Model-theoretic forcing in transition algebra

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- **Expressive power of TA (Categoricity)**

$$\mathfrak{B} \models T_{\mathfrak{A}} \iff \mathfrak{B} \simeq \mathfrak{A}$$

- **Forcing method**

- **Omitting Types Theorem (OTT) and its applications**

Cardinality assumptions

In general, results on infinitary logics (α-compactness, etc.) depend on cardinals. So to simplify the situation, we add some assumptions about cardinals.

Assumption 1

Note that there are two kinds of infinite cardinals:"successor cardinal α^+", and "limit cardinal".

1. *General Continuum Hypothesis. (GCH)*

$$\alpha = 2^\beta \qquad \textit{for all infinite successor cardinal } \alpha = \beta^+$$

2. *Non-existence of inaccessible cardinals. ($\neg$IC)*

$$\neg IC \Leftrightarrow cf(\alpha) < \alpha \textit{ for all uncountable limit cardinals } \alpha$$

where $cf(\alpha) = \min\{\beta \in On \mid \exists_{\gamma:\beta\to\alpha}\, \alpha = \cup_{i\in\beta}\gamma_i\}$.

These are useful for induction on infinite cardinals. This is because they allow us to rewrite any uncountable cardinal using smaller cardinals.

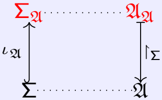
Categoricity

- A set of sentences is called a categorical theory if it has a unique model up to isomorphism.
- Let $\mathrm{Th}(\mathfrak{A}) := \{\phi \in \mathtt{Sen}(\Sigma) \mid \mathfrak{A} \models \phi\}$ for a Σ-model $\mathfrak{A}$.
  If $\mathrm{Th}(\mathfrak{A})$ is categorical, it means that $\mathfrak{A}$ can be represented by sentences.
- In general this is not true, but the following theorem says that it is possible if we allow signature expansions.

Theorem 1 (Categoricity)

Let $\mathfrak{A}$ be a Σ-model. There exist:

1. $\iota_{\mathfrak{A}} : \Sigma \hookrightarrow \Sigma_{\mathfrak{A}}$ – *sort-preserving inclusion,*
2. $\mathfrak{A}_{\mathfrak{A}}$ – *reachable*[1] $\iota_{\mathfrak{A}}$*-expansion of* $\mathfrak{A}$,



such that $\mathrm{Th}(\mathfrak{A}_{\mathfrak{A}})$ *is categorical (i.e. any* $\Sigma_{\mathfrak{A}}$*-model of* $\mathrm{Th}(\mathfrak{A}_{\mathfrak{A}})$ *is isomorphic to* $\mathfrak{A}_{\mathfrak{A}}$*).*

[1] That is, all elements of $\mathfrak{A}_{\mathfrak{A}}$ can be expressed using the symbols in $\Sigma_{\mathfrak{A}}$.

Proof.(sketch) For simplicity, let α be cardinal, $\Sigma = (\{s\}, \alpha, <)$, and $\mathfrak{A} = (\alpha, <_\alpha)$.

- Intuitively, we can do this by adding all the elements, functions, and relations on $\mathfrak{A}$ as symbols to Σ. [2]
- Generally, when two reachable models satisfy the same atomic sentences, they are isomorphic. Since $\mathrm{Th}(\mathfrak{A}_\mathfrak{A})$ includes atomic sentences, It is sufficient that:

$$\mathfrak{B} \models \mathrm{Th}(\mathfrak{A}_\mathfrak{A}) \text{ implies "}\mathfrak{B} \text{ is reachable" for all model } \mathfrak{B}.$$

$[\alpha < \omega]$ Let $\mathfrak{A}_\mathfrak{A} := \mathfrak{A}$. $\mathfrak{B} \models \mathrm{Th}(\mathfrak{A}_\mathfrak{A}) \ni \neg\exists x \cdot \wedge \{x \neq a \mid a \in \mathfrak{A}\}$ implies "$\mathfrak{B}$ is reachable".

$[\alpha = \omega]$ Here we use "*". We use the successor relation $suc^{\mathfrak{A}_\mathfrak{A}} := \{(i, i+1) \mid i \in \omega\}$.

