigvee\Gamma)$, for all $\Gamma \in S_i$.

Definition

The set of game sentences over tr is
$\Theta_{tr} = \{\varphi_{\Gamma_1} \wedge \cdots \wedge \varphi_{\Gamma_n} \mid \Gamma_1 \in S_1, \ldots, \Gamma_n \in S_n\}$.

Fraïssé-Hintikka theorem

Theorem

Let Δ be a finite signature.

1. *For all pointed models $(\mathfrak{M}, w)$ defined over Δ, and all gameboard trees tr with $root(tr) = \Delta$, there exists a unique game sentence $\varphi \in \Theta_{tr}$ such that $(\mathfrak{M}, w) \models \varphi$.*
2. *For all pointed models $(\mathfrak{M}, w)$ and $(\mathfrak{N}, v)$ defined over Δ and all gameboard trees tr with $root(tr) = \Delta$, the following are equivalent:*
   - 2.1 $(\mathfrak{M}, w) \approx_{tr} (\mathfrak{N}, v)$
   - 2.2 *There exists a unique game sentence $\varphi \in \Theta_{tr}$ such that $(\mathfrak{M}, w) \models \varphi$ and $(\mathfrak{N}, v) \models \varphi$.*
3. *For each sentence ϕ defined over Δ, there exists a gameboard tree tr with $root(tr) = \Delta$ and a set of game sentences $\Psi_\phi \subseteq \Theta_{tr}$ such that $\phi \leftrightarrow \bigvee \Psi_\phi$ is a theorem of $\mathcal{L}$.*

Finite EF games: exactly what you would expect

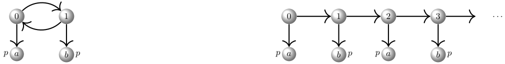
Corollary

Let $\mathfrak{M}$ and $\mathfrak{N}$ be two Kripke structures defined over a finite signature Δ. The following are equivalent:

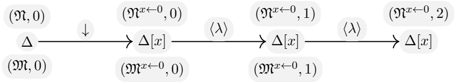
1. $(\mathfrak{M}, w)$ *and* $(\mathfrak{N}, v)$ *are* $\mathcal{L}$*-elementarily equivalent. In symbols* $(\mathfrak{M}, w) \equiv (\mathfrak{N}, v)$.
2. $\exists$*loise has a winning strategy for the EF game starting with* $(\mathfrak{M}, w)$ *and* $(\mathfrak{N}, v)$*, that is,* $(\mathfrak{M}, w) \approx_{tr} (\mathfrak{N}, v)$ *for all finite gameboard trees* tr.

Example 1

Let Δ be a signature with no nominals, one binary relation λ, and one propositional symbol p. Let $\mathfrak{M}$ and $\mathfrak{N}$ be the Δ-models shown below:



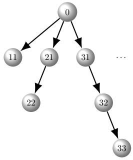
1. If $\mathcal{L}$ is PDL, then $(\mathfrak{M}, 0)$ and $(\mathfrak{N}, 0)$ are $\mathcal{L}$-elementarily equivalent.
2. If $\mathcal{L}$ is HPL, then $(\mathfrak{M}, 0) \not\approx_{tr} (\mathfrak{N}, 0)$ for any complete gameboard tree tr of height greater or equal than 3, as shown below:

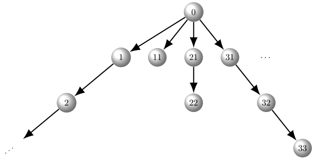


In the third round, $\exists$loise loses.

Example 2

Let Δ be the signature with one relation λ. Let $\mathfrak{M}$ be the model on the left, and $\mathfrak{N}$ the model on the right, with one infinite path.



