A glance at extensions of bi-intuitionistic logic

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Bi-intuitionistic logic

Bi-intuitionistic logic is intuitionistic logic with *co-implication*.

What is co-implication?

- Implication

$$\vdash \gamma \wedge \alpha \Rightarrow \beta \quad \text{iff} \quad \vdash \gamma \Rightarrow \alpha \to \beta.$$

- co-implication = dual of implication.

$$\vdash \beta \Rightarrow \alpha \vee \gamma \quad \text{iff} \quad \vdash \beta \prec \alpha \Rightarrow \gamma.$$

$$\vdash \beta \Rightarrow \alpha \vee \gamma \ \text{ iff } \ \vdash \beta \prec \alpha \Rightarrow \gamma.$$

- While the usual negation $\neg\alpha$ is an abbreviation of $\alpha \to \bot$, co-negation $\sim\alpha$ is defined to be an abbreviation of $\top \prec \alpha$.
- In classical logic Cl,

$$\vdash \beta \Rightarrow \alpha \vee \gamma \ \text{ iff } \ \vdash \beta \Rightarrow \alpha, \gamma \ \text{ iff } \ \vdash \beta \wedge \neg\alpha \Rightarrow \gamma.$$

Thus, $\beta \prec \alpha$ means $\beta \wedge \neg\alpha$, which is equal to $\neg(\beta \to \alpha)$. In particular, $\sim\alpha$ is equal to $\neg\alpha$. Thus, nothing new comes out of introducing co-implication in classical logic.

Almost 50 years ago, bi-intuitionistic logic BiInt is introduced and discussed by Cecylia Rauszer. BiInt is a conservative extension of intuitionistic logic Int, whose Kripke frames are the same as those of Int, where

$$w \models \alpha \prec \beta \text{ if } u \models \alpha \text{ and } u \not\models \beta, \text{ for some } u \leq w.$$

Can we expect some interesting outcome of introducing co-implication? In the present talk, we will discuss syntactic aspects and symmetric features of logics with co-implication.

Sequent systems for Cl and Int

From syntactic point of view, the best way of making comparisons among Cl, Int and BiInt can be carried out by using sequent formulation. Here, we consider multiple-succedents sequents, thus each sequent is of the following form for some $m, n \geq 0$, where $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n$ are formulas;

$$\alpha_1, \ldots, \alpha_m \Rightarrow \beta_1, \ldots, \beta_n$$

Now, Gentzen's sequent system **LK** for classical logic consists of initial sequents and inference rules. Initial sequents are;

$$(1)\ \alpha \Rightarrow \alpha, \quad (2)\ \bot \Rightarrow \ , \quad (3)\ \ \Rightarrow \top.$$

- Cut

$$\frac{\Gamma \Rightarrow \Delta, \alpha \quad \alpha, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi} \text{ (cut)}$$

- Structural rules: (exchange, weakening, contraction)

For instance, exchange rules;

$$\frac{\Gamma, \alpha, \beta, \Delta \Rightarrow \Pi}{\Gamma, \beta, \alpha, \Delta \Rightarrow \Pi} \qquad \frac{\Gamma \Rightarrow \Sigma, \alpha, \beta, \Pi}{\Gamma \Rightarrow \Sigma, \beta, \alpha, \Pi}$$

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Observe the "mirror" symmetry in these rules.

