Minimally undecidable reducts of Tarski's relation algebras

Marcel Jackson (joint with Robin Hirsch and Jaš Šemrl)

La Trobe University, Melbourne, Australia

Relation Algebras and results

Tarski relation algebras

Binary relations on a set X

The subsets $\wp(X \times X)$ of $X \times X$ forms:

- a Boolean algebra wrt $\cap, \cup, -, \varnothing, X \times X$
- a involuted monoid with respect to $\circ$, converse $^{-1}$ and $=_X$ (identity relation)
- some properties relating the “action” part to the “logic” part

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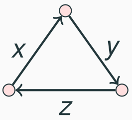
  - Distributivity of $\circ$ over $\cup$
  - Peircean law: $x \circ y \subseteq -z^{-1} \Leftrightarrow y \subseteq z \leq -y^{-1}$



$x\ ;\ y$ can't sit over z^{-1}

$\Leftrightarrow$

$y\ ;\ z$ can't sit over x^{-1}

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History

- 1800s: de Morgan, Peirce, Schröder (logic of relatives)
- 1940s: Tarski (relation algebras)

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(Tarski 1941) "*Is it the case that every sentence of the calculus of relations which is true in every domain of individuals is derivable from the axioms adopted under the second method? This problem presents some difficulties and still remains open. I can only say that I am practically sure that I can prove with the help of the second method, all of the hundreds of theorems to be found in Schröder's Algebra und Logic der Relative*"

Tarski relation algebras

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The next problem is the so-called representation problem. Is every model of the axiom system of the calculus of relations isomorphic with a class of binary relations which contains the relations $1, 0, 1', 0'$ *and is closed under all the operations considered in this calculus?*

Tarski relation algebras

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Fundamental representability question

When is an abstract algebra $\langle A, +, \cdot, -, ;, ^{\smile}, 0, 1, 1' \rangle$ isomorphic to an algebra of binary relations?

Tarski relation algebras

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…temporarily skipping 60 years of history…

Robin Hirsch and Ian Hodkinson 2001

This problem is undecidable for finite algebras

The question of representability of finite algebras of finite sets remains open

Examples

Point algebra

The 8-element relation algebra with atoms $<, >, =$

$;$	$<$	$>$	$=$
$<$	$<$	1	$<$
$>$	1	$>$	$>$
$=$	$<$	$>$	$=$

This is representable by giving these symbols their usual interpretation on a dense linear order such as $\langle \mathbb{Q}, < \rangle$*, but not on any finite set*

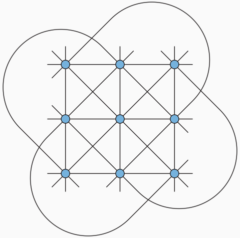
Examples

Lyndon algebras

Boolean atoms are $1', c_1, \ldots, c_n$ where $c_i ; c_i = 1' + c_i$ and $c_i ; c_j = -(c_i + c_j)$.

Representable only over affine plane of order $n - 1$

The question of which orders an affine plane exists remains open



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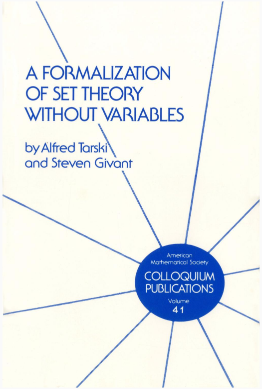
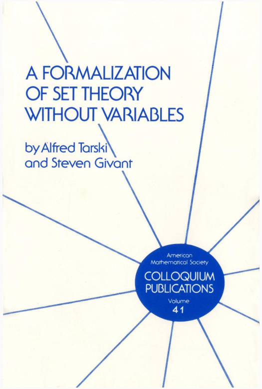


ZFC can be expressed equationally within the equational theory of relation algebras

Examples



ZFC can be expressed equationally within the equational theory of relation algebras

Axiom of Extensionality:

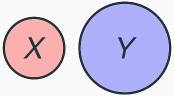
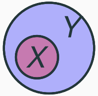
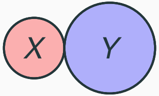
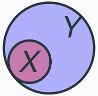
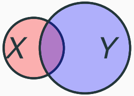
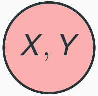
$$\forall x \forall y \big(\forall z (z \in x \leftrightarrow z \in y) \to x = y\big)$$

Relation Algebra:

$$(\in^{\smile} ; (-\in)) + ((-\in)^{\smile} ; \in) + 1' \approx 1$$

(In general it is known that the language of relation algebras captures the 3-variable fragment of the first order predicate calculus of binary relations)

Region Connection Calculus RCC8

Example	Relation	Example	Relation
	X dis Y		X in Y Y over X
	X et Y		X it Y Y it$^{\smile}$ X
	X po Y		$X = Y$

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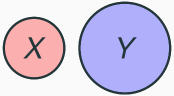
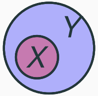
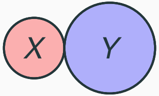
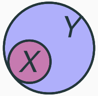
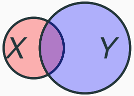
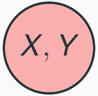
	X dis Y		X in Y Y over X
	X et Y		X it Y Y $\text{it}^{\smile}$ X
	X po Y		$X = Y$

