The Many Faces of Modal Logic
Day 4: Structural Proof Theory

Dirk Pattinson

Australian National University, Canberra

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Automated Reasoning, Or: Decidability and Complexity

Yesterday. Coalgebraic logics have the small model property.

However. Decidability is *not* automatic by FMP

Reason. If T maps finite sets to infinite sets, the set of transition structures

$$\{\gamma : C \to TC \mid \gamma \text{ a function}\}$$

may well be infinite even for finite C.

Example. Game frames, probabilistic frames, multigraph frames

Plus: Recall we know *nothing* about T and it may well encode the halting problem.

Wishful Thinking

Two Options for $\chi = \bigwedge_i \heartsuit_i \phi_i \to \bigvee_j \heartsuit_j \psi_k$:

Semantically $\neg\chi$ satisfiable $\implies$ $\forall \phi/\psi \in \mathscr{R}, \sigma : V \to \mathscr{F}(\Lambda)$. $\psi\sigma \vdash_{\mathrm{PL}} \chi \implies \neg\phi\sigma$ satisfiable $\dfrac{\text{Countermodels of the } \phi\sigma}{\text{Countermodel of } \chi}$	**Syntactically.** χ provable $\implies$ $\exists \phi/\psi \in \mathscr{R}, \sigma : V \to \mathscr{F}(\Lambda)$ $\phi\sigma$ provable and $\psi\sigma \vdash_{\mathrm{PL}} \chi$ $\dfrac{\text{Proof of some } \phi\sigma}{\text{Proof of } \chi}$

Problems.

1. checking single rules is insufficient, and
2. there may be infinitely many rules to check!

Example

Consequences of *multiple rules*

$$\dfrac{\dfrac{a \wedge (b \wedge c) \to a \wedge b \wedge c}{\Box a \wedge \Box(\mathbf{b} \wedge \mathbf{c}) \to \Box(a \wedge b \wedge c)} \qquad \dfrac{b \wedge c \to b \wedge c}{\Box b \wedge \Box c \to \Box(\mathbf{b} \wedge \mathbf{c})}}{\Box a \wedge \Box b \wedge \Box c \to \Box(a \wedge b \wedge c)}$$

Infinitely many *possible premises*

$$\dfrac{\vdash a \to b?}{\Box a \to \Box b \vdash \Box a \to \Box b} \qquad \dfrac{\vdash a \wedge a \to b?}{\Box a \wedge \Box a \to \Box b \vdash_{\mathrm{PL}} \Box a \to \Box b}$$

$$\dfrac{\vdash a \wedge a \wedge a \to b?}{\Box a \wedge \Box a \wedge \Box a \to \Box b \vdash_{\mathrm{PL}} \Box a \to \Box b} \qquad \text{etc.}$$

$\to$ Need admissibility of *cut* and *contraction* in a Sequent Calculus.

Syntactic and Semantic Approach

Problem 1. Eliminate the need for checking multiple rules.

Semantically	**Syntactically.**
$\chi = \bigwedge \heartsuit_i \phi_i \to \bigvee_j \phi_j$ valid $\Longrightarrow$ $\exists \phi / \psi \in \mathscr{R}$ and subst'n σ: ▸ PL $\vdash \psi\sigma \to \chi$ ▸ $\phi\sigma$ valid	$\chi = \bigwedge \heartsuit_i \phi_i \to \bigvee_j \phi_j$ provable $\Longrightarrow$ $\exists \phi / \psi \in \mathscr{R}$, subst'n σ ▸ PL $\vdash \psi\sigma \to \chi$ ▸ $\phi\sigma$ provable

Semantic Approach. Stronger coherence condition: valid clause consequence of *single* rule conclusion

Syntactically. Propositional combinations of rule conclusions are conclusions of *single* rule.

Formal Framework: Sequent Calculus

Sequents are *multisets* of formulas. Write $\Gamma, \Delta := \Gamma \cup \Delta$; $\quad \Gamma, A := \Gamma, \{A\}$

Propositional Rules

$$\frac{\Gamma, A}{\Gamma, \neg\neg A} \qquad \frac{\Gamma, A \quad \Gamma, B}{\Gamma, A \wedge B} \qquad \frac{\Gamma, \neg A, \neg B}{\Gamma, \neg(A \wedge B)}$$

Modal Rules from a one-step rule ϕ / ψ

$$\frac{\mathrm{Lit}(\phi_1)\sigma \quad \dots \quad \mathrm{Lit}(\phi_n)\sigma}{\mathrm{Lit}(\psi)\sigma, \Delta}$$

where σ subst'n, $\mathrm{cnf}(\phi) = \phi_1 \wedge \dots \wedge \phi_n$, and $\mathrm{Lit}(\cdot)$ is the *set* of literals occurring in a clause.