$\mathfrak{B} \models \mathrm{Th}(\mathfrak{A}_\mathfrak{A}) \ni \forall x \cdot\ 0 =(suc^*)\Rightarrow x$ implies "$\mathfrak{B}$ is reachable".

$[\alpha > \omega$ is successor cardinal$]$ From (GCH), we can write $\alpha = 2^\beta$.

By the induction hypothesis, β can already be expressed. Therefore, we can limit the number of elements by using the function $\cdot(\cdot) : \alpha \times \beta \simeq 2^\beta \times \beta \ni (f, x) \mapsto f(x) \in 2$.

$\mathfrak{B} \models \mathrm{Th}(\mathfrak{A}_\mathfrak{A}) \ni \forall f, g < \alpha\, (\forall x < \beta\, f(x) = g(x)) \to f = g$ implies "$\mathfrak{B}$ is reachable".

$[\alpha > \omega$ is limit cardinal$]$ From $(\neg\mathrm{IC})$ we can wright $\alpha = \cup_{i \in \beta} \gamma_i\, (\beta < \alpha,\ \gamma : \beta \to \alpha)$.

Since α is a limit of smaller cardinals, which can be expressed by the induction hypothesis, we can do this by using the symbol corresponding to γ.

$\mathfrak{B} \models \mathrm{Th}(\mathfrak{A}_\mathfrak{A}) \ni \forall n < \alpha\, \exists i < \beta \cdot n \leq \gamma_i$ implies "$\mathfrak{B}$ is reachable". □

[2] In reality, we don't need that many symbols; it is enough to add as many symbols as the cardinality of $\mathfrak{A}$.

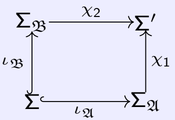
Applications of categoricity

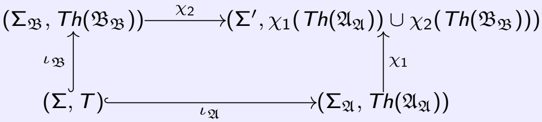
Corollary 2

In TA, *(α-)Compactness, and (α-)Upward Löwenheim-Skolem Property fails.*

Corollary 3 (Robinson Consistency)

Let (Σ, T) be a maximally consistent theory with at least two non-isomorphic[3] models $\mathfrak{A}$ and $\mathfrak{B}$. Construct a pushout as illustrated on the left side of the diagram below.





Then $\chi_1(Th(\mathfrak{A}_{\mathfrak{A}})) \cup \chi_2(Th(\mathfrak{B}_{\mathfrak{B}}))$ is not consistent.

- Since $\mathfrak{A} \models T$ and $\mathfrak{A}_{\mathfrak{A}} \restriction_{\Sigma} = \mathfrak{A}$, we have $T \subseteq Th(\mathfrak{A}_{\mathfrak{A}})$.
- Since $\mathfrak{B} \models T$ and $\mathfrak{B}_{\mathfrak{B}} \restriction_{\Sigma} = \mathfrak{B}$, we have $T \subseteq Th(\mathfrak{B}_{\mathfrak{B}})$.
- However, $\chi_1(Th(\mathfrak{A}_{\mathfrak{A}})) \cup \chi_2(Th(\mathfrak{B}_{\mathfrak{B}}))$ is not consistent, because any model of $\chi_1(Th(\mathfrak{A}_{\mathfrak{A}})) \cup \chi_2(Th(\mathfrak{B}_{\mathfrak{B}}))$, by Theorem 1, would imply that $\mathfrak{A}$ is isomorphic to $\mathfrak{B}$.

[1]For example, if Σ has only one label $\leq$ and "$\leq$ is order" $\in T$, then such models exist because the effect of "$*$" disappears.

In the case of ZFC

Categoricity and its results use GCH and $\neg IC$ in addition to ZFC. However, the story also has applications when using only ZFC.

Fact 4

$ZFC + GCH + \neg IC \nvdash \bot \iff ZFC \nvdash \bot$

Proposition 5

Let ZFC be consistent ($ZFC \nvdash \bot$). $ZFC + GCH + \neg IC \vdash \varphi \implies ZFC \nvdash \neg\varphi$

Proof. Suppose $ZFC \vdash \neg\varphi$, towards a contradiction.