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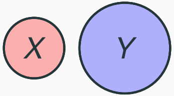
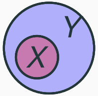
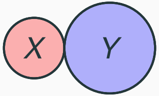
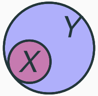
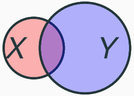
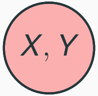
	X dis Y		X in Y Y over X
	X et Y		X it Y Y it$^{\smile}$ X
	X po Y		$X = Y$

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$\{po, it, in\}$

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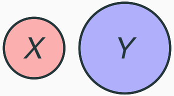
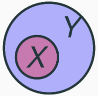
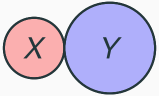
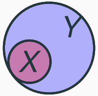
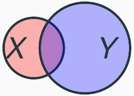
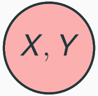
	X dis Y		X in Y Y over X
	X et Y		X it Y Y $\text{it}^{\smile}$ X
	X po Y		$X = Y$

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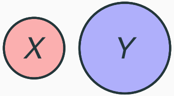
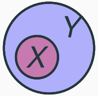
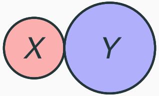
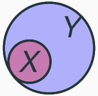
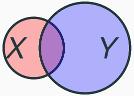
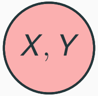
	X dis Y		X in Y Y over X
	X et Y		X it Y Y $\text{it}^{\smile}$ X
	X po Y		$X = Y$

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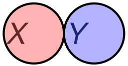
X *et* Y *in* Z



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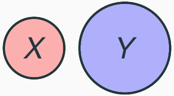
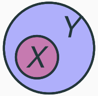
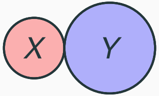
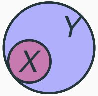
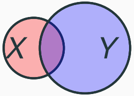
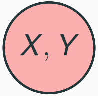
	X dis Y		X in Y Y over X
	X et Y		X it Y Y it$^{\smile}$ X
	X po Y		$X = Y$

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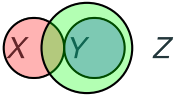

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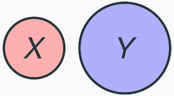
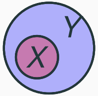
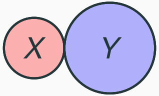
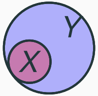
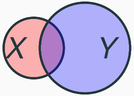
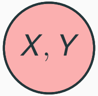
	X dis Y		X in Y Y over X
	X et Y		X it Y Y $\text{it}^{\smile}$ X
	X po Y		$X = Y$

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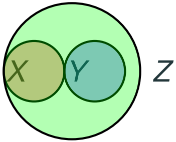

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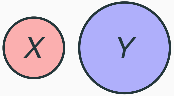
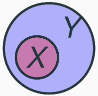
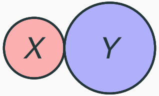
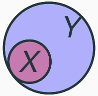
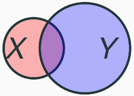
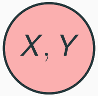
	X dis Y		X in Y Y over X
	X et Y		X it Y Y $\text{it}^{\smile}$ X
	X po Y		$X = Y$

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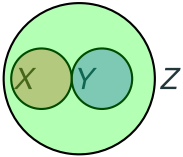
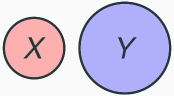
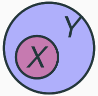
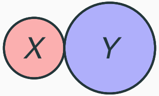
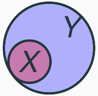
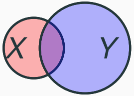
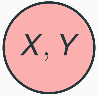


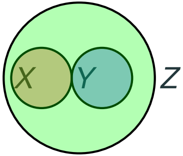
Region Connection Calculus RCC8

Example	Relation	Example	Relation
	X dis Y		X in Y Y over X
	X et Y		X it Y Y $\text{it}^{\smile}$ X
	X po Y		$X = Y$

Example

$$et \circ in$$
$$=$$
$$\{po, it, in\}$$

X *et* Y *in* Z



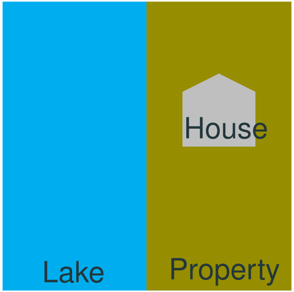
Qualitative reasoning example: the algebraic propagation algorithm

There is a house H

Qualitative reasoning example: the algebraic propagation algorithm

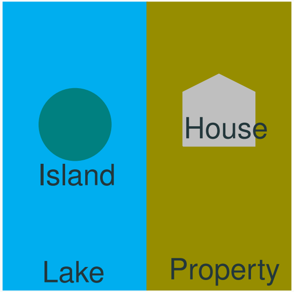
There is a house *H*
Properly within a property *P*

