Forcing,
Transition Algebras,
and Calculi

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In this talk

1. Short presentation of transition algebra
2. Applicability to process calculi
3. Proof system, soundness, completeness via forcing
4. Tool support and introduction to SpeX

Transition algebra (TA)

* at a glance, yet another logic used to reason about labelled transition systems

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* at a glance, yet another logic used to reason about labelled transition systems

* deserves further examination thanks to a blend of special features:

  - provides support both for the static, structural aspects of systems, via equations, and for the dynamic aspects of systems, via transitions

  - uniform treatment of states and transition labels (in particular, quantification over labels)

  - unrestricted use of equations and transitions

  - increased expressivity by employing actions similar to those found in dynamic logics

  - operational semantics (more to follow)

TA signatures

* ordinary algebraic signatures
* pairs (S, F), where:
  - S is a set of so-called sorts
  - F is an $S^* \times S$-indexed family of sets $F_{w \to s}$ of operation symbols of arity w and sort s
* as usual, we also write

$$\sigma : w \to s \in F$$

in place of $\sigma \in F_{w \to s}$

Models

* (S, F)-algebras A interpreting:
  - every sort $s \in S$ as a set A_s
  - every operation symbol $\sigma : w \to s \in F$ as a function $\sigma^A : A_w \to A_s$
  - for any sort $t \in S$, every element $e \in A_t$ as a binary S-sorted relation $(e_s \subseteq A_s \times A_s)_{s \in S}$

Sentences

* built using standard Boolean connectives and quantifiers from two kinds of atoms:

  equations $t = t'$

  transitions $t \, a \, t'$

  where t and t' are terms having the same sort and a is an action

* actions are built from terms using

  sequential composition $a \mathbin{;} b$

  choice $a \cup b$

  iteration a^*

Semantics

* $A \vDash t = t'$ when $t^A = t'^A$
* $A \vDash t\ a\ t'$ when $(t^A, t'^A) \in a^A$

and so on, where

* $(a \mathbin{⨾} b)^A = a^A \mathbin{⨾} b^A$
* $(a \cup b)^A = a^A \cup b^A$
* $(a^*)^A = (a^A)^* = \bigcup\{(a^A)^n \mid n \in \mathbb{N}\}$

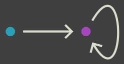
Quiz time

Which of the following models satisfies $\forall y \cdot \exists! z \cdot y \to z$?

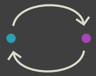
Quiz time

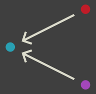
Which of the following models satisfies
$\forall y \cdot \exists! z \cdot y \rightarrow z$?

a) 

b) 

c) 

d) 

e) 

Quiz time

How do the models of $\forall y \cdot \exists! z \cdot y \to z \land \exists! x \cdot x \to y$ look like?

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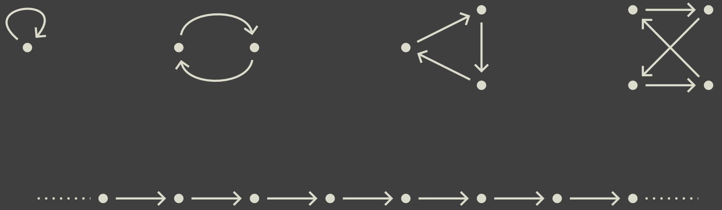

Quiz time

What if we add one of the following constraints?

* $\forall x, x' \cdot x \to^* x'$
* $\forall x, x' \cdot x \to^* x' \vee x' \to^* x$