- Rules for disjunction and conjunction

$$\frac{\alpha, \Gamma \Rightarrow \Delta \quad \beta, \Gamma \Rightarrow \Delta}{\alpha \vee \beta, \Gamma \Rightarrow \Delta} \ (\vee \Rightarrow)$$

$$\frac{\Gamma \Rightarrow \Delta, \alpha}{\Gamma \Rightarrow \Delta, \alpha \vee \beta} \ (\Rightarrow \vee 1) \qquad \frac{\Gamma \Rightarrow \Delta, \beta}{\Gamma \Rightarrow \Delta, \alpha \vee \beta} \ (\Rightarrow \vee 2)$$

$$\frac{\alpha, \Gamma \Rightarrow \Delta}{\alpha \wedge \beta, \Gamma \Rightarrow \Delta} \ (\wedge 1 \Rightarrow) \qquad \frac{\beta, \Gamma \Rightarrow \Delta}{\alpha \wedge \beta, \Gamma \Rightarrow \Delta} \ (\wedge 2 \Rightarrow)$$

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$$\frac{\Gamma \Rightarrow \Delta, \alpha \quad \Gamma \Rightarrow \Delta, \beta}{\Gamma \Rightarrow \Delta, \alpha \wedge \beta} \ (\Rightarrow \wedge)$$

Observe the "mirror" symmetry between rules for $\vee$ and $\wedge$.

- Rules for implication and negation (of **LK**)

$$\frac{\Gamma \Rightarrow \Delta, \alpha \quad \beta, \Sigma \Rightarrow \Pi}{\alpha \to \beta, \Gamma, \Sigma \Rightarrow \Delta, \Pi}\ (\to\Rightarrow) \qquad \frac{\alpha, \Gamma \Rightarrow \Delta, \beta}{\Gamma \Rightarrow \Delta, \alpha \to \beta}\ (\Rightarrow\to)$$

$$\frac{\Gamma \Rightarrow \Delta, \alpha}{\neg\alpha, \Gamma \Rightarrow \Delta}\ (\neg \Rightarrow) \qquad \frac{\alpha, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\alpha}\ (\Rightarrow \neg)$$

Next, consider the sequent system **LJ'** for intuitionistic logic, which is obtained from **LK** by restricting Δ to be empty in both $(\Rightarrow\to)$ and $(\Rightarrow \neg)$ (by S. Maehara 1954).

Sequent system **LBJ** for bi-intuitionistic logic BiInt

Lastly, **LBJ** is obtained from **LJ'** by adding rules for co-implication and co-negation.

$$\frac{\alpha \Rightarrow \Delta, \beta}{\alpha \prec \beta \Rightarrow \Delta}\ ({\prec}{\Rightarrow}) \qquad \frac{\Gamma \Rightarrow \Delta, \alpha \quad \beta, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi, \alpha \prec \beta}\ ({\Rightarrow}{\prec})$$

$$\frac{\Rightarrow \Delta, \beta}{\sim\beta \Rightarrow \Delta}\ ({\sim}{\Rightarrow}) \qquad \frac{\beta, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \sim\beta}\ ({\Rightarrow}{\sim})$$

Now, the "mirror" symmetry is established between rules for $\to$ and $\prec$, and also between $\neg$ and $\sim$.

$$\frac{\Gamma \Rightarrow \Delta, \alpha \quad \beta, \Sigma \Rightarrow \Pi}{\alpha \to \beta, \Gamma, \Sigma \Rightarrow \Delta, \Pi}\ ({\to}{\Rightarrow}) \qquad \frac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \to \beta}\ ({\Rightarrow}{\to})$$

$$\frac{\Gamma \Rightarrow \Delta, \alpha}{\neg\alpha, \Gamma \Rightarrow \Delta}\ (\neg \Rightarrow) \qquad \frac{\alpha, \Gamma \Rightarrow}{\Gamma \Rightarrow \neg\alpha}\ (\Rightarrow \neg)$$

♣ Cut elimination

Cut elimination for a sequent system **S** says:

- if a sequent is provable in **S** then it is also provable in **S** without any application of cut rule.

1. Cut elimination holds for **LK** and **LJ**. (G. Gentzen)
2. Cut elimination holds for **LJ**′. (S. Maehara)

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3. Cut elimination fails for **LBJ**. (Pinto-Uustalu 2009)

Here is a (slightly simplified) counterexample given by Uustalu:

$$q \Rightarrow p, r \rightarrow (q \prec p)$$

Still we can show the following. We say that a sequent system **S** has *analytic cut property*, when

- every sequent which is provable in **S** has a proof P in **S** such that each cut formula is a subformula of the lower sequent in every application of cut rule. (Consequently, the proof P has subformula property.)