Qualitative reasoning example: the algebraic propagation algorithm



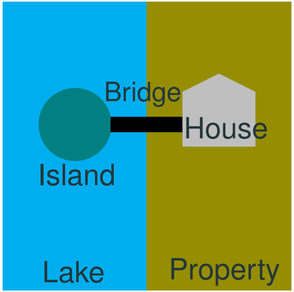
There is a house H
Properly within a property P
Bordering lake L

Qualitative reasoning example: the algebraic propagation algorithm



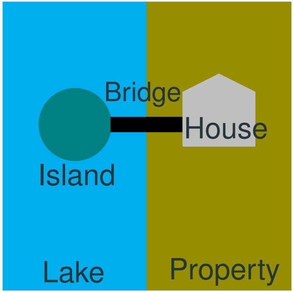
There is a house H
Properly within a property P
Bordering lake L
Containing island I

Qualitative reasoning example: the algebraic propagation algorithm



There is a house *H*
Properly within a property *P*
Bordering lake *L*
Containing island *I*
A bridge *B* connects *I* to *H*

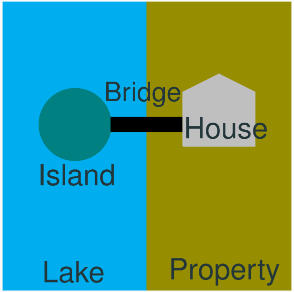
Qualitative reasoning example: the algebraic propagation algorithm



There is a house H
Properly within a property P
Bordering lake L
Containing island I
A bridge B connects I to H

How does B relate to L?

Qualitative reasoning example: the algebraic propagation algorithm



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Bordering lake L
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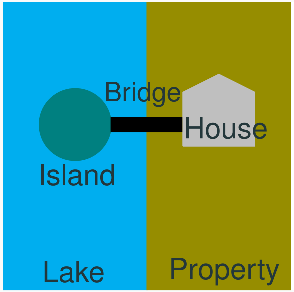
A bridge B connects I to H

How does B relate to L?

1. $B\ \{et, po\}\ I\ \{in\}\ L$

Qualitative reasoning example: the algebraic propagation algorithm



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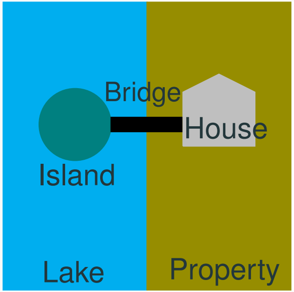
A bridge B connects I to H

How does B relate to L?

1. $B\ \{et, po\}\ I\ \{in\}\ L$
2. $\Rightarrow B\ \{po, it, in\}\ L$ (composing relations)

Qualitative reasoning example: the algebraic propagation algorithm



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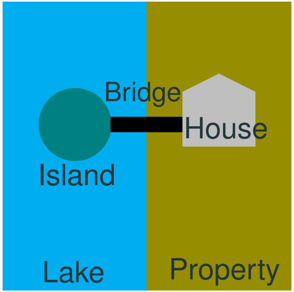

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3. $B\ \{et\}\ H\ \{dis\}\ L$

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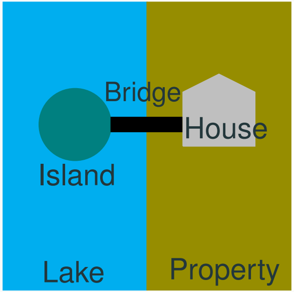

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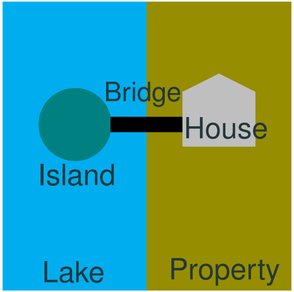
3. $B\ \{et\}\ H\ \{dis\}\ L$
4. $\Rightarrow B\ \{dis, et, po, over, it^{\smile}\}\ L$ (composing relations)

Qualitative reasoning example: the algebraic propagation algorithm



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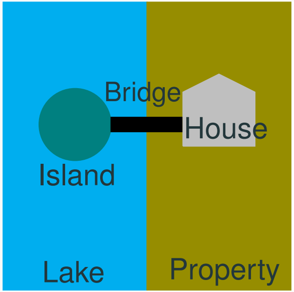
2. $\Rightarrow B\ \{po, it, in\}\ L$ (composing relations)
3. $B\ \{et\}\ H\ \{dis\}\ L$
4. $\Rightarrow B\ \{dis, et, po, over, it^{\smile}\}\ L$ (composing relations)
5. (2) and (4) give $B\ \{po, it, in\} \cap \{dis, et, po, over, it^{\smile}\}\ L$

Qualitative reasoning example: the algebraic propagation algorithm



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5. (2) and (4) give $B\ \{po, it, in\} \cap \{dis, et, po, over, it^{\smile}\}\ L$
6. $\Rightarrow B\ \{po\}\ L$

Hirsch Hodkinson proof

Tiling algebras

Algebras encoding the tiling problem for square tiles

Hirsch Hodkinson proof

Tiling algebras

Algebras encoding the tiling problem for square tiles

Jónsson signature (also *allegories* in the sense of Peter Freyd)

The argument can be carried through using only the operations $\cdot, ;, ^{\smile}$

How far down does undecidability of representability (UR) pervade?