Notation. $\mathrm{G}\mathscr{R} \vdash \Gamma$ if Γ can be derived using the propositional rules and the "imported" modal rules.

Rule Sets in Sequent Form (without side conditions)

Modal Logic E.

$$\frac{\neg p, q \quad p, \neg q}{\neg \Box p, \Box q}$$

Modal Logic K.

$$\frac{\neg p_1, \ldots, \neg p_n, p_0}{\neg \Box p_1, \ldots, \neg \Box p_n, p_0}$$

Graded Modal Logic.

$$\frac{\sum_{i=1}^{n} p_i \leq \sum_{j=1}^{m} q_j}{\neg \Diamond_{k_1} p_1, \ldots, \neg \Diamond_{k_n} p_n, \Diamond_{l_1} q_1, \ldots, \Diamond_{l_j} q_j}$$

Probabilistic Modal Logic.

$$\frac{\sum_{i=1}^{n} p_i + u \leq \sum_{j=1}^{m} q_j}{\neg L_{u_1} p_1, \ldots, \neg L_{u_n} p_n, L_{v_1} q_1, \ldots, L_{v_m} q_m}$$

Conditional Logic.

$$\frac{\neg p_0, p_1 \quad \neg p_1, p_0 \quad \ldots \quad \neg p_0, p_n \quad \neg p_n, p_0 \quad \neg q_1, \ldots, \neg q_n, q_0}{\neg (p_1 \Rightarrow q_1), \ldots, \neg (p_n \Rightarrow q_n), (p_0 \Rightarrow q_0)}$$

Coalition Logic.

$$\frac{\neg p_1, \ldots, \neg p_n}{\neg [C_1] p_1, \ldots, \neg [C_n] p_n} \qquad \frac{\neg p_1, \ldots, \neg p_n, q, r_1, \ldots, r_m}{\neg [C_1] p_1, \ldots, \neg [C_n] p_n, [D] q, [N] r_1, \ldots, [N] r_m}$$

Sequent Calculi: Simple Structural Properties

Structural Rules

$$(\mathrm{W})\frac{\Gamma}{\Gamma, \Delta} \qquad (\mathrm{Con})\frac{\Gamma, A, A}{\Gamma, A} \qquad (\mathrm{Cut})\frac{\Gamma, A \quad \Delta, \neg A}{\Gamma, \Delta}$$

Notation. $\mathrm{G}\mathscr{R}\{\mathrm{W}, \mathrm{Con}, \mathrm{Cut}\} \vdash \Gamma$ if Γ can be derived using indicated structural rules.

Weakening Lemma.

- $\mathrm{G}\mathscr{R}\mathrm{W} \vdash \Gamma$ implies $\mathrm{G}\mathscr{R} \vdash \Gamma$ (W is *admissible*)

Inversion Lemma.

- $\mathrm{G}\mathscr{R} \vdash \Gamma, A \wedge B$ implies $\mathrm{G}\mathscr{R} \vdash \Gamma, A$ and $\mathrm{G}\mathscr{R} \vdash \Gamma, B$
- $\mathrm{G}\mathscr{R} \vdash \Gamma, \neg(A \wedge B)$ implies $\mathrm{G}\mathscr{R} \vdash \Gamma, \neg A, \neg B$

(I.e. *inversion rules* are admissible)

Why Sequent Calculi?

Backwards Proof Search in Hilbert Calculi, e.g. modus ponens:

$$\frac{A \to B \qquad A}{B}$$

- infinitely many possibilities for A, failure of subformula property

in Sequent Calculi, e.g. $(\neg\wedge)$

$$\frac{\Gamma, \neg A, \neg B}{\Gamma, \neg(A \wedge B)}$$

- premiss mentions only subformulae of conclusion, finitely many possible applications

The Enemies.

$$(\mathrm{Cut})\frac{\Gamma, A \qquad \Delta, \neg A}{\Gamma, \Delta} \qquad (\mathrm{Con})\frac{\Gamma, A, A}{\Gamma, A}$$

Goal: Cut and Contraction-Free Calculi

Suppose we've already succeeded.

$$\mathsf{G}\mathscr{R} \vdash \Gamma \iff T \models \bigvee \Gamma$$

Decidability.

- proof trees are of finite depth and finitely branching
- decidability as long as rules are effective

Complexity.

- proof trees are polynomially deep
- PSPACE if rules are 'in NP'

The Easy Part

Soundness. $G\mathscr{R} \vdash \Gamma$ implies $T \models \Gamma$

Completeness with Cut. If $\mathscr{R}$ is one-step complete, then $G\mathscr{R}\mathsf{ConCut} \vdash \Gamma$ whenever $T \models \bigvee \Gamma$.

