The Many Faces of Modal Logic
Day 1: Examples

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The Plan for Today

- Brief recap of propositional logic
- Good old modal logic
- A zoo of modal logics
  - Syntax
  - Semantics
  - Reasoning principles

Propositional Logic

That is, classical propositional logic:

$$\phi ::= \bot \mid p \mid \neg\phi \mid \phi_1 \wedge \phi_2 \qquad (p \in V)$$

($\vee$, $\to$, $\leftrightarrow$, $\top$ defined).

Semantics: *valuations* $\kappa : V \to 2 = \{\bot, \top\}$

$$\begin{aligned}
\kappa &\not\models \bot \\
\kappa \models p \quad &\text{iff} \quad \kappa(p) = \top \\
\kappa \models \neg\phi \quad &\text{iff} \quad \kappa \not\models \phi \\
\kappa \models \phi_1 \wedge \phi_2 \quad &\text{iff} \quad \kappa \models \phi_1 \text{ and } \kappa \models \phi_2
\end{aligned}$$

Satisfiability and All That

- ϕ *satisfiable* if $\kappa \models \phi$ for some κ
- ϕ *valid* or *tautology* ($\models \phi$) if $\kappa \models \phi$ for all κ
  - ϕ valid iff $\neg\phi$ unsatisfiable
- Φ set of formulas:

$$\Phi \models \psi \quad \text{iff} \quad \forall \kappa. (\kappa \models \Phi \to \kappa \models \psi)$$
$$\text{iff} \quad \exists \phi_1, \dots, \phi_n \in \Phi. \models \phi_1 \wedge \dots \wedge \phi_n \to \psi$$

(*compactness*)

Boolean Algebra

Alternative semantics: valuations $\kappa : V \to A$, A Boolean algebra;

$$\kappa \models \phi \quad \text{iff} \quad \kappa(\phi) = \top$$

Stone duality:

A is a Boolean subalgebra of some 2^X.

This implies:

Validity in Boolean algebras = validity over 2.

Normal Forms

Negation normal form (NNF): $\neg$ only in front of atoms –

$$\phi ::= \bot \mid \top \mid p \mid \neg p \mid \phi_1 \wedge \phi_2 \mid \phi_1 \vee \phi_2.$$

Conjunctive normal form (CNF):

- Literal = atom or negated atom
- Clause = finite disjunction (set) of literals
- CNF = finite conjunction (set) of clauses.

Dual: conjunctive clause, DNF.

Modal Logic

$$\phi ::= \bot \mid p \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid \Box\phi \qquad (p \in P)$$

with $\Box\phi$ read

- 'necessarily ϕ'
- 'it is believed that ϕ' (doxastic)
- 'it is known that ϕ' (epistemic)
- 'it is obligatory that ϕ' (deontic)
- ...

Dual: $\Diamond = \neg\Box\neg$ 'possibly'

Kripke Models

Kripke models (X, R, π), where

- (X, R) *Kripke frame*
  - X set of *states* / *worlds*
  - $R \subseteq X \times X$ *accessibility* / *transition* relation
- $\pi : P \to \mathcal{P}(X)$ *valuation*

Satisfaction in Kripke Models

$x \models_M \phi$ 'state x in model M satisfies ϕ'; $[\![\phi]\!]_M = \{x \in M \mid x \models_M \phi\}$.

$$
\begin{aligned}
x &\not\models_M \bot \\
x \models_M p &\iff x \in V(p) \\
x \models_M \phi \wedge \psi &\iff x \models_M \phi \text{ and } x \models_M \psi \\
x \models_M \neg\phi &\iff x \not\models_M \phi \\
x \models_M \Box\phi &\iff y \models_M \phi \text{ for all } y \in X \text{ such that } (x, y) \in R
\end{aligned}
$$

hence

$$x \models_M \Diamond\phi \iff y \models_M \phi \text{ for some } y \in X \text{ such that } (x, y) \in R.$$

The Modal Logic K

Classes of frames induce modal logics; for now: all frames.
Axiomatization:

- *Propositional Reasoning*:
  - Modus ponens $\phi; \phi \to \psi / \psi$
  - all instances of propositional tautologies
- *Necessitation*

$$\frac{\phi}{\Box \phi}$$

- All instances of axiom

$$(K) \qquad \Box(\phi \to \psi) \to \Box\phi \to \Box\psi.$$

Soundness and Completeness

$\Phi \vdash \psi$ if $\phi_1 \wedge \cdots \wedge \phi_n \rightarrow \psi$ is derivable for some $\phi_1, \ldots, \phi_n \in \Phi$.

Soundness If $\Phi \vdash \psi$ then $\Phi \models \psi$ (Proof: Induction over $\vdash$)

(Strong) Completeness If $\Phi \models \psi$ then $\Phi \vdash \psi$

Reword: Φ *consistent* if $\Phi \nvdash \bot$

- Soundness: Φ satisfiable $\rightarrow$ Φ consistent
- (Strong) Completeness: Φ consistent $\rightarrow$ Φ satisfiable
  - Proof: Satisfy consistent Φ in *canonical model*, with worlds = maximally consistent sets.

