The Many Faces of Modal Logic
Day 2: Unifying Semantics

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(Slides based on a NASSLLI 2014 Tutorial and are joint work with Lutz Schröder)

LAC 2018

Modal Logics are Coalgebraic

Why Coalgebras?

▶ because they provide a *uniform* semantics
  for a large class of modal logics

So which logics are amenable to colgebraic semantics?

▶ Of course, K. But also: neighbourhood logic, coalition logic,
  probabilistic logic, graded logic, conditional logic . . .

And what can you do at this level of generality?

▶ The usual stuff: completeness, complexity, Hennessy-Milner,
  interpolation – but *generically* and *compositionally*

Yesterday's Cook's Tour Through Modal Logics

Kripke Modal Logic
- $\Diamond \phi$
- ϕ can be true

Conditional Logic
- $\phi \Rightarrow \psi$
- ψ if ϕ

Coalition Logic
- $[C]\phi$
- C can force ϕ

Graded Modal Logic
- $\Diamond_k \phi$
- ϕ in $> k$ successors

Probabilistic Modal Logic
- $L_p \phi$
- ϕ holds with probability $\geq p$

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Similarities
- same *questions:* Completeness, decidability, complexity, . . .
- *combinations:* probabilities and non-determinism, uncertainty in games
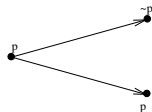

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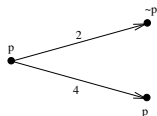- ϕ holds with probability $\geq p$

A Cook's Tour Through Modal Semantics

Kripke Frames.
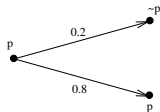

$$C \rightarrow \mathcal{P}(C)$$

Multigraph Frames.

$$C \rightarrow \mathcal{B}(C)$$

$$\mathcal{B}(X) = \{f : X \rightarrow \mathbb{N} \mid \text{supp}(f) \text{ finite}\}$$

Probabilistic Frames.

$$C \rightarrow \mathcal{D}(C)$$

$$\mathcal{D}(X) = \{\mu : X \rightarrow [0,1] \mid \sum_{x \in X} \mu(x) = 1\}$$

More Examples
Neighbourhood Frames.

$$C \to \mathcal{P}\mathcal{P}(C) = \mathcal{N}(C)$$

mapping each world $c \in C$ to a set of neighbourhoods

Game Frames over a set N of agents

$$C \to \{((S_n)_{n \in N}, f) \mid f : \prod_n S_n \to C\} = \mathcal{G}(C)$$

associating to each state $c \in C$ a *strategic game* with strategy sets S_n and outcome function f

Conditional Frames.

$$C \to \{f : \mathcal{P}(C) \to \mathcal{P}(C) \mid f \text{ a function}\} = \mathcal{C}(C)$$

where every state yields a *selection function* that assigns successor sets to conditions

Coalgebras and Modalites: A Non-Definition

Coalgebras are about *successors*. T-coalgebras are pairs (C, γ) where

$$\gamma : C \to TC$$

maps states to successors. $\mathrm{Coalg}(T)$ is the class of T-coalgebras.

states: elts $c \in C$	prop's of states: subsets $P \subseteq C$
succ's: elts $\gamma(c) \in TC$	prop's of successors: subsets $[\![\heartsuit]\!](P) \subseteq TC$

Modalities are about *properties* of successors: *predicate liftings*

$$[\![\heartsuit]\!]_C : \mathcal{P}(C) \to \mathcal{P}(TC)$$

Intended use:

$$c \models \heartsuit\phi \quad \text{iff} \quad \gamma(c) \in [\![\heartsuit]\!]_C([\![\phi]\!]_C)$$

Example: Kripke Frames

Intuition. In $\gamma : C \to \mathcal{P}(C)$ think of $\gamma(c)$ as "the" successor. Then:

$$c \models \Box\phi \iff \text{all elements of "the" successor } \gamma(c) \text{ of } c \text{ satisfy } \phi$$
$$\iff \text{"the" successor } \gamma(c) \text{ of } c \text{ is a subset of } [\![\phi]\!]$$
$$\iff \gamma(c) \in \{B \subseteq C \mid B \subseteq [\![\phi]\!]\}$$

$\to$ **Predicate lifting**

$$[\![\Box]\!]_C : \mathcal{P}(C) \to \mathcal{PP}(C)$$
$$A \mapsto \{B \subseteq C \mid B \subseteq A\}.$$