- Since $ZFC \subseteq ZFC + GCH + \neg IC$, $ZFC + GCH + \neg IC \vdash \neg\varphi$.
- From the assumption, $ZFC + GCH + \neg IC \vdash \bot$.
- By the fact $ZFC \vdash \bot$.

But this contradicts $ZFC \nvdash \bot$. □

Therefore, the results shown so far cannot be disproven from ZFC as long as ZFC is consistent (i.e., as long as modern mathematics is not broken).

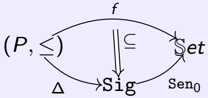
From here on, GCH and $\neg IC$ will not be used.

- **Expressive power of TA (Categoricity)**
  - $T \Leftarrow$representation$= \mathfrak{A}$
  - Negative results.
- **Forcing method**
  - $T =$implementation$\Rightarrow \mathfrak{A}$
  - Positive results.
- **Omitting Types Theorem (OTT) and its applications**

Forcing property

Definition 6

A forcing property is a tuple $\mathbb{P} = (P, \leq, \Delta, f)$, where:



1. $(P, \leq)$ is a partially ordered set with a least element 0.
2. $\Delta : (P, \leq) \to \mathtt{Sig}$ is a functor, which maps each $(p \leq q) \in (P, \leq)$ to $\Delta(p) \subseteq \Delta(q)$.
3. $f : (P, \leq) \to \mathbb{S}et$ is a functor such that $\subseteq : f \implies \Delta; \mathtt{Sen}_0$ is a natural transformation.
4. If $f(p) \models \varphi$ then $\varphi \in f(q)$ for some $q \geq p$, for all atomic sentences $\varphi \in \mathtt{Sen}_0(\Delta(p))$.

Forcing relation

Definition 7 (Forcing relation)

The forcing relation $\Vdash$ between conditions and sentences is defined by:

- $p \Vdash \varphi$ if $\varphi \in f(p)$, for all atomic sentences $\varphi \in \mathtt{Sen}_0(\Delta(p))$.
- $p \Vdash (\mathfrak{a}_1 \mathbin{;} \mathfrak{a}_2)(t_1, t_2)$ if $p \Vdash \mathfrak{a}_1(t_1, t)$ and $p \Vdash \mathfrak{a}_2(t, t_2)$ for some term $t \in T_{\Delta(p)}$.
- $p \Vdash (\mathfrak{a}_1 \cup \mathfrak{a}_2)(t_1, t_2)$ if $p \Vdash \mathfrak{a}_1(t_1, t_2)$ or $p \Vdash \mathfrak{a}_2(t_1, t_2)$.
- $p \Vdash \mathfrak{a}^*(t_1, t_2)$ if $p \Vdash \mathfrak{a}^n(t_1, t_2)$ for some natural number $n < \omega$.
- $p \Vdash \neg\phi$ if there is no $q \geq p$ such that $q \Vdash \phi$.
- $p \Vdash \vee\Phi$ if $p \Vdash \phi$ for some $\phi \in \Phi$.
- $p \Vdash \exists x \cdot \phi$ if $p \Vdash \phi[x \leftarrow t]$ for some ground term t.

Generic set and generic model

Definition 8 (Generic set)

A subset of conditions $G \subseteq P$ is generic if

1. G is an ideal, i.e.,
   - $G \neq \emptyset$,
   - for all $p \in G$ and all $q \leq p$ we have $q \in G$, and
   - for all $p, q \in G$ there exists $r \in G$ such that $p \leq r$ and $q \leq r$ (directedness);
2. G determines the truth of sentences, i.e.,
   - for all conditions $p \in G$ and all sentences $\phi \in \mathtt{Sen}(\Delta(p))$ there exists a condition $q \in G$ such that $q \geq p$ and either $q \Vdash \phi$ or $q \Vdash \neg\phi$ holds.

Theorem 9 (Generic model theorem)

Let Δ_G be signature of G, i.e., the "union" of all signatures of the conditions in a generic set G There is a reachable Δ_G-model $\mathfrak{A}_G$ which is generic for G i.e.,

$$\mathfrak{A}_G \models \phi \iff r \Vdash \phi \text{ for some } r \in G$$

for all Δ_G-sentences ϕ.