Proof. Use completeness of Hilbert system;
cut rule simulates modus ponens.

The Hard Part. Get rid of cut and contraction.

Semantic Cut Elimination

Stronger **Coherence Condition:** $\mathscr{R}$ is *one-step cutfree complete* if

$$TX, \sigma \models \chi \implies \exists \phi/\psi \in \mathscr{R}, \tau : V \to V \text{ s.t. } X, \sigma \models \phi\tau \text{ and } \psi\tau \vdash_{\mathsf{PL}} \chi$$

(χ clause over $\Lambda(V)$, $\sigma : V \to \mathscr{P}(X)$).

Intuition. Valid modalized formulas are derivable using a *single* rule.

Cutfree complete rule sets exist. The set of all one-step sound one-step rules is one-step cutfree complete (as seen earlier).

Cut-Free Completeness

Examples. All rule sets we have seen today are one-step cutfree complete.

Thm. Suppose $\mathscr{R}$ is one-step cutfree complete. Then

$$T \models \bigvee \Gamma \quad \text{iff} \quad \mathsf{G}\mathscr{R}\mathsf{Con} \vdash \Gamma.$$

Proof Sketch. By contraposition: If $\mathsf{G}\mathscr{R}\mathsf{Con} \nvdash \Gamma$

- all rules applicable to Γ have a premiss that is not derivable
- countermodel construction with $C =$ modal nodes (no top-level propositional connectives)
- construct $\gamma : C \to TC$ using cutfree completeness
- by induction: $T \not\models \bigvee \Gamma$

Syntactic Cut Elimination

Defn. A rule set $\mathscr{R}$ *absorbs cut* if for all $\chi = \bigwedge \heartsuit_i p_i \to \bigvee \heartsuit_j p_j$

- if $\psi_1\sigma_1, \dots, \psi_n\sigma_n \vdash_{\mathsf{PL}} \chi$ for rules ϕ_i/ψ_i, renamings σ_i
- then there is $\phi/\psi \in \mathscr{R}$, renaming σ such that
- $\phi_1\sigma_1, \dots, \phi_n\sigma_n \vdash_{\mathsf{PL}} \phi\sigma$ and $\psi\sigma \vdash_{\mathsf{PL}} \chi$

(combinations of rule conclusions are subsumed by by single rules)

Thm. Let $\mathscr{R}$ be one-step complete and absorb cut. Then

$$T \models \bigvee \Gamma \text{ iff } \mathsf{G}\mathscr{R}\mathsf{Con} \vdash \Gamma.$$

Proof Sketch. If $T \models \bigvee \Gamma$ then $\mathscr{R} \vdash \bigvee \Gamma$. Consequently:

- for every component χ of the cnf of $\bigvee \Gamma$
- we can find $\phi/\psi \in \mathscr{R}$ and $\sigma : V \to \mathscr{F}(\Lambda)$ s.t. $\psi\sigma \vdash_{\mathsf{PL}} \chi$ and $\mathscr{R} \vdash \phi\sigma$

All these steps can be simulated in $\mathsf{G}\mathscr{R}\mathsf{Con}$.

Cut Elimination Proof

Recall. $\mathscr{R} \vdash \bigvee \Gamma \iff \mathsf{G}\mathscr{R}\mathsf{CutCon} \vdash \Gamma$.

Alternative Approach. Cut admissibility.

Thm. If $\mathscr{R}$ absorbs cut then $\mathsf{G}\mathscr{R}\mathsf{CutCon} \vdash \Gamma$ iff $\mathsf{G}\mathscr{R}\mathsf{Con} \vdash \Gamma$.

Proof Sketch. By induction on proofs with cut, show that cut can be eliminated.

- cuts between propositional rules: propagate upwards in proof tree
- cuts between modal rules: by cut-free completeness, cut on $\heartsuit A$ can be replaced by cuts on A
- cuts between modal / propositional rules: don't occur on $\heartsuit A$ and propagate.

Technically: double induction on size of proof and size of cut formula.

Cut-Free Completeness vs Strict Completeness

Recall. The set of all valid one-step rules is

- one-step cut-free complete (by inspection of proof)
- absorbs cut (by a simple semantical argument)

Thm. TFAE:

1. $\mathscr{R}$ is one-step complete for T and absorbs cut
2. $\mathscr{R}$ is one-step cutfree complete for T

Elimination of Contraction

Our Second Enemy.

$$(\text{Con})\frac{\Gamma, A, A}{\Gamma, A}$$

- *multisets* make contraction explicit, and contraction gives arbitrarily large proof trees

Question. Can we circumnavigate the problem by having sequents as *sets* instead? (think about it . . .)