$\to$ **Satisfaction**

$$c \models \Box\phi \iff \gamma(c) \in [\![\Box]\!]_C([\![\phi]\!])$$

Another Example: Neighbourhood Frames

Intuition. In $\gamma : C \to \mathcal{P}\mathcal{P}(C)$, $\gamma(c)$ contains the *neighbourhoods* of c

$$c \models \Box\phi \iff [\![\phi]\!] \in \gamma(c)$$
$$\iff \gamma(c) \in \{N \in \mathcal{N}(C) \mid [\![\phi]\!] \in N\}.$$

→ **Predicate lifting**

$$[\![\Box]\!]_C : \mathcal{P}(C) \to \mathcal{P}(\mathcal{N}(C)), \quad A \mapsto \{N \in \mathcal{N}(C) \mid A \in N\}$$

→ **Satisfaction**

$$c \models \Box\phi \iff \gamma(c) \in [\![\Box]\!]_C([\![\phi]\!]_C)$$

(Recall the definition for Kripke Frames?)

Probabilistic Frames

Intuition. In $\gamma : C \to \mathcal{D}(C)$, $\gamma(c)$ is a *random successor* of c.

$$c \models L_p \phi \iff \gamma(c)(\llbracket \phi \rrbracket) \geq p$$
$$\iff \gamma(c) \in \{\mu \in \mathcal{D}(C) \mid \mu(\llbracket \phi \rrbracket) \geq p\}.$$

$\to$ **Predicate lifting**

$$\llbracket L_p \rrbracket_C : \mathcal{P}(C) \to \mathcal{P}(\mathcal{D}C), \quad A \mapsto \{\mu \in \mathcal{D}(C) \mid \mu(A) \geq p\}$$

$\to$ **Satisfaction**

$$c \models L_p \phi \iff \gamma(c) \in \llbracket L_p \rrbracket_C(\llbracket \phi \rrbracket_C)$$

(Recall Kripke frames and neighbourhood frames?)

Conditional Frames

Intuition. In $\gamma : C \to (\mathcal{P}(C) \to \mathcal{P}(C))$, $\gamma(c)(A)$ are the most typical alternatives to c under (non-monotonic) condition A

$$c \models \phi \Rightarrow \psi \iff \gamma(c)(\llbracket \phi \rrbracket) \subseteq \llbracket \psi \rrbracket$$
$$\iff \gamma(c) \in \{f \in \mathcal{C}W \mid f(\llbracket \phi \rrbracket) \subseteq \llbracket \psi \rrbracket\}.$$

$\to$ **Binary predicate lifting**

$$\llbracket \Rightarrow \rrbracket_W : \mathcal{P}(W) \times \mathcal{P}(W) \to \mathcal{P}(\mathcal{C}W), \quad (A, B) \mapsto \{f \in \mathcal{C}W \mid f(A) \subseteq B\}$$

$\to$ **Satisfaction**

$$c \models \phi \Rightarrow \psi \iff \gamma(c) \in \llbracket \Rightarrow \rrbracket_c(\llbracket \phi \rrbracket_c, \llbracket \psi \rrbracket_c)$$

More Examples

Graded Modal Logic over *multigraph frames*

$$\gamma : C \to \mathcal{B}C = \{f : C \to \mathbb{N} \mid \text{supp}(f) \text{ finite}\}$$

Predicate lifting for "more than k successors validate ..."

$$[\![\Diamond_k]\!]_C(A) = \{f : C \to \mathbb{N} \mid \sum_{a \in A} f(a) \geq k\}$$

Coalition Logic over *game frames*

$$\gamma : C \to \mathcal{G}C = \{(f, (S_n)_{n \in N} \mid f : \prod_n S_n \to C\}$$

Predicate lifting for "coalition $K \subseteq N$ can force ..."

$$[\![[K]]\!]_C(A) = \{(f, (S_n)) \in \mathcal{G}C \mid$$
$$\exists \sigma_K \in \prod_K S_i . \forall \sigma_{N-K} \in \prod_{N-K} S_k . (f(\sigma_K, \sigma_{N-K}) \in A)\}$$

Satisfaction in either case:

$$c \models \heartsuit\phi \iff \gamma(c) \in [\![\heartsuit]\!]_C([\![\phi]\!]_C)$$

Kripke Frames: Bisimilarity

$S \subseteq C \times D$ *simulation* if for all cSd

► $c \in \pi(p) \implies d \in \pi(p)$

► $\forall c' \in \gamma(c). \exists d' \in \gamma(d). c'Sd'$

S *bisimulation* $\iff S, S^-$ simulations

c, d *bisimilar* $\iff cSd$ for some bisimulation S.