Application to process calculi

Demo in

```
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Syntactic entailment

* get to $\Gamma \vDash \varphi$ via $\Gamma \vdash \varphi$
* where $\vdash$ is defined by proof rules of the following form:

$$\frac{\Gamma \vdash t = t'}{\Gamma \vdash t' = t} \qquad \frac{\Gamma \vdash t\ a\ t', \quad \Gamma \vdash t'\ b\ t''}{\Gamma \vdash t\ (a \mathbin{;} b)\ t''} \qquad \frac{\Gamma \vdash t\ a\ t'}{\Gamma \vdash t\ (a \cup b)\ t'}$$

$$\frac{\Gamma \vdash t\ (a \mathbin{;} b)\ t'', \quad \Gamma \cup \{t\ a\ x, x\ b\ t'\} \vdash \varphi}{\Gamma \vdash \varphi}$$

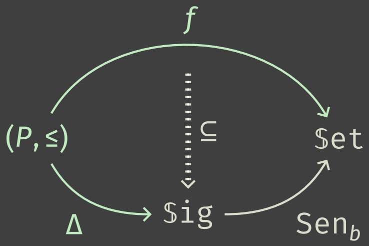
$$\frac{\Gamma \vdash \neg\neg\varphi}{\Gamma \vdash \varphi} \qquad \frac{\Gamma \cup \{\varphi\} \vdash \bot}{\Gamma \vdash \neg\varphi} \qquad \dots$$

Nota bene

* $\vdash$ is ω_1-compact whereas $\models$ is not so for uncountable signatures
* therefore, we cannot hope for a general completeness result for TA

* for at most countable signatures, neither $\vdash$ nor $\models$ is compact
* so we cannot tackle completeness using the classical Henkin method
* which leads us to forcing…

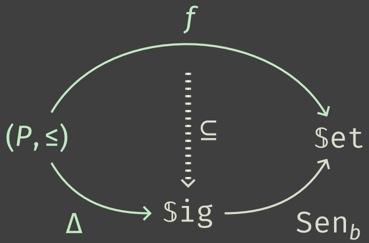
Forcing properties



where

* $(P, \leq)$ is a poset of so-called conditions
* Δ maps $p \leq q$ to $\Delta_p \subseteq \Delta_q$, and similarly for f
* $f(p) \subseteq \mathrm{Sen}_b(\Delta_p)$
* $f(p) \vDash \varphi$ implies $\varphi \in f(q)$ for some $q \geq p$

Syntactic forcing



where

* $p = (\Delta_p, \Gamma_p)$ where $\Delta_p = \Sigma \cup C_p$ and $\Gamma_p \subseteq \mathrm{Sen}(\Delta_p)$ consistent
* $p \leq q$ iff $\Delta_p \subseteq \Delta_q$ and $\Gamma_p \subseteq \Gamma_q$
* $\Delta(p) = \Delta_p$
* $f(p) = \Gamma_p \cap \mathrm{Sen}_b(\Delta_p)$

Forcing relation

* $p \Vdash \varphi$ when $\varphi \in f(p)$ for atomic sentences
* $p \Vdash t\,(a \fatsemi b)\,t'$ when $p \Vdash t\,a\,\tau$ and $p \Vdash \tau\,b\,t'$
  for some Δ_p-term τ
* $p \Vdash t\,(a \cup b)\,t'$ when $p \Vdash t\,a\,t'$ or $p \Vdash t\,b\,t'$
* $p \Vdash t\,a^*\,t'$ when $p \Vdash t\,a^n\,t'$ for some $n \in \mathbb{N}$
* $p \Vdash \neg\varphi$ when $q \nVdash \varphi$ for all $q \geq p$
* $p \Vdash \bigvee \Phi$ when $p \Vdash \varphi$ for some $\varphi \in \Phi$
* $p \Vdash \exists x \cdot \varphi$ when $p \Vdash \theta(\varphi)$ for some substitution θ

Forcing properties

* $p \Vdash \neg\neg\varphi$ iff for all $q \geq p$
  there is $r \geq q$ such that $r \Vdash \varphi$
* if $p \leq q$ and $p \Vdash \varphi$ then $q \Vdash \varphi$
* if $p \Vdash \varphi$ then $p \Vdash \neg\neg\varphi$
* we cannot have both $p \Vdash \varphi$ and $p \Vdash \neg\varphi$
* $\Gamma_p \vdash \varphi$ iff $p \Vdash \neg\neg\varphi$

Generic sets and models

Theorem

1. Every $p \in P$ belongs to a generic set $G \subseteq P$.

* G is an ideal
* for all $q \in G$ and sentences φ there is $r \in G$ such that $r \geq q$ and either $r \Vdash \varphi$ or $r \Vdash \neg\varphi$

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2. If $p \in G$ and G is generic, then $\bigcup\{\Gamma_q \mid q \in G\}$ is a maximally consistent set that includes Γ_p.

3. G admits a countable and reachable generic model A.

* $A \vDash \varphi$ iff $q \Vdash \varphi$ for some $q \in G$

Completeness

Theorem
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* suppose ad absurdum $\Gamma \nvdash \varphi$
* we get $\Gamma \nvdash \neg\neg\varphi$, hence $\Gamma \cup \{\neg\varphi\} \nvdash \bot$
* it follows that $\Gamma \cup \{\neg\varphi\}$ is consistent, so it has a model
* thus contradicting $\Gamma \vDash \varphi$

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Term rewriting as a substructure for specification-language interpreters

* solid mathematical foundation
* algebraic, close to standard notation used by working theoretical computer scientists
* great for rapid prototyping

but

* still somewhat rigid as a meta-language
* limited support for modularization

Object-based programming

* we rewrite configurations
  multisets of ↲

  1. objects   < Id : Class | Attributes >
  2. messages   message(To, From, Arguments)

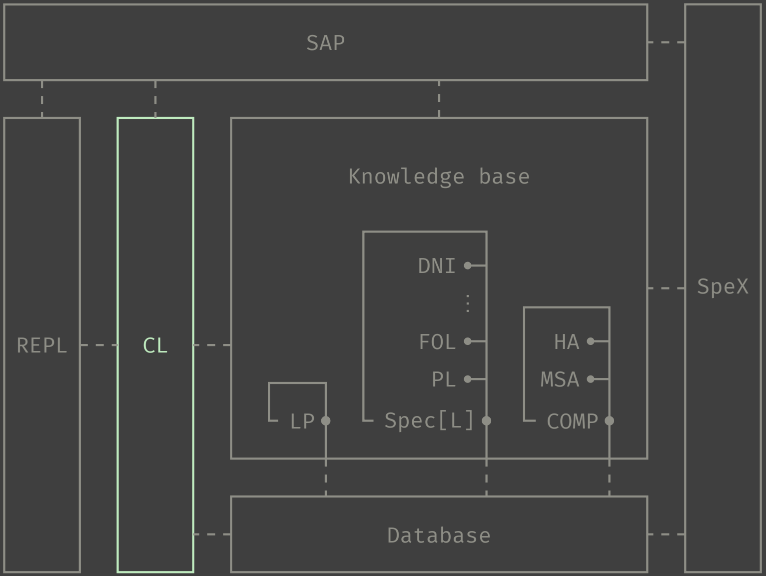
* using rules of the form