4. Analytic cut property holds for **LBJ**. (Kowalski-O 2017)

Some immediate consequences:

- **LBJ** is a conservative extension of **LJ'**.
- BiInt is decidable.

Note that different from Int, disjunction property doesn't hold for BiInt, since the formula $\alpha \vee \sim\alpha$ is always provable.

Moreover,

Theorem

Craig interpolation property holds for BiInt.

- This is shown first in (Kowalski-O 2017) by extending Maehara's method (1960).
- Another proof is given in (O-Sano 2022) by extending Mints' method (Mints 2001) to **LBJ**, originally applied to **LJ'** (in which "imaginary" interpolants is used in addition to "real" ones).
- What is essential in these proofs lies in the fact that any existing analytic cut in a given proof will not disturb the implementation of these methods, as subformula property still holds.

Going further on BiInt

In the calculus **LBJ**, $(\Rightarrow\prec)$ and $(\neg\Rightarrow)$ can be applied without any restriction, while $(\prec\Rightarrow)$ and $(\Rightarrow\neg)$ are not.

- If a sequent $\alpha_1, \ldots, \alpha_m \Rightarrow \beta$ is provable, then $\sim\beta \Rightarrow \sim\alpha_1, \ldots, \sim\alpha_m$ is provable.
- If a sequent $\beta \Rightarrow \alpha_1, \ldots, \alpha_m$ is provable, then $\neg\alpha_1, \ldots, \neg\alpha_m \Rightarrow \neg\beta$ is provable.

- Hence, by applying the above in a consequtive way, we can show that if a sequent $\alpha_1, \ldots, \alpha_m \Rightarrow \beta$ is provable, then $\odot\alpha_1, \ldots, \odot\alpha_m \Rightarrow \odot\beta$ is provable, where $\odot\delta$ means $\neg\sim\delta$.
- In particular, if δ is provable, so is $\odot\delta$.

Duality function

For each formula α, a formula α^d is defined inductively as follows:

- $p^d = p$ for every propositional variable p,
- $\top^d = \bot$ and $\bot^d = \top$,
- $(\alpha \wedge \beta)^d = \alpha^d \vee \beta^d$ and $(\alpha \vee \beta)^d = \alpha^d \wedge \beta^d$,
- $(\alpha \to \beta)^d = \beta^d \prec \alpha^d$ and $(\alpha \prec \beta)^d = \beta^d \to \alpha^d$,
- $(\neg\alpha)^d = {\sim}\alpha^d$ and $({\sim}\alpha)^d = \neg\alpha^d$.

The mapping d defined by $d(\alpha) = \alpha^d$ is called the *duality function*. It is naturally extended to a mapping over finite sequences of formulas. Obviously, $(\alpha^d)^d = \alpha$ holds.

(Discussed also in (Restall 1997) and (Wolter 1998).)

From our observation on "mirror" symmetry of initial sequents and rules of **LBJ**, we can easily show the following by using the induction on the length of a given proof.

Theorem

A sequent $\Gamma \Rightarrow \Delta$ is provable in **LBJ** *(without cut) iff $\Delta^d \Rightarrow \Gamma^d$ is provable in* **LBJ** *(without cut).*

- In particular, a formula α is provable in **LBJ** iff $\neg(\alpha^d)$ is provable in it.
- From our result it follows that $p, (p \to q) \prec r \Rightarrow q$ is another counterexample of cut eliminability of **LBJ**, as this is the dual to Uustalu's one; $q \Rightarrow p, r \to (q \prec p)$.