Lemma: Bisimilar states satisfy the same modal formulae

Kripke Frames: *p*-Morphisms and Bisimilarity

p-morphisms = functions $f : C \to D$ s.t.

$$\begin{array}{ccc} C & \xrightarrow{\ f\ } & D \\ \gamma \downarrow & & \downarrow \delta \\ \mathcal{P}(C) & \xrightarrow{\ \mathcal{P}(f)\ } & \mathcal{P}(D) \end{array}$$

where $\mathcal{P}(f)(A) = f[A] = \{f(a) \mid a \in A\}$.

Lemma: Two states are bisimilar if and only if they can be identified by *p*-morphisms:

$$c \in C \qquad\qquad\qquad D \ni d$$
$$\xrightarrow{\ f\ } \qquad\qquad \xleftarrow{\ g\ }$$
$$f(c) = g(d) \in E$$

Behavioural Equivalence, Coalgebraically

Defn. $f : (C, \gamma) \to (D, \delta)$ *(coalgebra homo-)morphism* if

States c, d are *behaviourally equivalent* ($c \simeq d$) if they can be identified by a morphism.

Ooops! How is $T(f)$ defined *in general*?

Answer: Require T to be a *functor*, i.e. $Tf : TA \to TB$ if $f : A \to B$ and

$$T(id_A) = id_{TA} \quad T(g \circ f) = Tg \circ Tf$$

for $A \in$ **Set** and composable functions f, g.

Good for us. The action of T on functions is usually canonical.

A Wee Bit of Structure Theory

Abstract Coalgebra: Prove statements about T-coalgebras *without knowing T*.

Simple Stuff: The identity is a morphism and morphisms compose.

Harder Stuff Behavioural equivalence is transitive and preserved by morphisms. (Try to prove it!)

Modalities and Behavioural Equivalence

"Good" Modalities are **compatible**. This one is from hell:

$$\llbracket \Box \rrbracket_C(A) = \begin{cases} \emptyset & C = \mathbb{N} \\ TC & \text{o/w} \end{cases}$$

Defn. An *n-ary predicate lifting* for $T : \mathbf{Set} \to \mathbf{Set}$ is a family $(\lambda_X)_{X \in \mathbf{Set}}$ of functions $\lambda_X : \mathcal{P}(X)^n \to \mathcal{P}(TX)$ such that

$$
\begin{array}{ccc}
(\mathcal{P}X)^n & \xrightarrow{\ \lambda_X\ } & \mathcal{P}(TX) \\
{\scriptstyle (f^{-1})^n} \Big\uparrow & & \Big\uparrow {\scriptstyle (Tf)^{-1}} \\
(\mathcal{P}Y)^n & \xrightarrow[\ \lambda_Y\ ]{} & \mathcal{P}(TY)
\end{array}
$$

commutes for all $f : X \to Y$.

($\lambda : \mathcal{Q}^n \to \mathcal{Q} \circ T^{op}$ is *natural* for $\mathcal{Q} : \mathbf{Set}^{op} \to \mathbf{Set}$ contravariant powerset)

Naturality: Examples

▶ **Kripke frames / K**:

$$\mathcal{P}(f)(A) \in [\![\Box]\!](B) \iff f[A] \subseteq B$$
$$\iff A \subseteq f^{-1}[B] \iff A \in [\![\Box]\!](f^{-1}[B]).$$

▶ **Kripke frames / Graded Modal Logic**: $f : \{0,1\} \to \{0\}$:

$$\mathcal{P}(f)(\{0,1\}) \notin [\![\Diamond_1]\!](\{0\}) \quad \text{but} \quad \{0,1\} \in [\![\Diamond_1]\!](f^{-1}[\{0\}])$$

but

$$\mathcal{B}(f)(\mu) \in [\![\Diamond_k]\!](B) \iff \mathcal{B}(f)(\mu)(B) = \mu(f^{-1}[B]) > k$$
$$\iff \mu \in [\![\Diamond_k]\!](f^{-1}[B])$$