```
rl < Id : Class | ... >
   message(Id, ... )
 ⇒ ...
```

SpeX

* not a plain interpreter, but an ‘environment’
* integrates specification-language processors
* language agnostic
* offers a basic system UI ‘for free’
* based on Maude 3 (OBP with external rewrites)

System overview (overly simplified)



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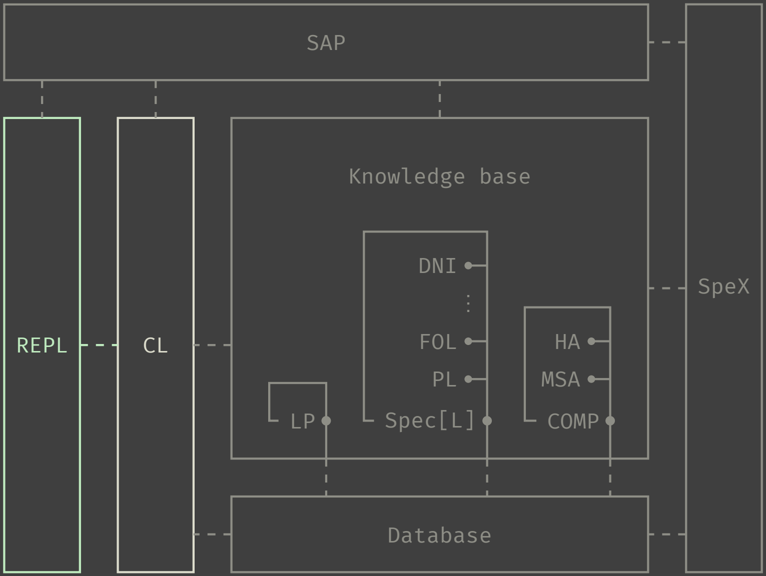

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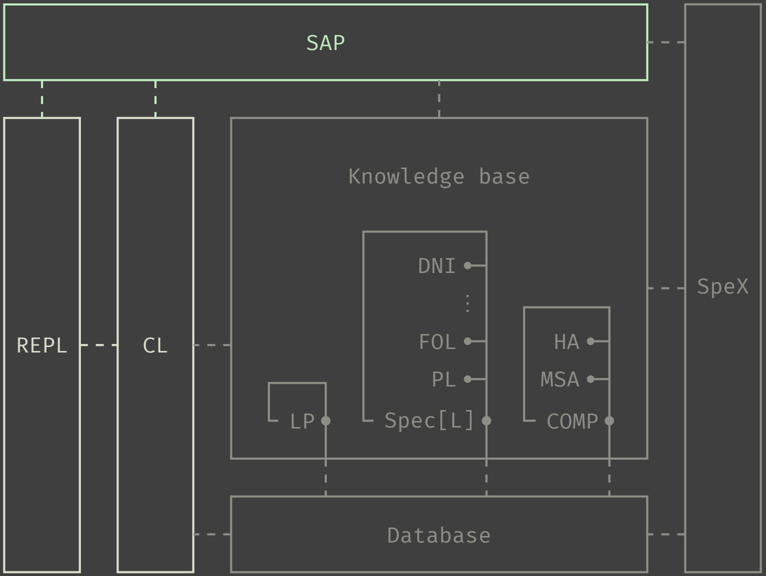