Graded Modal Logic

(Fine 1972) *Count* successors:

$$x \models_M \Box_k \phi \iff |\{y \in X \mid (x,y) \in R \text{ and } y \models_M \neg\phi\}| \leq k$$
$$x \models_M \Diamond_k \phi \iff |\{y \in X \mid (x,y) \in R \text{ and } y \models_M \phi\}| > k.$$

(Note $\Box = \Box_0$, $\Diamond_0 = \Diamond$).

Alternative semantics: *Multigraphs* (D'Agostino/Visser 2002)

$$(X, \mu : X \times X \to \mathbb{N} \cup \{\infty\}, \pi)$$

$$x \models \Box_k \phi \iff \mu(x, [\![\neg\phi]\!]) \leq k$$
$$x \models \Diamond_k \phi \iff \mu(x, [\![\phi]\!]) > k.$$

Equivalent, i.e. makes the same (sets of) formulas satisfiable.

Axiomatizing Graded Modal Logic

(Fine 1972)

Propositional reasoning plus

$$\phi / \Box_0 \phi$$

$$\Box_0(\phi \to \psi) \to \Box_0 \phi \to \Box_0 \psi$$

$$\Diamond_k \phi \to \Diamond_l \phi \qquad (l < k)$$

$$\Diamond_k \phi \leftrightarrow \bigvee_{i=-1}^{k} (\Diamond_i(\phi \land \psi) \land \Diamond_{k-1-i}(\phi \land \neg \psi))$$

$$\Box_0(\phi \to \psi) \to \Diamond_k \phi \to \Diamond_k \psi$$

(with $\Diamond_{-1}\phi \equiv \top$)

Description Logics

…feature ingredients from graded modal logic:

$$\exists R \equiv \Diamond^R$$
$$\forall R \equiv \Box^R$$
$$\geq n\ R \equiv \Diamond^R_{n-1}$$
$$\leq n\ R \equiv \neg\Diamond^R_n$$

E.g.

$$\text{Elephant} = (= 2\ \text{hasPart}.\,\text{tusk}) \sqcap (= 1\ \text{hasPart}.\,\text{trunk}) \sqcap (= 4\ \text{hasPart}.\,\text{leg})$$

Probabilistic Modal Logic

Exchange $\mu : X \times X \to \mathbb{N} \cup \{\infty\}$ for *Markov Chain* (X, μ, π)

- $\mu(x, \cdot)$ *probability measure on X* for each $x \in X$

Operators L_p 'with probability $\geq p$' $\quad (p \in \mathbb{Q} \cap [0,1])$

$$x \models L_p \phi \iff \mu(x, [\![\phi]\!]) \geq p$$

Define 'with probability $\leq p$':

$$M_p = L_{1-p} \neg$$

E.g.

- Probabilistic concurrent systems: $\quad \text{finished} \vee M_{0.1} \text{error}$
- Uncertain belief: $\quad L_{1/2}^{\text{Draghi}} L_{1/3}^{\text{Yellen}} \text{Recession}$

(No) Compactness

- Logic $\mathcal{L}$ *compact* if every finitely satisfiable set of $\mathcal{L}$-formulas is satisfiable.
- Strong completeness implies compactness.

Probabilistic modal logic fails to be compact:

$$\{L_{p-1/n}a \mid n \in \mathbb{N}\} \cup \{\neg L_p a\}$$

is finitely satisfiable but not satisfiable.

Axiomatizing Probabilistic Modal Logic: The Problem

Have

$$L_p(\phi \wedge \psi) \wedge L_q(\phi \wedge \neg\psi) \rightarrow L_{p+q}\phi$$

but no reasonable converse

Axiomatizing Probabilistic Modal Logic: The Easy Part

$$L_o \phi$$

$$L_p \top$$

$$L_p \phi \to \neg L_q \neg \phi (p + q > 1)$$

$$\neg L_p \phi \to M_p \phi$$

$$\phi \to \psi / L_p \phi \to L_p \psi$$

Axiomatizing Probabilistic Modal Logic: The Hard Part

(Heifetz/Mongin 2001)

For $\phi = (\phi_1, \dots, \phi_m)$, $\psi = (\psi_1, \dots, \psi_n)$ put

$$\phi^{(k)} \equiv \bigvee_{1 \leq l_1 < \dots < l_k \leq m} (\phi_{l_1} \vee \dots \vee \phi_{l_k})$$

$$\phi \leftrightarrow \psi \equiv \bigwedge_{k=1}^{\max(m,n)} \phi^{(k)} \leftrightarrow \psi^{(k)}.$$