Finally: Proper Definitions

Defn. *(Modal) signature* Λ = set of finitary modal operators (for readability: unary). Λ-*formulas*:

$$\mathcal{F}(\Lambda) \ni \phi, \psi ::= p \mid \bot \mid \neg\phi \mid \phi \wedge \psi \mid \heartsuit(\phi_1, \ldots, \phi_n) \qquad (p \in V, \heartsuit \in \Lambda \; n\text{-ary})$$

Λ-*structure*:

- Functor $T : \mathbf{Set} \to \mathbf{Set}$
- n-ary predicate lifting $[\![\heartsuit]\!]$ for $\heartsuit \in \Lambda$ n-ary

Modal Semantics over $\gamma : C \to TC$, $\sigma : \mathcal{V} \to \mathcal{P}(C)$:

$$C, c, \sigma \models p \text{ iff } c \in \sigma(p)$$
$$C, c, \sigma \models \heartsuit(\phi_1, \ldots, \phi_n) \text{ iff } \gamma(c) \in [\![\heartsuit]\!]_C([\![\phi_1]\!]_{C,\sigma}, \ldots, [\![\phi_n]\!]_{C,\sigma})$$
$$C, \sigma \models \phi \text{ iff } C, c\sigma \models \phi \text{ for all } c \in C$$
$$C \models \phi \text{ iff } C, \sigma \models \phi \text{ for all } \sigma : \mathcal{V} \to \mathcal{P}(C)$$

where $[\![\phi]\!]_{C,\sigma} = \{ c \in C \mid C, c, \sigma \models \phi \}$

Logical Equivalence vs Behavioural Equivalence

Defn. States c, d *logically equivalent* if

$$c \models \phi \iff d \models \phi \qquad \text{for all } \phi \in \mathcal{F}(\Lambda)$$

Lemma. Morphisms f preserve semantics: $c, f(c)$ logically equivalent
Proof:

$$f(c) \models \heartsuit\phi \iff \delta(f(c)) = Tf(\gamma(c)) \in \llbracket\heartsuit\rrbracket_D\llbracket\phi\rrbracket_D$$
$$\iff \gamma(c) \in \llbracket\heartsuit\rrbracket_D f^{-1}[\llbracket\phi\rrbracket_D] \overset{\text{IH}}{=} \llbracket\heartsuit\rrbracket_D\llbracket\phi\rrbracket_C \iff c \models \heartsuit\phi$$

Cor. Modal logic is invariant under behavioural equivalence:

$$c \simeq d \implies c, d \text{ logically equivalent}$$

(*Converse*, i.e. the *Hennessy-Milner property*, holds
over *finitely branching* systems if there are *enough* modalities)

E.g. graded modalities are invariant over $\mathcal{B}$ (but not over $\mathcal{P}$!)

Coalgebras and Their Logics

What are the laws for reasoning about
probabilities / strategic games / non-monotonic conditionals . . . ?

Defn. A *logic* over a signature Λ is a set $L \subseteq \mathcal{F}(\Lambda)$ of formulae that

▶ contains all propositional tautologies

▶ is closed under modus ponens, uniform substitution, and *congruence*

$$\frac{\phi \leftrightarrow \psi \in L}{\heartsuit\phi \leftrightarrow \heartsuit\psi \in L}$$

$$\mathsf{Log}(T) = \{\phi \in \mathcal{F}(\Lambda) \mid C \models \phi \text{ for all } T\text{-coalgebras } C\}$$

is a logic. Write $T \models \phi$ for $\phi \in \mathsf{Log}(T)$.

Logics, Syntactically Defined

Defn. A Λ-*rule* is a pair ϕ/ψ of Λ-formulae, and ϕ/ψ is *admissible* in a logic L if $\psi\sigma \in L$ whenever $\phi\sigma \in L$ and σ is a substitution.

$$\mathsf{Log}(\mathcal{R}) = \bigcap \{L \subseteq \mathcal{F}(\Lambda) \mid L \text{ a logic}, \phi/\psi \text{ admissible in } L \text{ for all } \phi/\psi \in \mathcal{R}\}$$

is the logic generated by the set $\mathcal{R}$ of rules. Write $\mathcal{R} \vdash \phi$ if $\phi \in \mathsf{Log}(\mathcal{R})$.