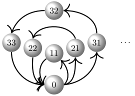


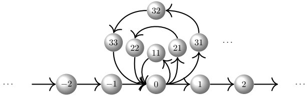
1. In all fragments of HPL, $\exists$loise has a winning strategy over finite gameboard trees starting with $(\mathfrak{M}, 0)$ and $(\mathfrak{N}, 0)$.
2. In any fragment of HDPL closed under the Booleans and actions, $\exists$loise loses the game played over $\Delta \xrightarrow{\langle\lambda\rangle} \Delta \xrightarrow{\langle\lambda^*\rangle} \Delta \xrightarrow{\langle\lambda\rangle} \Delta$ starting with $(\mathfrak{M}, 0)$ and $(\mathfrak{N}, 0)$:

$$\begin{array}{ccccccc} (\mathfrak{N},0) & & (\mathfrak{N},1) & & (\mathfrak{N},m) & & (\mathfrak{N},m+1) \\ \Delta & \xrightarrow{\langle\lambda\rangle} & \Delta & \xrightarrow{\langle\lambda^*\rangle} & \Delta & \xrightarrow{\langle\lambda\rangle} & \Delta \\ (\mathfrak{M},0) & & (\mathfrak{M},n1) & & (\mathfrak{M},nn) & & \end{array}$$

Example 3

Let Δ be a signature with one binary relation λ, and let $\mathfrak{M}$ and $\mathfrak{N}$ be model on the left and right, respectively.





1. $(\mathfrak{M}, 0) \equiv (\mathfrak{N}, 0)$ in any fragment $\mathcal{L}$ of HPL closed under the Booleans.
2. $(\mathfrak{M}, 0) \not\equiv (\mathfrak{N}, 0)$ in any fragment of HDPL closed under the Booleans, actions and store. For ∀belard can win in three steps:

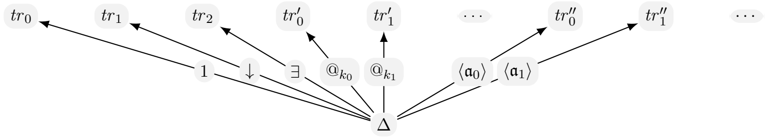
$$\begin{array}{ccccccc} (\mathfrak{N},0) & & (\mathfrak{N}^{x\leftarrow 0},0) & & (\mathfrak{N}^{x\leftarrow 0},1) & & \\ \Delta & \xrightarrow{\quad\downarrow\quad} & \Delta[x] & \xrightarrow{\quad\langle\lambda\rangle\quad} & \Delta[x] & \xrightarrow{\quad\langle\lambda^*\rangle\quad} & \Delta[x] \\ (\mathfrak{M},0) & & (\mathfrak{M}^{x\leftarrow 0},0) & & (\mathfrak{M}^{x\leftarrow 0},n1) & & (\mathfrak{M}^{x\leftarrow 0},0) \end{array}$$

Interlude: a glossary of equivalences

	already occurred
$\cong$	isomorphism
$\equiv$	elementary equivalence
$\approx_{tr}$	$\exists$loise has a winning strategy in a finite EF game
	to come shortly
$\approx_{\omega}$	$\exists$loise has a winning strategy in a countable EF game
$\equiv_{\mathcal{I}}$	back-and-forth equivalence
$\equiv_{B}$	bisimulation equivalence

Gameboard trees

A countable gameboard tree tr is of countable height and countable width, and is defined recursively as follows. Start with:



where

- $F = \{k_i \mid i < \alpha\}$ is an enumeration of all Δ-nominals and α is the cardinal of $\mathrm{Sen}(\Delta)$,
- $\mathcal{A}(\Delta) = \{\mathfrak{a}_i \mid i < \alpha\}$ is an enumeration of all actions defined over Δ.

Extend so that each of tr_0, tr_1, tr_2, tr'_i and tr''_i are trees of the same form as tr.