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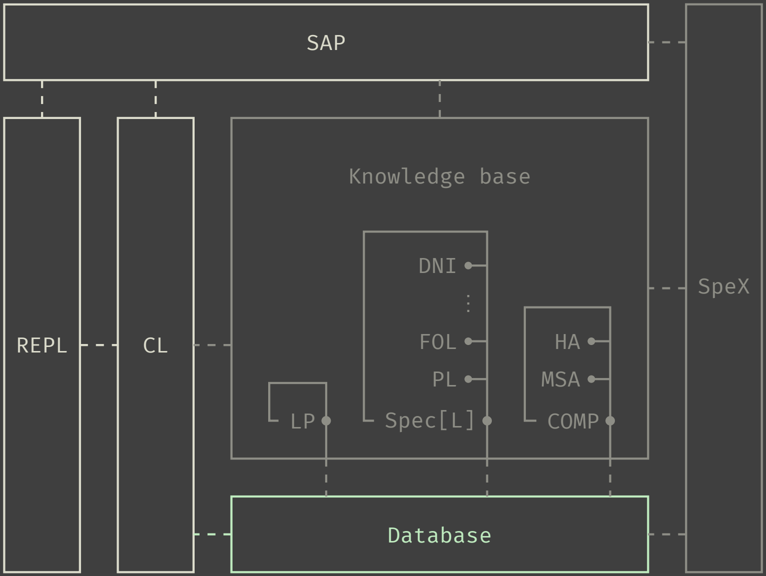

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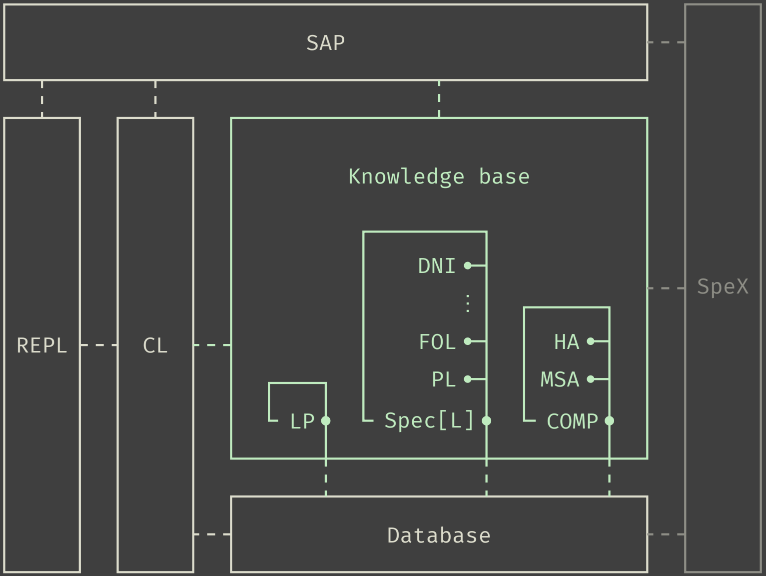

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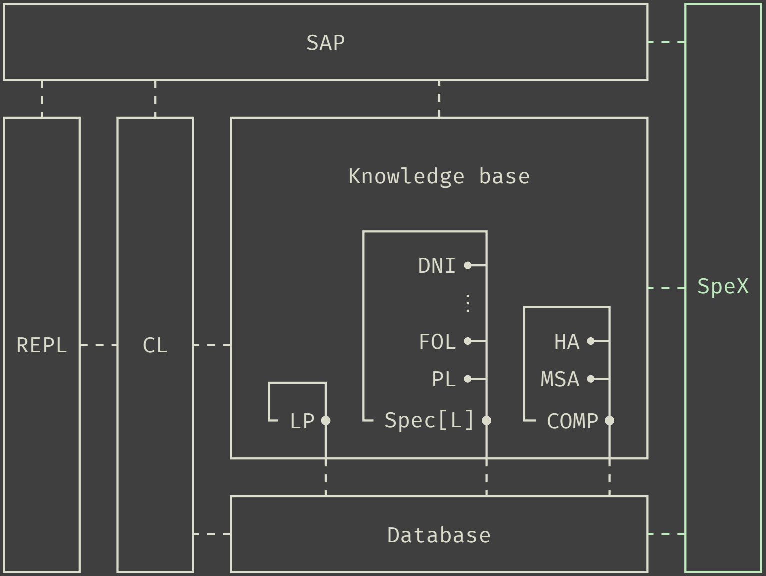


Integrating new languages into SpeX

* by means of processors
  - objects of class PROC
  - interact with SpeX (the object)

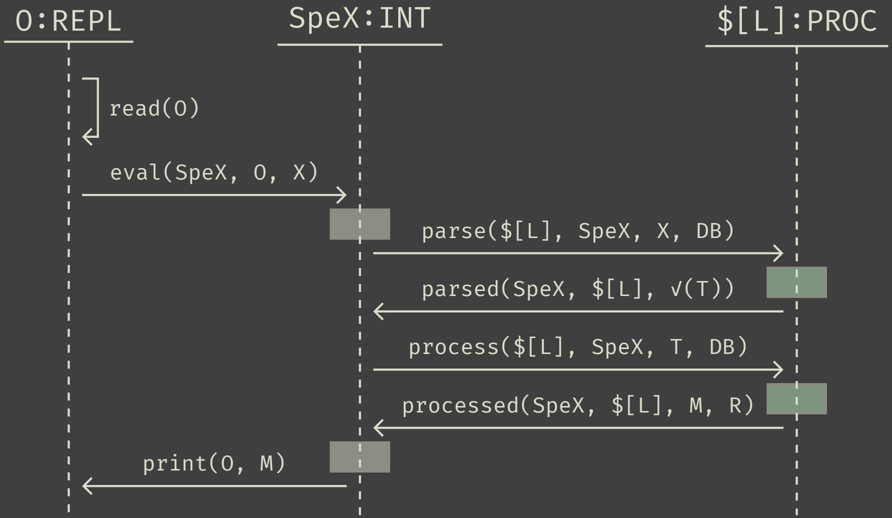
* receive messages of the form