Goal. Given a Λ-structure T, find rules $\mathcal{R}$ such that $\mathsf{Log}(T) = \mathsf{Log}(\mathcal{R})$.

Roadmap.

▶ solve this *generically*, i.e. without looking into T

▶ instead, postulate *coherence conditions* linking $\mathcal{R}$ and T.

Soundness

$\mathcal{R}$ *sound* : $\iff$ Everything that is provable is true: $\mathrm{Log}(\mathcal{R}) \subseteq \mathrm{Log}(T)$

Lemma. $\mathcal{R}$ is sound if

$$C, \sigma \models \psi \text{ whenever } C, \sigma \models \phi \text{ for all } \phi/\psi \in \mathcal{R}, \sigma : \mathcal{V} \to \mathcal{P}(C)$$

Slogan. The system is sound if each rule is sound

Aside on $\mathrm{Log}(\mathcal{R})$**.** $\mathrm{Log}(\mathcal{R})$ = *provable* formulae, inductively:

▶ all instances of propositional tautologies are provable.

▶ if $\mathcal{R} \vdash \phi\sigma$, then $\mathcal{R} \vdash \psi\sigma$ for all $\phi/\psi \in \mathcal{R}$

▶ if $\mathcal{R} \vdash \phi \to \psi$, $\mathcal{R} \vdash \phi$, then $\mathcal{R} \vdash \psi$

▶ if $\mathcal{R} \vdash \phi \leftrightarrow \psi$ then $\mathcal{R} \vdash \heartsuit\phi \leftrightarrow \heartsuit\psi$

Examples

Modal Logic K

$$\frac{p}{\Box p} \qquad \frac{p \wedge q \to r}{\Box p \wedge \Box q \to \Box r}$$

Neighbourhood Frames

$$\frac{p \leftrightarrow q}{\Box p \to \Box q}$$

Probabilistic Modal Logic

$$\frac{}{L_0 p} \qquad \frac{p}{L_u p} \qquad \frac{\neg p \vee \neg q}{\neg L_u p \vee \neg L_v q}\,(u+v>1) \qquad \frac{p \vee q}{L_u p \vee L_v q}\,(u+v=1)$$

$$\frac{\sum_{i=1}^{r} 1_{p_i} = \sum_{j=1}^{s} 1_{\bar{q}_j}}{\bigwedge_{i=1}^{r} L_{u_i} p_i \wedge \bigwedge_{j=2}^{s} L_{(1-v_j)} q_j \to L_{v_1} q_1}(\textstyle\sum_{j=1}^{s} v_j = \sum_{i=1}^{r} u_i)$$

where $\bar{d}_1 = d_1$ and $\bar{d}_j = \neg d_j$ for $j \geq 2$.

Observation. Propositional premiss, modal conclusion.

More Examples

Coalition Logic for pairwise disjoint sets C_i of coalitions:

$$\frac{\bigvee_{i=1,\ldots,n} \neg p_i}{\bigvee_{i=1,\ldots,n} \neg [C_i] p_i} \qquad \frac{p}{[C]p} \qquad \frac{p \vee q}{[\emptyset]p \vee [N]q} \qquad \frac{\bigwedge_{i=1,\ldots,n} p_i \to q}{\bigwedge_{i=1,\ldots,n} [C_i] p_i \to [\bigcup C_i] q}$$

Graded Modal Logic

$$\frac{p \to q}{\diamond_{n+1} p \to \diamond_n q} \qquad \frac{r \to p \vee q}{\diamond_{n+k} r \to \diamond_n p \vee \diamond_k q} \qquad \frac{p \leftrightarrow q}{\diamond_k p \to \diamond_k q}$$

$$\frac{(p \vee q \to r) \wedge (p \wedge q \to s)}{\diamond_n p \wedge \diamond_k q \to \diamond_{n+k} r \vee \diamond_0 s} \qquad \frac{\neg p}{\neg \diamond_0 p}$$

Conditional Logic

$$\frac{q}{p \Rightarrow q} \qquad \frac{q_1 \wedge q_2 \to q_0}{(p \Rightarrow q_1) \wedge (p \Rightarrow q_2) \to (p \Rightarrow q_0)} \qquad \frac{p_1 \leftrightarrow p_2}{(p_1 \Rightarrow q) \to (p_2 \Rightarrow q)}$$

Observation. Again, propositional premiss, modal conclusion

Examples: Soundness

Propn. All above rule sets are sound, i.e. $\mathrm{Log}(\mathcal{R}) \subseteq \mathrm{Log}(T)$.