Semantic forcing

Example 10

- Depending on the logical property being studied, a condition may take on different forms.
- For OTT and DLS, a condition is a triple $(\Sigma[C], \Phi, \mathcal{M})$: [a]
  - Φ - set of $\Sigma[C]$-sentences
  - $\mathcal{M}$ - class of $\Sigma[C]$-models

[a] Strictly speaking, this is not a tuple, since $\mathcal{M}$ can be a proper class.

- Syntactic forcing uses $p = (\Sigma_p, \Phi_p)$.

$$p \Vdash \neg\neg\phi \quad \Longleftrightarrow \quad \Phi_p \vdash_{\Sigma_p} \phi$$

- Semantic forcing uses $p = (\Sigma_p, \Phi_p, \mathcal{M}_p)$.

$$p \Vdash \neg\neg\phi \quad \Longleftrightarrow \quad \Phi_p \models_{\Sigma_p, \mathcal{M}_p} \phi$$

where $\Phi \models_{\Sigma,\mathcal{M}} \phi :\Longleftrightarrow \mathfrak{A} \models \Phi$ implies $\mathfrak{A} \models \phi$ for all $\mathfrak{A} \in \mathcal{M}$.
Note: $\Phi \models_{\Sigma} \phi :\Longleftrightarrow \mathfrak{A} \models \Phi$ implies $\mathfrak{A} \models \phi$ for all $\mathfrak{A}$.

- **Expressive power of TA (Categoricity)**
- **Forcing method**
- **Omitting Types Theorem (OTT) and its applications**

$$(\mathfrak{A} \models \Phi) + \begin{cases} \mathfrak{A} \models T(x) \text{ for some } x \in |\mathfrak{A}| & (\textit{realize}) \\ \mathfrak{A} \not\models T(x) \text{ for each } x \in |\mathfrak{A}| & (\textit{omit}) \quad \leftarrow \text{OTT} \end{cases}$$

Examples

Example 11 (1-type)

- $\Sigma = (\{Nat\}, \{0 :\to Nat, s : Nat \to Nat, \_+\_ : Nat\ Nat \to Nat\})$
- $\Phi = \{\forall x \cdot x + 0 = 0, \forall x, y \cdot x + s(y) = s(x + y)\}$
- $T(x) = \{x \neq 0, x \neq s(0), x \neq s^2(0), \dots\}$
- If $\mathfrak{A}$ omits [a] $T \iff \mathfrak{A}$ is reachable by $0 :\to Nat$ and $s : Nat \to Nat$
- Lemma: Φ locally omits [b] T

[a] $\mathfrak{A}$ omits $T :\iff$ there is no element (solution) $x \in |\mathfrak{A}|$ that satisfies $T(x)$
[b] Φ locally omits $T :\iff \Phi \cup \Gamma \not\models_{\Sigma[x]} T$ for all $\Gamma \in \mathcal{P}(\mathtt{Sen}(\Sigma[x]))$, such that $\mathtt{card}(\Gamma) < \mathtt{card}(\mathtt{Sen}(\Sigma[x]))$.

Example 12 (ω-type)

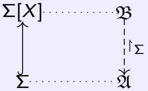
- $\Sigma = (\{s\}, \{r :\to s \mid r \in \mathbb{R}\})$
- $X = \{x_i \mid i \in \mathbb{N}\}$ a set of variables for Σ (Note that $\mathbb{R}^{\mathbb{N}} \approx 2^{\mathbb{N}\times\mathbb{N}} \approx 2^{\mathbb{N}} \approx \mathbb{R}$)
- $T(X) = \cup_{i\in\mathbb{N}} T(x_i) \cup \{x_i \neq x_j$ [a] $\mid i \neq j\ i, j \in \mathbb{N}\}$ where $T(x) = \{x \neq r \mid r :\to s \in F\}$:
- $\mathfrak{A}$ omits $T(X) \iff$ There are at most finite solutions to $T(x)$ (though they may exist).
- Lemma: Any countable and satisfiable set of sentences Φ locally omits $T(X)$.

[a] If we replace "$\neq$" with another relation, we can allow finite chains of solutions while eliminating infinite chains.