Undecidability of representability for subreducts

Hirsch and J 2012: undecidability of representability for reducts

$\{+, \cdot, ;, 1'\}, \{\backslash, ;, 1'\}, \{\Rightarrow, ;, 1'\}, \{\leq, -, 1'\}$

Here, as usual, $x \backslash y := x \cdot (-y)$, $x \Rightarrow y := -x \vee y$ and $x \leq y$ means $x \cdot y = x$

Undecidability of representability for subreducts

Hirsch and J 2012: undecidability of representability for reducts

$\{+, \cdot, ;, 1'\}, \{\backslash, ;, 1'\}, \{\Rightarrow, ;, 1'\}, \{\leq, -, 1'\}$

Neuzerling 2016

$\{+, \cdot, ;\}, \{;, \leq, -\}$

Undecidability of representability for subreducts

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Neuzerling 2016

$\{+, \cdot, ;\}, \{;, \leq, -\}$

Very small cases

$\{;, \Rightarrow\}$ (Lewis-Smith, Semrl 2023), $\{;, -\}$ (Hirsch, J, Šemrl 2022)

Undecidability of representability for subreducts

Hirsch and J 2012: undecidability of representability for reducts

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Neuzerling 2016

$\{+, \cdot, ;\}, \{;, \leq, -\}$

Main result (Hirsch-J-Šemrl, Semigroup Forum 111 (2025) 469–489)

1. The signature $\{;, -\}$ is a minimal subset of Tarski's having UR

Undecidability of representability for subreducts

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Neuzerling 2016

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2. But there is an infinite chain of increasingly weak term reduct signatures, each with UR, but whose limit is $\{;\}$ with DR

Undecidability of representability for subreducts

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Neuzerling 2016

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Main result (Hirsch-J-Šemrl, Semigroup Forum 111 (2025) 469–489)

1. The signature $\{;, -\}$ is a minimal subset of Tarski's having UR
2. But there is an infinite chain of increasingly weak term reduct signatures, each with UR, but whose limit is $\{;\}$ with DR
3. Moreover, UR holds for a term reduct of a term reduct with DR

Some example open problems on decidability of representability

- The finite representability problem for $\{;, \cdot\}$ (Bredikhin and Schein 1978)
- $\{;, +\}$ (Andreka 1990s)
- $\{;, \smile\}$ (Schein 1974)
- $\{;, \leq, 1'\}$ (Hirsch, 2005)

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Theorem from Schein 1974

The free involuted semigroup is representable as binary relations

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Theorem from Schein 1974

The free involuted semigroup is representable as binary relations

These either involve $^\smile$ but don't seem amenable to the tiling method, or avoid $^\smile$ but seem incapable of fully encoding the the partial group embedding problem. All are nontrivial

Methods

Partial group embedding problem

Trevor Evans 1953

The uniform word problem in a class is Turing equivalent to the problem of deciding if partial algebras complete to full algebras in the class

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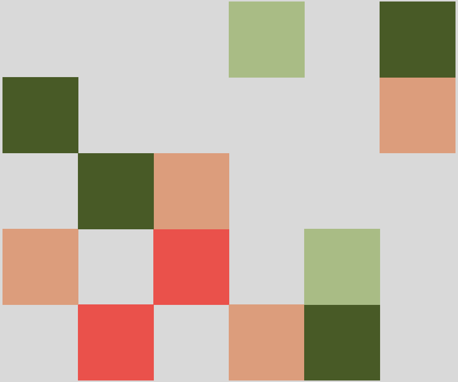
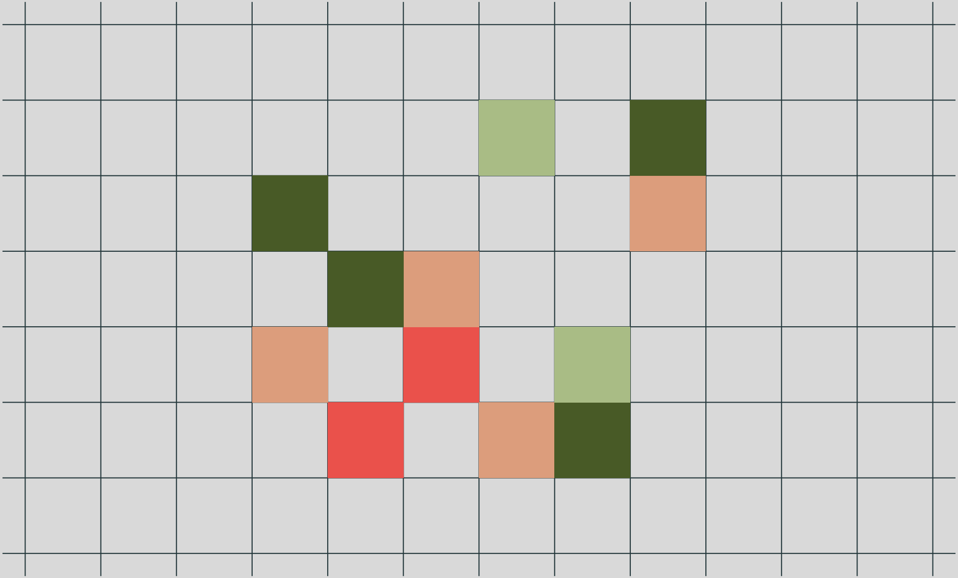
Groups

