Tight inference and real-valued logic

Workshop on Logic, Algebra and Category Theory (LAC)

Fukuoka, 2025

This talk is based on ongoing work with Guillermo Badia, Ronald Fagin, Alexander Gray, Carles Noguera, Ryan Riegel, and
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Motivation: deduction in real-valued logics

Working in logics with truth-values in $[0, 1]$.

Assume we have the following background knowledge:

- value of $\varphi \to \theta$ is in $[0.7, 0.8]$
- value of φ is in $[0.4, 0.5]$

What can we deduce about θ? What is the tightest set of values that θ can take?

Real-valued logic

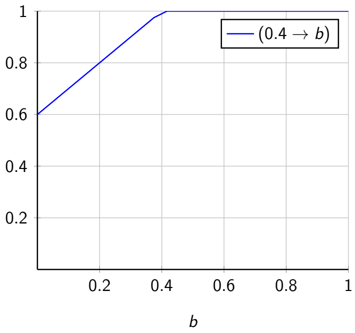
Syntax

- atomic variables: $V = \{p, q, r, \dots\}$
- constants $\overline{0}, \overline{1}$ and connectives $\neg, \rightarrow, \&$

Semantics of Łukasiewicz logic

- Assignment $M : V \rightarrow [0, 1]$
- $M(\overline{0}) = 0, M(\overline{1}) = 1$
- $M(\neg\varphi) = 1 - M(\varphi)$
- $M(\varphi \,\&\, \theta) = \max\{0, M(\varphi) + M(\theta) - 1\}$
- $M(\varphi \rightarrow \theta) = 1 - \max\{0, M(\varphi) - M(\theta)\}$

Write $\to, \&, \neg, \overline{0}, \overline{1}$ for both syntax and semantics.



$a \to b$ equals $1 - \max\{0, a - b\}$

if $a \leq b$ then $a \to b$ has value 1

if $a > b$ then $a \to b$ has value $1 - (a - b)$

Sentences

- Usually 1 is a favored value, e.g., M satisfies φ if $M(\varphi) = 1$.
- Instead, for a set $S \subseteq [0,1]$ of truth-values, we will express statements

$$\text{“}M(\varphi) \in S\text{”}$$

  - $(\varphi; S)$ is called a **sentence**
  - S is called the **information set**
  - $M \models (\varphi; S)$ if $M(\varphi) \in S$ Say **“M satisfies $(\varphi; S)$”**
  - $M \models B$ if M satisfies every sentence in B

Foundations of reasoning with uncertainty via real-valued logics, Fagin, Riegel, Gray (PNAS 2024) introduced a logic of multi-dimensional sentences $(\varphi_1, \cdots, \varphi_d; R)$ with $R \subseteq [0,1]^d$

Tight-values set

Given:

- a finite set B of background sentences
- a target formula φ

Define the tight-values set T of $\langle B, \varphi \rangle$ as

$$T = \bigcup \{ M(\varphi) : M \text{ satisfies } B \}$$

(all possible values of φ under assignments that satisfy B)

$$T = \bigcap \{ X : B \vDash (\varphi; X) \}$$

(smallest set X of values such that $B \vDash (\varphi; X)$)

We say: from B tightly infer $(\varphi; T)$

Deduction illustrated

From $(\varphi \to \theta; [c, c'])$ and $(\varphi; [d, d'])$

tightly infer

$(\theta; [c \,\&\, d, c' \,\&\, d'])$ if $c' < 1$

$(\theta; [c \,\&\, d, 1])$ otherwise

Recall: $a \,\&\, b = \max\{0, a + b - 1\}$.

- If the value of $\varphi \to \theta$ is in $[0.7, 0.8]$, and
- if the value of φ is in $[0.4, 0.5]$, then
- we tightly infer the value of θ is in $[0.1, 0.3]$

Computational problems

Let T denote the tight-values set of $\langle B, \varphi \rangle$.

1. Decide $s \in T$ where s is a given rational number.
2. Compute (some representation of) T.

Assumption for this talk: Information sets in the background B are closed intervals with rational endpoints.