Omitting Types Theorem

Definition 13 (Type)

- $T \subseteq \mathtt{Sen}(\Sigma[X])$ is a type, where $\alpha = \mathtt{card}(\mathtt{Sen}(\Sigma))$ and $\alpha^{\mathtt{card}(X)} \leq \alpha$.
- A Σ-model $\mathfrak{A}$ realizes T if $\mathfrak{B} \models T$ for some expansion $\mathfrak{B}$ of $\mathfrak{A}$ to $\Sigma[X]$.
- $\mathfrak{A}$ *omits* T if $\mathfrak{A}$ does not realize T.



Definition 14 (Isolated type)

- Let Σ be a signature of power $\mathtt{card}(\mathtt{Sen}(\Sigma)) = \alpha$.
- $\Phi \subseteq \mathtt{Sen}(\Sigma)$ isolates $T \subseteq \mathtt{Sen}(\Sigma[X])$ iff $\Phi \cup \Gamma \models_{\Sigma[X]} T$ for some $\Gamma \in \mathcal{P}_\alpha(\mathtt{Sen}(\Sigma[X]))$.
- Φ *locally omits* T if Φ does not isolate T.

Theorem 15 (Omitting Types Theorem)

- Σ *- signature of power* α.
- $\mathcal{L}$ *is a (syntactic) fragment of* TA.
- *If* $\alpha > \omega$ *then we assume that fragment* $\mathcal{L}$ *is compact.*

If Φ has a model, and Φ locally omits $T_i \subseteq \mathtt{Sen}(\Sigma[X_i])$ for all $i < \alpha$, then Φ has a model which omits T_i for all $i < \alpha$.

Proof

- The proof of OTT is based on forcing.
- The key is to construct a generic set G that forces all negations of formulas in the type T.

$\Sigma \setminus T$	$\phi_0(x)$	$\phi_1(x)$	$\phi_2(x)$	$\phi_3(x)$	$\phi_4(x)$	$\phi_5(x)$	$\cdots$	add counterexamples
c_0	true	true	true	true	not yet	true	$\cdots$	$\Phi_0 \leftarrow \neg\phi_4(c_0)$
c_1	true	not yet	false	true	false	true	$\cdots$	$\Phi_1 \leftarrow \neg\phi_1(c_1)$
c_2	false	false	false	true	true	true	$\cdots$	$\Phi_2 \vDash \neg\phi_0(c_2)$
c_3	true	true	true	not yet	false	true	$\cdots$	$\Phi_3 \leftarrow \neg\phi_3(c_3)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$

- Then Generic Model Theorem delivers a model $\mathfrak{A}$ such that

$$\mathfrak{A} \models \phi \iff G \Vdash \phi, \text{ for all sentences } \phi.$$

Hence, $\mathfrak{A}$ omits T.

Below is an example of a type that cannot be omitted if the Henkin constants are added all at once. Note that the Kleene star "$*$" is not used in this example. In the case of the proof of OTT, the classical method would run into problems if there were just an infinite number of sorts.

Example 16 (0-type)

- $\Sigma = (\{s_n \mid n \in \mathbb{N}\}, F, P)$
- $T = \{\exists x_0 : s_0, \ldots, x_n : s_0 \cdot \wedge_{i \neq j} x_i \neq x_j \mid n \in \mathbb{N}\}$: there infinitely many elements of sort s_0
- $\mathfrak{A}$ omits $T \iff \mathfrak{A}_{s_0}$ is finite
- $\phi_n = \exists z_n : s_n \cdot \top \implies \exists x_1 : s_0, \ldots, x_n : s_0 \cdot \wedge_{i \neq j} x_i \neq x_j$ for all $n > 0$: if there exists an element of sort s_n then there exist at least n elements of sort s_0
- Lemma: $\Phi = \{\phi_n \mid n > 0\}$ locally omits T.

Applications of OTT

- OTT + compactness $\Longrightarrow$ Löwenheim-Skolem Properties
- OTT + compactness $\Longrightarrow$ Joint Robinson Consistency $\xLeftrightarrow{\text{negation}}$ Interpolation
- OTT for uncountable languages of regular cardinality $\Longrightarrow$ inf-compactness [4]
- Completeness of $\mathcal{L}$ $\xRightarrow{OTT}$ completeness of the fragment obtained from $\mathcal{L}$ by restricting the semantics to models which omit a certain type.
  - type = "infinity" $\Longrightarrow$ completeness of finite model theory
  - type is based on constructors $\Longrightarrow$ completeness of constructor-based logics

[4] Each set of sentences Φ has an infinite model whenever each finite subset $\Phi_f \subseteq \Phi$ has an infinite model.