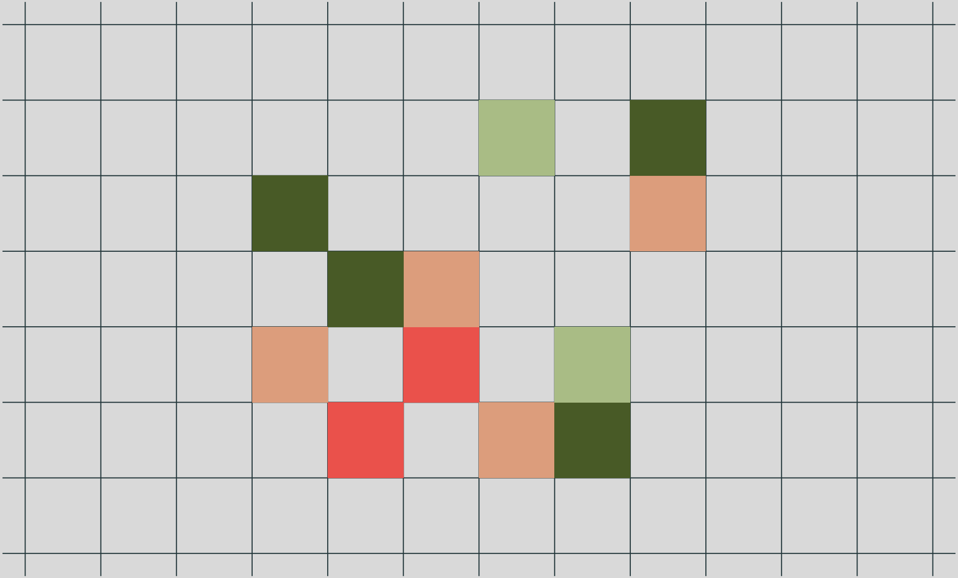
The uniform word problem for groups is undecidable (Novikov 1955, Boone 1958). The uniform word problem for finite groups is undecidable (Slobodskoĭ 1982)

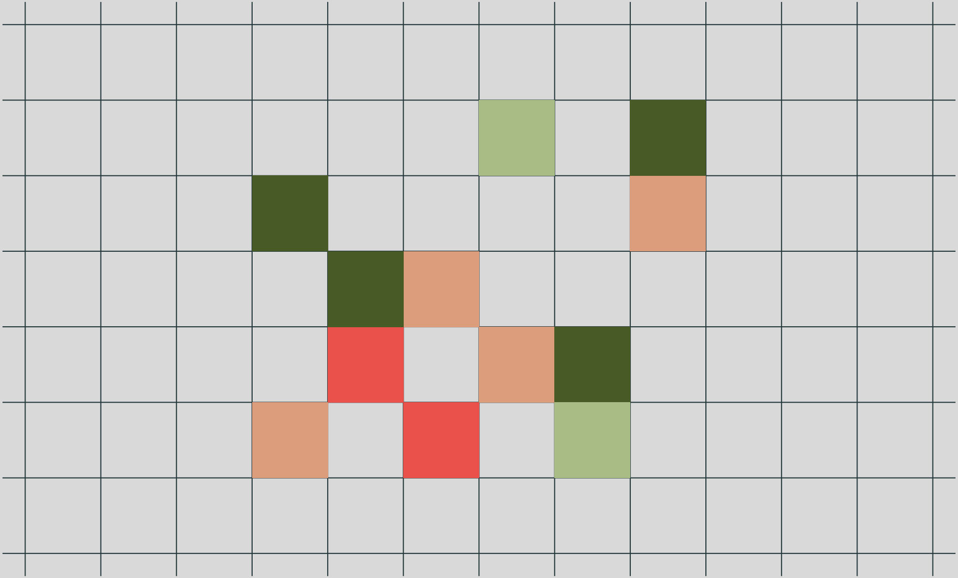
In the particular case of groups, we may interpret “complete to full algebras” in a very flexible way