ω-bisimulations

Definition

Let $\mathfrak{M}$ and $\mathfrak{N}$ be Kripke structures. A relation $B_\ell \subseteq (|\mathfrak{M}|^\ell \times |\mathfrak{M}|) \times (|\mathfrak{N}|^\ell \times |\mathfrak{N}|)$ is an ℓ-bisimulation from $\mathfrak{M}$ to $\mathfrak{N}$ if for all $(\overline{w}, w)\ B_\ell\ (\overline{v}, v)$ the following hold:

(prop) $p \in M(w)$ iff $p \in N(v)$ for all propositional symbols $p \in \texttt{Prop}$;

(nom) $w = k^{\mathfrak{M}}$ iff $v = k^{\mathfrak{N}}$ for all nominals $k \in F$;

(wvar) $\overline{w}(j) = w$ iff $\overline{v}(j) = v$ for all $1 \leq j \leq \ell$;

(forth) if $\Diamond \in \mathcal{O}$ then for all actions $\mathfrak{a} \in \mathcal{A}(\Delta)$ and all states $w' \in \mathfrak{a}^{\mathfrak{M}}(w)$ there exists $v' \in \mathfrak{a}^{\mathfrak{N}}(v)$ such that $(\overline{w}, w')\ B_\ell\ (\overline{v}, v')$;

(back) if $\Diamond \in \mathcal{O}$ then for all actions $\mathfrak{a} \in \mathcal{A}(\Delta)$ and all states $v' \in \mathfrak{a}^{\mathfrak{N}}(v)$ there exists $w' \in \mathfrak{a}^{\mathfrak{M}}(w)$ such that $(\overline{w}, w')\ B_\ell\ (\overline{v}, v')$;

(atv) if $@ \in \mathcal{O}$ then $(\overline{w}, \overline{w}(j))\ B_\ell\ (\overline{v}, \overline{v}(j))$ for all $1 \leq j \leq \ell$;

(atn) if $@ \in \mathcal{O}$ then $(\overline{w}, k^{\mathfrak{M}})\ B_\ell\ (\overline{v}, k^{\mathfrak{N}})$ for all nominals $k \in F$;

An ω-bisimulation from $\mathfrak{M}$ to $\mathfrak{N}$ is a family of ℓ-bisimulations $B = (B_\ell)_{\ell \in \omega}$ from $\mathfrak{M}$ to $\mathfrak{N}$ such that for all natural numbers $\ell \in \omega$ and all tuples $(\overline{w}, w) \in |\mathfrak{M}|^\ell \times |\mathfrak{M}|$ and $(\overline{v}, v) \in |\mathfrak{N}|^\ell \times |\mathfrak{N}|$ the following conditions are satisfied:

(st) if $\downarrow \in \mathcal{O}$ and $(\overline{w}, w)\ B_\ell\ (\overline{v}, v)$ then $(\overline{w}\ w, w)\ B_{\ell+1}\ (\overline{v}\ v, v)$, where the juxtaposition stands for the concatenation of sequences; and

(ex) if $\exists \in \mathcal{O}$ and $(\overline{w}, w)\ B_\ell\ (\overline{v}, v)$ then:

- (ex-f) for all $w' \in |\mathfrak{M}|$ there is $v' \in |\mathfrak{N}|$ such that $(\overline{w}\ w', w)\ B_{\ell+1}\ (\overline{v}\ v', v)$,
- (ex-b) for all $v' \in |\mathfrak{N}|$ there is $w' \in |\mathfrak{M}|$ such that $(\overline{w}\ w', w)\ B_{\ell+1}\ (\overline{v}\ v', v)$.

ω-bisimilar models

Definition

Two pointed models $(\mathfrak{M}, w)$ and $(\mathfrak{N}, v)$ are ω-bisimilar if there exists an ω-bisimulation B from $\mathfrak{M}$ to $\mathfrak{N}$ such that $w\ B_0\ v$.

- Notation for ω-bisimilarity: $(\mathfrak{M}, w) \equiv_B (\mathfrak{N}, v)$.
- An ω-bisimulation is a relation between $|\mathfrak{M}|^*$ and $|\mathfrak{N}|^*$.
- Very roughly: ω-bisimulations are bisimulations with memory.