Logics over BiInt

Definition

A set L of formulas is a *logic over* BiInt (or, an extension of BiInt) iff

- it is closed under substitution,
- it is closed under the provability in **LBJ**, i.e. if $\alpha_1, \ldots, \alpha_m \Rightarrow \beta$ is provable in **LBJ** for $\alpha_1, \ldots, \alpha_m \in$ L then β is also in L,
- it is closed under $\odot$, i.e. if $\alpha \in$ L then $\odot\alpha \in$ L.

The smallest logic including a set U of formulas is denoted by BiInt[U], and also by BiInt$[\gamma_1, \ldots, \gamma_k]$ when $U = \{\gamma_1, \ldots, \gamma_k\}$.

■ **Q.1** Is BiInt[K] always conservative over any K over Int? (Wotler 1998)

When K has the finite model property, the answer is positive, as every finite Heyting algebra can be naturally extended to a bi-Heyting algebra.

■**Q.2** Which logical property of a logic K over Int is preserved by BiInt[K]?

Symmetric lattice structure of logics over BiInt

Let $\mathcal{L}$ be the set of all logics over BiInt, which in fact forms a lattice. For a logic L in $\mathcal{L}$, let S_{L} be the set of formulas $\{\neg(\alpha^d) : \alpha \in \mathsf{L}\}$. We define L^{m} to be the smallest logic in $\mathcal{L}$ which includes S_{L}. L^{m} is called the *mirror image of* L (cf. (Wolter 1998)).

Lemma

A sequent $\Gamma \Rightarrow \Delta$ *is provable in* L *iff* $\Delta^d \Rightarrow \Gamma^d$ *is provable in* L^{m}.

If a logic L is axoimatized by axiom schemes $[\varphi_1, \ldots, \varphi_k]$, then the logic L^{m} can be axoimatized by axiom schemes $[\neg(\varphi_1^d), \ldots, \neg(\varphi_k^d)]$.

Examples:

- Classical logic can be expressed as BiInt$[p \vee \neg p]$, which is equal to its mirror image BiInt$[\neg(p \wedge \sim p)]$.

- The logic BiInt$[(p \to q) \vee (q \to p)]$ is a bi-intuitionistic extension of Gödel logic, Int$[(p \to q) \vee (q \to p)]$. Meanwhile, its mirror image BiInt$[\neg((p \prec q) \wedge (q \prec p))]$ is a conservative extension of Int (Wolter 1998).

Let m be a mapping on $\mathcal{L}$ defined by $\mathrm{m}(\mathrm{L}) = \mathrm{L}^{\mathrm{m}}$.

Theorem

The mapping m *is a (complete) lattice isomorphism over* $\mathcal{L}$*, which is moreover involutive, i.e.* $\mathrm{m} \circ \mathrm{m} = \mathrm{id}$.

■**Q.3** Which logical property is preserved by the mapping m?

Theorem

The following properties are preserved by the mapping m*.*

- *decidability,*
- *Kripke completeness,*
- *finite model property,*
- *Craig's interpolation property,*
- *Maksimova's variable separation property.*

Bi-intuitionistic tense logics

What we have shown so far can be extended to bi-intuitionistic tense logics. The basic system BiSKt (Stell, Schmidt and Rydeheard 2016) is an extension of BiInt with two monotone, tense operators ⧫ and □ satisfying that

$$\vdash \blacklozenge\alpha \Rightarrow \beta \quad \text{iff} \quad \vdash \alpha \Rightarrow \Box\beta.$$

Obviously, we can see that the "mirror" symmetry holds between ⧫ and □.

Together with K. Sano, we are preparing a paper in which sufficient conditions are given in terms of Kripke frames, for logics over BiSkt (and also over BiInt) to have Craig and also Lyndon interpolation properties.

A key notion here is bisimulation products, whose idea goes back to (Maksimova 1980), (O 1986) and (Marx 1998).

■**Q.4** How many logics are there over BiInt that have Craig Interpolation property? Finitely many?