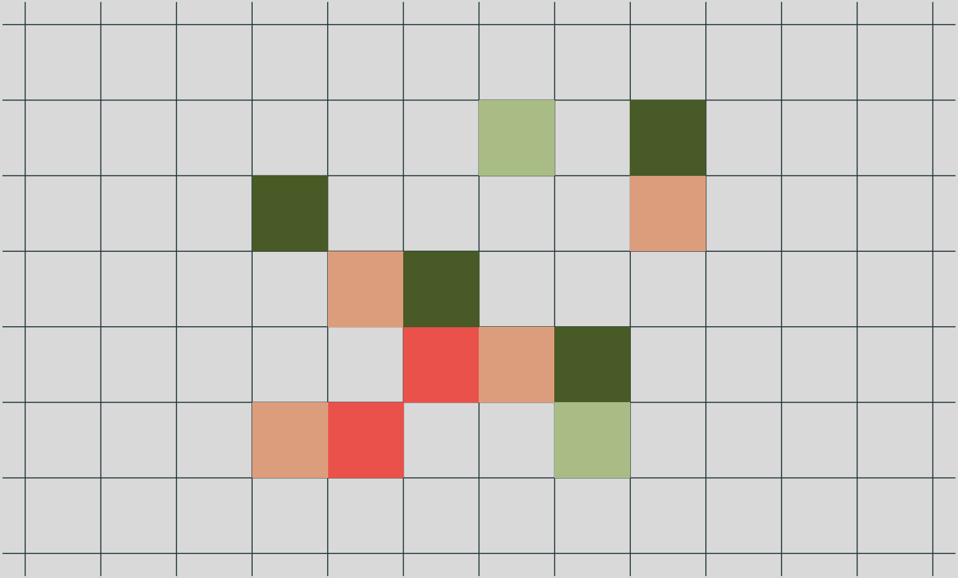


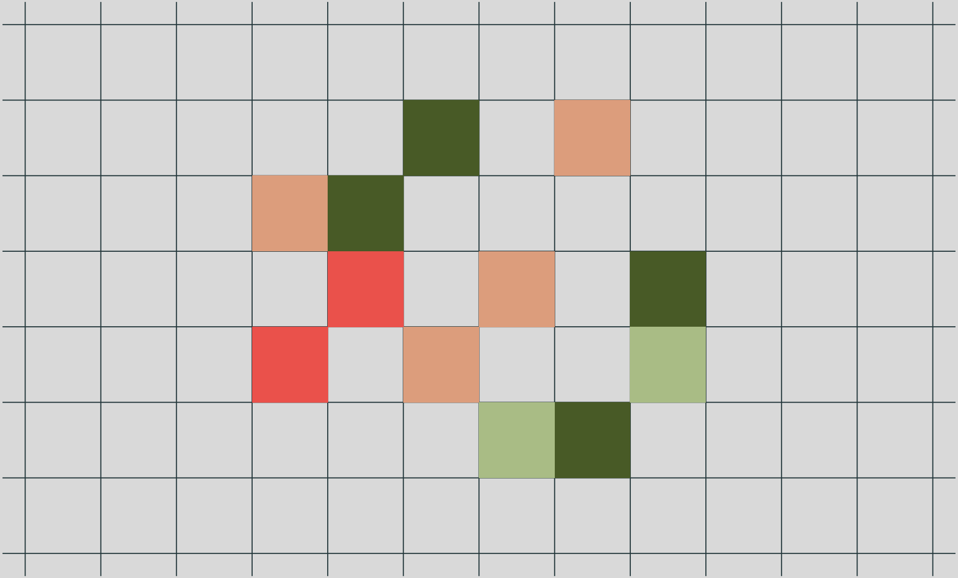


That 7-element example with 12 coloured squares is from Dietrich and Wanless 2018, after a 10-element example with 26 coloured squares in Hirsch and J (2012)







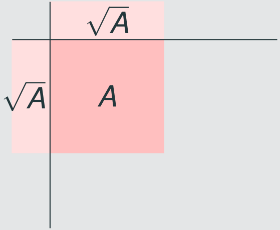


Square partial groups

Square partial group A

There is $e \in A$ and a subset $\sqrt{A} \subseteq A$ with

1. $e \cdot x = x \cdot e = x$ for $x \in \sqrt{A}$
2. $x \cdot y$ if and only if $x, y \in \sqrt{A}$ or $e \in \{x, y\}$
3. for each $x \in \sqrt{A}$ there is x' with $xx' = e = x'x$
4. $\sqrt{A} \cdot \sqrt{A} = A$



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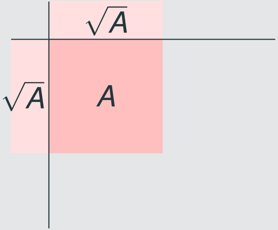

Theorem: the following are undecidable

Input: a square partial group A

- does A embed into a group?
- does A embed into a finite group?

Square partial groups

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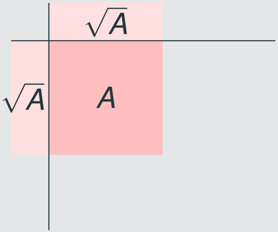

Theorem: the following are recursively inseparable

- finite square partial groups A that *do not* embed into a group
- finite square partial groups A that embed into finite groups

Green's relations

Definition: $\mathcal{L}$ (with $\mathcal{R}$ defined dually)

$a \leq_{\mathcal{L}} b$ if $\exists x \; xb = a$. Define the binary relation $\mathcal{L}$ by

$$a \mathrel{\mathcal{L}} b \iff a \leq_{\mathcal{L}} b \text{ and } b \leq_{\mathcal{L}} a$$