Lemma

Let B be an ω-bisimulation between $(\mathfrak{M}, \mu)$ and $(\mathfrak{N}, \nu)$ such that $(\mu, \mu')\ B_1\ (\nu, \nu')$ for some $(\mu, \mu') \in |\mathfrak{M}| \times |\mathfrak{M}|$ and $(\nu, \nu') \in |\mathfrak{N}| \times |\mathfrak{N}|$. Let $\chi : \Delta \to \Delta[x]$ be a signature extension with a variable x. Then $B^x = (B^x_\ell)_{\ell \in \omega}$ defined by

$$(\overline{w}, w)\ B^x_\ell\ (\overline{v}, v) \text{ iff } (\mu\, \overline{w}, w)\ B_{\ell+1}\ (\nu\, \overline{v}, v),$$

$$\text{for all } \ell \in \omega, \text{ all } (\overline{w}, w) \in |\mathfrak{M}|^\ell \times |\mathfrak{M}| \text{ and all } (\overline{v}, v) \in |\mathfrak{N}|^\ell \times |\mathfrak{N}|,$$

is a bisimulation between $(\mathfrak{M}^{x \leftarrow \mu}, \mu')$ and $(\mathfrak{N}^{x \leftarrow \nu}, \nu')$.

Back-and-forth systems

A partial isomorphism $h : \mathfrak{M} \nrightarrow \mathfrak{N}$ is a bijection between a subset of $|\mathfrak{M}|$ and a subset of $|\mathfrak{N}|$, preserving and reflecting all accessibility relations λ.

Definition

A back-and-forth system between two Kripke structures $\mathfrak{M}$ and $\mathfrak{N}$ over a signature $\Delta = ((F, P), \texttt{Prop})$ is a non-empty family $\mathcal{I}$ of basic partial isomorphisms between $\mathfrak{M}$ and $\mathfrak{N}$ satisfying the following:

- (@-extension) If $\mathcal{L}$ is closed under retrieve, then for all $h \in \mathcal{I}$ and all $k \in F$, there exists a $g \in \mathcal{I}$ such that $h \subseteq g$ and $k^{\mathfrak{M}} \in \mathrm{dom}(g)$.
- ($\Diamond$-extension) If $\mathcal{L}$ is closed under possibility over an action $\mathfrak{a}$, then:
  - (forth) for all $h \in \mathcal{I}$, all $w_1 \in \mathrm{dom}(h)$ and all $w_2 \in |\mathfrak{M}|$ such that $w_1 \ \mathfrak{a}^{\mathfrak{M}} \ w_2$, there exists a $g \in \mathcal{I}$ such that $h \subseteq g$, $w_2 \in \mathrm{dom}(g)$, and $g(w_1) \ \mathfrak{a}^{\mathfrak{N}} \ g(w_2)$;
  - (back) for all $h \in \mathcal{I}$, all $v_1 \in \mathrm{rng}(h)$ and all $v_2 \in |\mathfrak{N}|$ such that $v_1 \ \mathfrak{a}^{\mathfrak{N}} \ v_2$, there exists a $g \in \mathcal{I}$ such that $h \subseteq g$, $v_2 \in \mathrm{rng}(g)$, and $g^{-1}(v_1) \ \mathfrak{a}^{\mathfrak{M}} \ g^{-1}(v_2)$.
- ($\exists$-extension) If $\mathcal{L}$ is closed under existential quantifiers, then:
  - (forth) for all $h \in \mathcal{I}$ and all $w \in |\mathfrak{M}|$, there exists a $g \in \mathcal{I}$ such that $h \subseteq g$ and $w \in \mathrm{dom}(g)$;
  - (back) for all $h \in \mathcal{I}$ and all $v \in |\mathfrak{N}|$, there exists a $g \in \mathcal{I}$ such that $h \subseteq g$ and $v \in \mathrm{rng}(g)$.