Contraction

$\mathscr{R}$ *absorbs contraction* if for each $\phi/\psi \in \mathscr{R}$ over V and each $\sigma : V \to W$, there exist $\phi'/\psi' \in \mathscr{R}$ over V' and $\sigma' : V' \to W$ s.t.

$$\phi\sigma \vdash_{\mathsf{PL}} \phi'\sigma' \qquad\qquad \psi'\sigma' \vdash_{\mathsf{PL}} \psi\sigma$$

and

$$\sigma' : V' \to W' \quad \textit{injective}.$$

Example: K

$$\frac{\phi}{\psi} = \frac{\neg a_1, \neg a_2, b}{\neg\Box a_1, \neg\Box a_2, \Box b} \qquad \sigma(a_1) = \sigma(a_2) = a$$

may be replaced by

$$\frac{\phi'}{\psi'} = \frac{\neg a, b}{\neg\Box a, \Box b} \qquad \sigma' = id$$

Example: GML

$$\frac{\sum_{i=1}^{n} p_i \leq \sum_{j=1}^{m} q_j}{\neg\Diamond_{k_1} p_1, \dots, \neg\Diamond_{k_n} p_n, \Diamond_{l_1} q_1, \dots, \Diamond_{l_j} q_j} \quad (\sum_{i=1}^{n}(k_i + 1) > \sum_{j=1}^{m} l_j)$$

fails to absorb contraction $\rightarrow$ extend the rule set:

$$\frac{\sum_{i=1}^{n} r_i p_i \geq 0}{sgn(r_1)\Diamond_{k_1} p_1, \dots, sgn(r_n)\Diamond_{k_n} p_n} \; (r_i \in \mathbb{Z} - \{0\};\ \sum_{r_i<0} |r_i|(k_i + 1) > \sum_{r_i>0} r_i k_i)$$

Complexity

Suppose we have a rule set absorbing cut and contraction

PSPACE Bounds via non-deterministic proof search:

- Have polynomial bound on the height of the proof tree

Require additionally $\mathscr{R}$ *PSPACE-tractable*:

- for every sequent Γ, the (codes of) rules that entail Γ can be guessed and checked in (nondet.) polynomial time
  - *Main point:* polynomially large codes must suffice
- for every (code of a) rule, its premises can be (guessed and) checked in (nondet.) polynomial time.

PSPACE Bounds

Thm. If $\mathscr{R}$ is PSPACE-tractable and absorbs cut and contraction then $\mathscr{R}$-provability is in PSPACE.

Proof Sketch.

- Implement proof search on an alternating Turing machine:
  - existentially guess a matching proof rule
  - universally check provability of all premises
- Proof trees are of polynomial height
- Runs in APTIME = PSPACE.

Examples

Modal Logics K, E, coalition logic and conditional logic:

- earlier rule sets are already contraction closed, and clearly PSPACE-tractable

Graded Modal Logic. Recall rule:

$$\frac{\sum_{i=1}^{n} r_i p_i \geq 0}{sgn(r_1)\Diamond_{k_1} p_1, \ldots, sgn(r_n)\Diamond_{k_n} p_n} \; (r_i \in \mathbb{Z} - \{0\}; \sum_{r_i<0} |r_i|(k_i+1) > \sum_{r_i>0} r_i k_i)$$

PSPACE-tractability: polysize solutions of systems of linear inequalities $\rightarrow$ polysize r_i suffice

Generic Complexity Bounds

Thm. Provability in the logics E, M, K, graded modal logic, probabilistic modal logic, coalition logic (and many others!) is decidable in polynomial space.

Remarks.

- all results just *instances* of the same, general theory
- results by just *inspecting* corresponding rule sets
- generation of rule-sets semi-automatic

Extensions.

- accommodate non-iterative rules: more logics (not covered)
- accommodate other logical primitives: fixpoints, nominals, etc.

Interlude: Parametric Implementation

Example Language: Haskell

Parametric Formulas.

```
data L a
  =  F | T | Atom Int
    | Neg (L a) | And (L a) (L a) | M a (L a)
```

Example. The logic *K* and graded modal logic

```
data K = K deriving (Eq,Show)
data G = G Int
```

Logic. Type-class that supports matching

```
class (Eq a,Show a) => Logic a where
  match :: Clause a -> [[L a]]
```

(double lists as rule premises are generally in cnf)

Matching and Provability

Example. Syntax of K (again)

```
data K = K
```

Matching: representation of resolution closed rule sets

```
instance Logic K where
  match (Clause (pl,nl)) =
    let (nls,pls) = (map neg (stripany nl), stripany pl)
    in map disjlst (map (\x -> x:nls) pls)
```

Generic Provability Predicate.

```
provable :: (Logic a) => L a -> Bool
provable phi = all (\c -> any (all provable) ( match c))
                   (cnf phi)
```

(lazyness of Haskell guarantees polynomial space)