Definition: $\mathcal{H}$

$\mathcal{H} = \mathcal{L} \cap \mathcal{R}$

Split systems

Given a square partial group **A**

Split system $\mathcal{A}$

$$\{a_{12} \mid a \in \sqrt{A}\} \cup \{a_{23} \mid a \in \sqrt{A}\} \cup \{a_{13} \mid a \in A\} \cup \{a_{ii} \mid a = e\}$$

with multiplication $a_{ij} \cdot a_{jk} = a_{ik}$

Theorem (Sapir, 1997)

It is undecidable to determine, given a split system $\mathcal{A}$, if there is a semigroup embedding $\mathcal{A}$ in which $\{a_{ij} \mid a \in A\}$ lie within an $\mathcal{H}$ class for each $i, j \in \{1, 2, 3\}$

Split system as a semigroup

Split system $\mathcal{A}$

$$\{a_{12} \mid a \in \sqrt{A}\} \cup \{a_{23} \mid a \in \sqrt{A}\} \cup \{a_{13} \mid a \in A\} \cup \{a_{ii} \mid a = e\}$$

with multiplication $a_{ij} \cdot a_{jk} = a_{ik}$

Note that if A is an actual group, then $\sqrt{A} = A$ and we could define a_{21}, a_{32}, a_{31} as well and obtain a Brandt groupoid $B_3(A)$ with inverses: $(a_{ij})^{\smile} = a_{ji}^{-1}$

Split system as a semigroup

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As a semigroup $S(\mathcal{A})$

Add a 0 and let all undefined products be 0. Note that $a_{ij} \leq_{\mathcal{L}} e_{jj}$ and $a_{ij} \leq_{\mathcal{R}} e_{ii}$

Example

$\cdot$	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

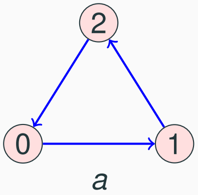
Example

$\cdot$	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

becomes

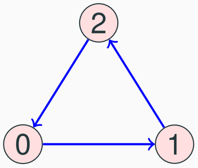
$\cdot$	e_1	a_1	b_1	e_2	a_2	b_2	e_3	a_3	b_3
e_1	0	0	0	e_3	a_3	b_3	0	0	0
a_1	0	0	0	a_3	b_3	e_3	0	0	0
b_1	0	0	0	b_3	e_3	c_3	0	0	0

As binary relations



$\cdot$	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

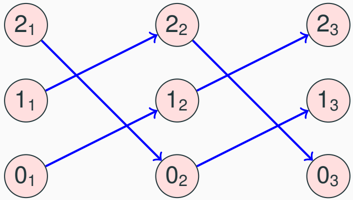
As binary relations



a

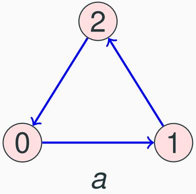
$\cdot$	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

becomes



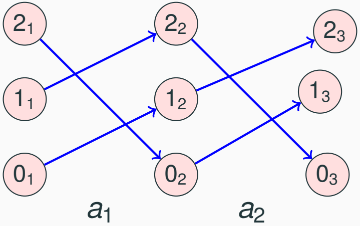
a_1 a_2

As binary relations

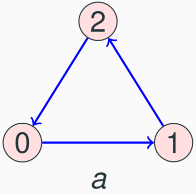


$\cdot$	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

becomes



As binary relations



$\cdot$	e	a	b
e	e	a	b
a	a	b	e
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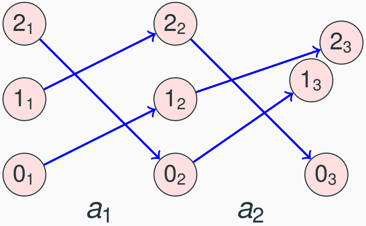
becomes



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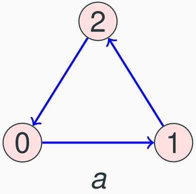
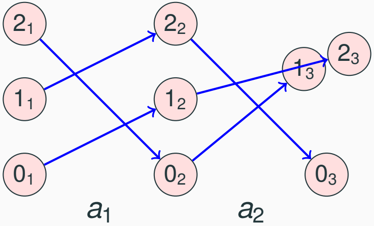
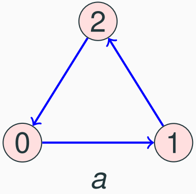


$\cdot$	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

becomes

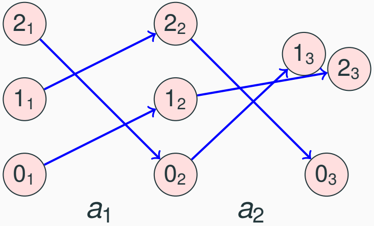


As binary relations



$\cdot$	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

becomes



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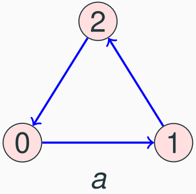

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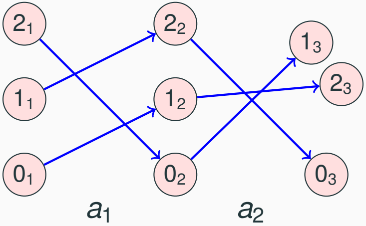