Back-and-forth systems

Definition

- Two Kripke structures $\mathfrak{M}$ and $\mathfrak{N}$ are back-and-forth equivalent, if there is a back-and-forth system $\mathcal{I}$ between $\mathfrak{M}$ and $\mathfrak{N}$, in symbols, $\mathfrak{M} \equiv_{\mathcal{I}} \mathfrak{N}$.
- Two pointed models $(\mathfrak{M}, w)$ and $(\mathfrak{N}, v)$ are back-and-forth equivalent, if there is a back-and-forth system $\mathcal{I}$ between $\mathfrak{M}$ and $\mathfrak{N}$ such that $h(w) = v$ for some $h \in \mathcal{I}$, in symbols, $(\mathfrak{M}, w) \equiv_{\mathcal{I}} (\mathfrak{N}, v)$.

Bisimulations, back-and-forth systems, ω-EF games

Theorem

Let $(\mathfrak{M}, w)$ *and* $(\mathfrak{N}, v)$ *be two pointed models over a signature* Δ*. Then:*

$$(\mathfrak{M}, w) \equiv_B (\mathfrak{N}, v) \quad \textit{iff} \quad (\mathfrak{M}, w) \approx_\omega (\mathfrak{N}, v).$$

Theorem

Let $(\mathfrak{M}, w)$ *and* $(\mathfrak{N}, v)$ *be two pointed models over a signature* Δ*.*

1. *If* $(\mathfrak{M}, w) \equiv_{\mathcal{I}} (\mathfrak{N}, v)$ *for some back-and-forth system* $\mathcal{I}$*, then* $(\mathfrak{M}, w) \approx_\omega (\mathfrak{N}, v)$.
2. *Assume that (i)* $\downarrow \in \mathcal{O}$*, and (ii)* $@ \in \mathcal{O}$ *whenever* $\Diamond \in \mathcal{O}$ *or* $\exists \in \mathcal{O}$*. Then the converse holds as well, that is:*

   If $(\mathfrak{M}, w) \approx_\omega (\mathfrak{N}, v)$*, then* $(\mathfrak{M}, w) \equiv_{\mathcal{I}} (\mathfrak{N}, v)$ *for some back-and-forth system* $\mathcal{I}$*.*

Image-finite models

A model $\mathfrak{M}$ is image-finite if each state has a finite number of direct successors.

Theorem

Let $(\mathfrak{M}, w)$ and $(\mathfrak{N}, v)$ be two image-finite pointed models defined over a signature Δ. Then:

$$(\mathfrak{M}, w) \equiv (\mathfrak{N}, v) \quad \textit{iff} \quad (\mathfrak{M}, w) \approx_\omega (\mathfrak{N}, v)\ .$$

Corollary (Hennessy-Milner theorem)

Let $(\mathfrak{M}, w)$ and $(\mathfrak{N}, v)$ be two image-finite pointed models. Then:

$$(\mathfrak{M}, w) \equiv (\mathfrak{N}, v) \quad \textit{iff} \quad (\mathfrak{M}, w) \equiv_B (\mathfrak{N}, v)\ .$$

Countable rooted models

Theorem

Assume that $\{\Diamond, @, \downarrow\} \subseteq \mathcal{O}$. *Let* $(\mathfrak{M}, w_0)$ *and* $(\mathfrak{N}, v_0)$ *be two rooted pointed models that are countable. Then:*

$$(\mathfrak{M}, w_0) \cong (\mathfrak{N}, v_0) \quad \textit{iff} \quad (\mathfrak{M}, w_0) \equiv_B (\mathfrak{N}, v_0) \ .$$

Corollary

Assume that $\{\Diamond, @, \downarrow\} \subseteq \mathcal{O}$. *Let* $(\mathfrak{M}, w_0)$ *and* $(\mathfrak{N}, v_0)$ *be two rooted image-finite pointed models. Then:*

$$(\mathfrak{M}, w_0) \cong (\mathfrak{N}, v_0) \quad \textit{iff} \quad (\mathfrak{M}, w_0) \equiv (\mathfrak{N}, v_0) \ .$$

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