Example 17 (Natural numbers)

(Σ, Φ):

- $\Sigma = (\{Nat\}, \{0 :\to Nat, s : Nat \to Nat, \_+\_ : Nat\ Nat \to Nat\})$
- $F^c = \{0 :\to Nat, s : Nat \to Nat\}$ – constructors
- Nat is a sort interpreted as finite
- $\Phi = \{\forall x \cdot x + 0 = 0, \forall x, y \cdot x + s(y) = s(x + y)\}$

$$(CB)\frac{\Phi \vdash \psi[x \leftarrow s^n(0)] \text{ for all } n \in \mathbb{N}}{\Phi \vdash \forall x \cdot \psi} \qquad (FN)\frac{\Phi \cup \{\forall x_0, \ldots, x_n \cdot \vee_{i \neq j} x_i = x_j\} \vdash \psi \text{ for all } n \in \mathbb{N}}{\Phi \vdash \psi}$$

Completeness of TA

$\Phi \not\vdash \bot \implies$ there is a model $\mathfrak{A}$ s.t. $\mathfrak{A} \models \Phi$

Completeness of TA + CB + FN

$\Phi \not\vdash_{CB,FN} \bot \implies$ there is a model $\mathfrak{A}$ s.t. $\mathfrak{A} \models \Phi$ and omits infiniteness and inreachability.

Example 18 (Lists)

(Σ, Φ):

1. $\Sigma = (\{Elt, List\}, \{empty :\to List, \_;\_ : List\ Elt \to List, add : List\ List \to List\})$
2. $F^c = \{empty :\to List, \_;\_ : List\ Elt \to List\}$ – constructors
3. $\Phi = \{\forall x \cdot add(x, empty) = x, \forall x, y, e \cdot add(x, y; e) = add(x, y); e\}$

$$(CB)\frac{\Phi \vdash \forall e_1, \ldots, e_n \cdot \psi[x \leftarrow e_1; \ldots; e_n; empty] \text{ for all } n \in \mathbb{N}}{\Phi \vdash \forall x \cdot \psi}$$

Conclusion and current work

- **Expressive power of TA (Categoricity)**
  - TA is strong enough to uniquely determine models, by using sentences.
  - Categoricity can be used to show negative results.
- **Forcing method**
  - Forcing makes it possible to construct models even for infinitary logic.
  - Forcing can be used to show positive results.
- **Omitting Types Theorem (OTT) and its applications**
  - Omitting Types is a way to avoid adding unnecessary elements to models.
  - OTT has applications such as proving the completeness of constructor-based proofs.

Monoidal categories + TA = MTA (Monoidal Transition Algebra)

1. Monoidal categories can be used to discuss quantum systems.

   Quantum system + Transition = Dynamic quantum system [5]

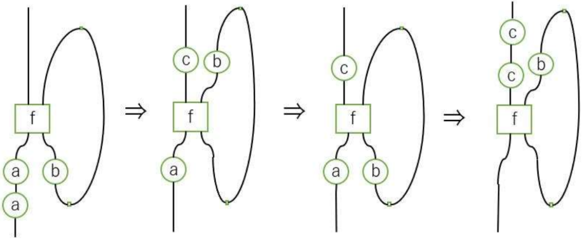
2. Since second-order logical expressions are possible, differential equations can be handled grammatically.

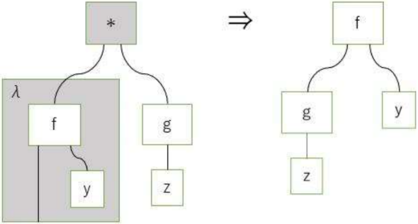
   Differential equation + Transition = Hybrid system

Since monoidal categories generate new objects (sorts) through tensor products, the forcing approach that adapts to the infinite number of sorts is effective.

$$\{s,\ s\otimes s,\ s\otimes s\otimes s,\ s\otimes s\otimes s\otimes s,\ s\otimes s\otimes s\otimes s\otimes s,\ \ldots\}$$

[5] e.g., a system that changes its next action depending on the observation result





$$(\lambda x.f(x,y))g(z) =(\beta)\Rightarrow f(g(z),y)$$