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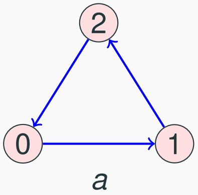

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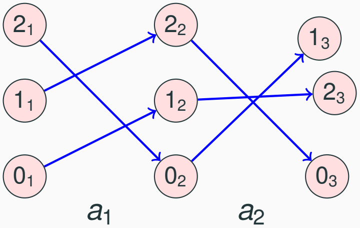

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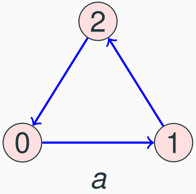

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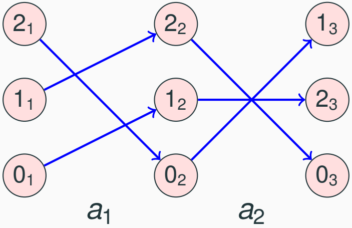

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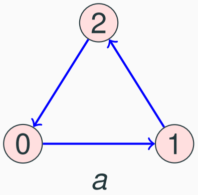

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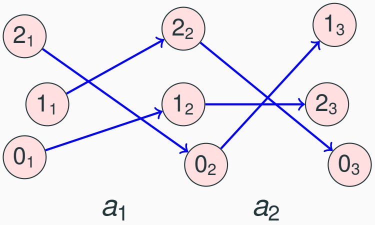

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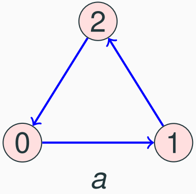

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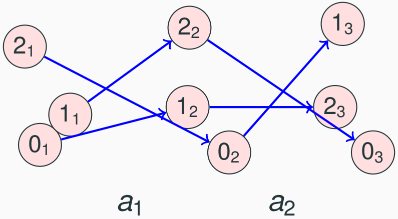

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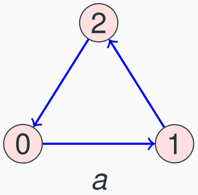

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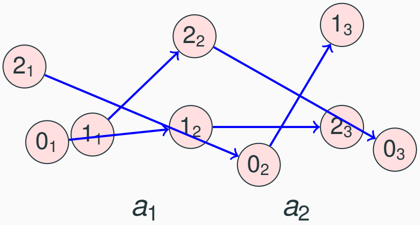

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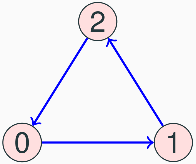

a

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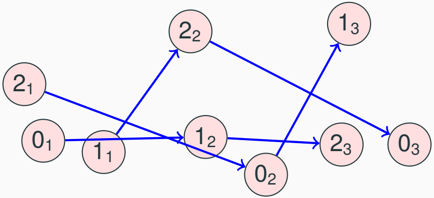

a_1 a_2

Critical observation for some elements e, a

If $e; a = e$ then $e \leq_{\mathcal{L}} a$.

If $e; 1 = a; 1$ and e, a are known to represent as injective partial functions, then $e \, \mathcal{L} \, a$

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- $((-(a\,;-1'))\,;1\,;(-(-1'\,;a)) = 1) \;\;\&\;\; 1\,;1 = 1 \;\;\&\;\; -1\,;-1 = -1 \;\;\&\;\; \ldots$ (new trick for Hirsch, J and Šemrl 2025)

Generalised kernels

Consider the relation algebraic terms $K_{L,n}(x) := (x \,;\, x^{\smile})^n$ and $K_{R,n} := (x^{\smile} \,;\, x)^n$

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Idea: if $(a_{ij})^{\smile}$ *were to equal* $(a^{-1})_{ji}$, *then* $a_{ij} \,;\, (a_{ij})^{\smile} = a_{ij} \,;\, (a^{-1})_{ji} = (aa^{-1})_{ii} = e_{ii}$

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If $S(\mathcal{A})$ is isomorphic to a system of binary relations respecting $K_{L,n}$ and $K_{R,n}$, then $\mathcal{A}$ embeds into a group (an undecidable problem)

Proof: we know that $e_{ii} \geq_{\mathcal{R}} a_{ij}$ from before.

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If and only if

An $\mathcal{H}$-embedding (from previous slide)

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Converse direction

$S(\mathcal{A})$ is isomorphic to a system of binary relations respecting $K_{L,n}$ and $K_{R,n}$, if $\mathcal{A}$ embeds into a group

Proof: this is just because if A completes to G, then $S(\mathcal{A})$ embeds in the Brandt semigroup $B_3(G)$ which is representable, even as injective partial functions

Weaker and weaker signatures

Obviously $\{K_{L,2^{n+1}}, K_{R,2^{n+1}}, ;\}$ are term functions in $\{K_{L,2^n}, K_{R,2^n}, ;\}$, so we have an infinite descending chain of weaker (?) and weaker signatures having undecidability of representability

Weaker and weaker signatures

Theorem

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So a term expressible in $\{K_{L,2^n}, K_{R,2^n}, ;\}$ for all n must involve ; only