Observation. All rules have the form ϕ / ψ where $\phi \in \mathrm{Prop}(\mathcal{V})$ and $\psi \in \mathrm{Prop}(\Lambda(\mathcal{V}))$ where

- $\mathrm{Prop}(F)$ are propositional combinations of elements of $F \subseteq \mathcal{F}(\Lambda)$

- $\Lambda(F) = \{\heartsuit \phi \mid \heartsuit \in \Lambda, \phi \in F\}$

Defn. Rules of the above form are *one-step rules*.

Coherence Conditions 1: Soundness

> **Goal.** Using the form of the rules, can we find a simpler condition that ensures soundness?

One-Step Logics = sets $L \subseteq \mathsf{Prop}(\Lambda(\mathsf{Prop}(\mathcal{V})))$ that

▶ contain all instances of propositional tautologies

▶ are closed under modus ponens

▶ contain $\heartsuit\phi \leftrightarrow \heartsuit\psi$ whenever $\phi \leftrightarrow \psi$ is a propositional tautology

One-Step Semantics. Given $\sigma : \mathcal{V} \to \mathcal{P}(X)$ we have

▶ $[\![\phi]\!]\sigma \subseteq X$ for $\phi \in \mathsf{Prop}(\mathcal{V})$

▶ $[\![\psi]\!]\sigma \subseteq TX$ for $\psi \in \mathsf{Prop}(\Lambda(\mathsf{Prop}(\mathcal{V})))$ where
$$[\![\heartsuit\phi]\!]\sigma = [\![\heartsuit]\!]_X([\![\phi]\!]\sigma)$$

(NB: one-step semantics doesn't refer to models)

One-Step Soundness

$$\text{Log}_1(T) = \{\psi \in \text{Prop}(\Lambda(\text{Prop}(\mathcal{V}))) \mid \llbracket \psi \rrbracket \sigma = TX \text{ for all } \sigma : \mathcal{V} \to \mathcal{P}(X)\}$$

is a one-step logic.

Coherence for Soundness. One-step rule ϕ/ψ *admissible* in one-step logic L if

$$\phi\sigma \text{ tautology} \implies \psi\sigma \in L.$$

For $\mathcal{R}$ set of one-step rules,

$$\text{Log}_1(\mathcal{R}) := \bigcap\{L \text{ one-step logic} \mid \phi/\psi \text{ admissible in } L \text{ for all } \phi/\psi \in \mathcal{R}\}$$

$\mathcal{R}$ *one-step sound* if $\text{Log}_1(\mathcal{R}) \subseteq \text{Log}_1(T)$.

Soundness Theorem. One-step soundness implies soundness, that is $\text{Log}(\mathcal{R}) \subseteq \text{Log}(T)$ whenever $\text{Log}_1(\mathcal{R}) \subseteq \text{Log}_1(T)$.

One-Step Soundness: Examples

Equivalent Characterisation. $\mathcal{R}$ one-step sound iff

$$[\![\phi]\!]\sigma = X \implies [\![\psi]\!]\sigma = TX$$

for all $\phi/\psi \in \mathcal{R}$, $\sigma : \mathcal{V} \to \mathcal{P}(X)$.

Example. For $T = \mathcal{P}$, the rule $p/\Box p$ is one-step sound: pick $\sigma : \mathcal{V} \to \mathcal{P}(X)$ such that $\sigma(p) = \top = X$. Then

$$[\![\Box p]\!]\sigma = \{ Y \subseteq X \mid Y \subseteq [\![p]\!]\sigma \} = \top = \mathcal{P}(X)$$

Observations.

- all rules that we have presented are one-step sound
- one-step soundness (a little) easier to check than soundness
- N.B.: Sound $\not\Rightarrow$ one-step sound: E.g. $\bot/\bot$ or probabilistic modal logic without propositional atoms

Tomorrow

> **Key Question.** Do we have enough rules to axiomatise *all* valid formulae?

Tomorrow: Completeness

▶ algebraic semantics: completeness is easy

▶ duality: from algebraic to coalgebraic models

▶ coherence conditions for completeness