Use Mixed Integer Linear Programming

MILP

- $P(\bar{x}; \bar{z})$
- real-variables $\bar{x}$, Boolean variables $\bar{z}$
- set of linear constraints $t_1(\bar{x}, \bar{z}) \leq t_2(\bar{x}, \bar{z})$
- deciding if $P(\bar{x}; \bar{z})$ has a solution is NP-complete

For a set X of sentences, build a MILP P_X whose solutions capture the models satisfying X:

- For every subformula ψ, add variable x_ψ and constraint $0 \leq x_\psi \leq 1$
- If $\psi = \overline{0}$, add constraint $x_{\overline{0}} = 0$ (similarly $x_{\overline{1}} = 1$).
- If ψ is $\neg\varphi$, add constraint $x_\psi = 1 - x_\varphi$
- If ψ is $\varphi \to \theta$, add constraints for $x_\psi = 1 - \max\{0, x_\varphi - x_\theta\}$

  we handle this non-linearity by using a Boolean variable to distinguish if the max is 0 or not, and two additional real variables
- If ψ is $\varphi \,\&\, \theta$, is handled similarly.
- If $(\psi; [c, d]) \in X$ then add constraints $c \leq x_\psi \leq d$

Prop. The solutions of P_X, projected onto x_p for atoms p, are exactly the models M that satisfy X.

Let T be the tight-values set of $\langle B, \varphi \rangle$.

Cor. T is a finite union of closed intervals with rational endpoints, computable in EXPTIME.

- Take X to consist of B and $(\varphi; [0,1])$
- Instantiate the Boolean variables in P_X, to get (exp-many) linear programs.
- LP solution-sets are convex, their projection onto x_φ are closed intervals.

Prop. T may consist of exponentially many disjoint intervals in the worst case.

- Cantor-set like construction.

Computational complexity

Let T be the tight-values set of $\langle B, \varphi \rangle$.

Cor. Deciding if a given rational is in T is NP-complete.

- NP: check if there is a solution to the MILP P_X where $X = B \cup \{ (\varphi; \{s\}) \}$ and s is the given rational
- NP-hard: via a simple reduction from Boolean satisfiability.

Thm. Deciding if a given rational is in the kth disjoint interval of T is in PSPACE.

- Find the ith disjoint interval, $i = 1, 2, \cdots, k$, only storing the last one ("on-the-fly")

Inference in neural networks

- $NN : [0,1]^k \to [0,1]$
- Inference: Given NN and output value s, find input values $\bar{a}$ s.t. $NN(\bar{a}) = s$.
- Some NNs can be represented as Łukasiewicz logic formulas.
  "Neural Networks and Rational McNaughton Functions", Amato, Di Nola, Gerla (MVLSC 2005)
- Inference: Given ψ and truth-value s, find truth-values $\bar{a}$ s.t. $\psi(\bar{a}) = s$.
- The set of such $\bar{a}$ is exactly the tight-values set of $\langle B, \bar{p} \rangle$ where B consists of $(\psi; \{s\})$, and $\bar{p}$ are the atoms in ψ.

Ongoing work

Extend analysis/techniques to handle:

- multi-dimensional sentences
- other real-valued logics, including probabilistic logics

A logic for reasoning about probabilities, Fagin, Halpern, Megiddo (I&C 1990)

Prop. The tight-values set may consist of an exponential number of disjoint intervals.

$$\left((\frac{1}{5} \to x_0) \wedge (x_0 \to \frac{2}{5})\right) \vee \left((\frac{3}{5} \to x_0) \wedge (x_0 \to \frac{4}{5})\right)$$

means

$$x_0 \in [0.2, 0.4] \cup [0.6, 0.8]$$

Adding

$$(x_0 \leftrightarrow (x_1 \oplus x_1)) \vee (x_0 \leftrightarrow (\neg x_1 \oplus \neg x_1))$$

shrinks the intervals into the left-half and into the right-half:

$$x_1 \in [0.1, 0.2] \cup [0.3, 0.4] \cup [0.6, 0.7] \cup [0.8, 0.9]$$

Repeat.