```
parse($[L], SpeX, Input, DB)
process($[L], SpeX, AnnotatedTerm, DB)
```

* reply with messages of the form

```
parsed(SpeX, $[L], ParsingOutcome)
processed(SpeX, $[L], Text, Record)
```

A basic execution scenario



Example: Spec[DNI] (codev'd J. Fiadeiro and C. Chiriță)

```
spec Bind is
  including Base .
  mod __bind_ : Protein Organelle ⋋ Coat .
  ...
  ax store k:Nominal
     forall-local {p:Protein, o:Organelle}
     [ p o bind z:Coat ]
     (forall-local {o':Organelle}
       brane(o') = @(k) brane(o))
     and
     (forall-local {c':Coat}
       c' = z implies brane(c') = @(k) brane(o))
     [label: bind-effect] .
endspec
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Example: COMP (codev'd R. Diaconescu)

```
bobj WATCH is
  syncing (UP-TO-24-COUNTER as HOUR)
      and (UP-TO-60-COUNTER as MINUTE)
      and (UP-TO-60-COUNTER as SECOND) .
  op _:_:_ : Nat Nat Nat → State .
  act tick_ : State → State .
  act inc-min_ : State → State .
  …
endbo

open WATCH
  check tick inc-min (H:Nat : M:Nat : S:Nat)
      ~ inc-min tick (H:Nat : M:Nat : S:Nat)
  forall M:Nat < 60 = true
     and S:Nat < 60 = true .
close
```

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close
```

Example: IPDL (dev'd K. Sojakova,
M. Codescu,
and J. Gancher)

```
protocol real =
  newfamily SendInShare[bound N + 2 bound N + 2
                        dependentBound I]
            indices: m, n, i ... : bool in
  newfamily OTMsg-0[bound N + 2 bound N + 2
                    bound K ]
            indices: n, m, k ... : bool in
   ...
  parties || 1OutOf4OTReal
  where parties = ...
    and 1OutOf4OTReal = ...
```

See https://arxiv.org/abs/2507.22705

Example: Spec[TA] (codev'd D. Găină and A. Riesco)

```
spec Institute is
  including CCS .

  ops theorem, coffee, coin ...
  ops Institute, Mathematician, CoffeeVM ...

  ax Institute
     = (Mathematician | CoffeeVM) \ coin \ coffee .
  ax Mathematician
     =(tau)> (snd(coin) . rcv(coffee) .
              snd(theorem) . Mathematician) .
  ax CoffeeVM
     =(tau)> (rcv(coin) . snd(coffee) . CoffeeVM) .
endspec
```

Obtaining SpeX and TATP

* from the git repositories:

  https://gitlab.com/ittutu/spex

  https://github.com/Transition-Algebra/TATP

* then, provided Maude 3(>.2) is installed:

```
./configure
make
[sudo] make install
```

Happy hacking!