Rule (B):

$$\frac{(\phi_1, \dots, \phi_m) \leftrightarrow (\psi_1, \dots, \psi_n)}{\bigwedge_{i=1}^{m} L_{p_1} \phi_i \wedge \bigwedge_{j=2}^{n} M_{q_j} \psi_j \rightarrow L_{p_1 + \dots + p_m - q_2 - \dots - q_n} \psi_1}$$

Linear Inequalities

(Halpern/Fagin/Megiddo 1990) *n-ary modal operators*

$$\sum_{i=1}^{n} a_i l(\phi_i) \geq b \qquad (a_i, b \in \mathbb{Q})$$

interpreted by

$$[\![l(\phi)]\!]_x = \mu(x, [\![\phi]\!])$$

E.g.

$$l(\text{MunichChampion}) \geq 10 \cdot l(\text{NurembergChampion})$$

Same for graded logic $\rightarrow$ *Presburger modal logic* (Demri/Lugiez 2006):

$$3 \cdot \#_{\text{hasGame}}\text{won} + 1 \cdot \#_{\text{hasGame}}\text{tied} \geq 37 \rightarrow \neg\text{relegated}.$$

E.g. *majority logic* (Pacuit/Salame 2004)

$$W\phi = \#(\phi) \geq \#(\neg\phi)$$

Neighbourhood Semantics

Compositional semantics of $\Box$ over X requires

$$[\![\Box]\!] : 2^X \to 2^X$$

or, transposing,

$$\mathfrak{N} : X \to 2^{(2^X)}$$

– that's a *neighbourhood frame*:

$$A \text{ neighbourhood of } x \iff A \in \mathfrak{N}(x).$$

Then

$$x \models \Box\phi \iff [\![\phi]\!] \in \mathfrak{N}(x)$$

Axiomatizing Neighbourhood Logic

Propositional reasoning plus *replacement of equivalents:*

$$\frac{\phi \leftrightarrow \psi}{\Box\phi \leftrightarrow \Box\psi}$$

Monotone Neighbourhoods

Impose *monotonicity*

$$\frac{\phi \to \psi}{\Box\phi \to \Box\psi}$$

$\to$ *monotone neighbourhood frames*:

$$A \in \mathfrak{N}(x) \wedge A \subseteq B \to B \in \mathfrak{N}(x)$$

Game Logic

Monotonicity plus *seriality:*

$$\Box\top \qquad \Diamond\top$$

Semantically:

$$X \in \mathfrak{N}(x), \emptyset \notin \mathfrak{N}(x).$$

Temporalized multi-agent version: *game logic* (Parikh 1983)

$[a]\phi$ ‘Angel can enforce ϕ in game a’

$\langle a\rangle\phi$ ‘Demon can enforce ϕ in game a’

Conditional Logics

Binary operator $\cdot \Rightarrow \cdot$ for non-material implication,
e.g. ‘if – then normally’ (*default implication*):

$$\models (\text{Monday} \Rightarrow \text{work}) \not\to ((\text{Monday} \wedge \text{sick}) \Rightarrow \text{work})$$

(*non-monotonic conditional*)

Selection Function Semantics

Conditional frame $(X, (R_A), \pi)$:

$$R_A \subseteq X \times X \qquad (A \subseteq X)$$

xR_Ay 'at x, y is most typical for condition A'.

$\phi \Rightarrow \cdot$ is box over $R_{[\![\phi]\!]}$:

$$x \models \phi \Rightarrow \psi \quad \text{iff} \quad \forall y. (xR_\phi y \to y \models \psi).$$

The Conditional Logic *CK*

= The logic of all conditional frames

- replacement of equivalents on the left:

$$\frac{\phi \leftrightarrow \phi'}{(\phi \Rightarrow \psi) \leftrightarrow (\phi' \Rightarrow \psi)}$$

- *normality* on the right:

$$(\phi \Rightarrow (\psi \to \chi)) \to (\phi \Rightarrow \psi) \to (\phi \Rightarrow \chi).$$

Reasoning Principles = Frame Conditions

ID	$\phi \Rightarrow \phi$	$xR_\phi y \to y \models \phi$
CEM	$(\phi \Rightarrow \psi) \vee (\phi \Rightarrow \neg\psi)$	$xR_\phi y \wedge xR_\phi y' \to y = y'$
MP	$(\phi \Rightarrow \psi) \to \phi \to \psi$	$xR_\phi x$

KLM

(Burgess 1981; Kraus/Lehmann/Magidor 1990)

ID plus

DIS	$(\phi \Rightarrow \chi) \to (\psi \Rightarrow \chi) \to ((\phi \vee \psi) \Rightarrow \chi)$
CM	$(\phi \Rightarrow \chi) \to (\phi \Rightarrow \psi) \to ((\phi \wedge \chi) \Rightarrow \psi)$

Cautious Monotony:

$$(\text{Monday} \Rightarrow \text{work}) \to (\text{Monday} \Rightarrow \text{sick}) \to ((\text{Monday} \wedge \text{sick}) \Rightarrow \text{work})$$

Preferential Models

KLM complete for *preferential models* (X, R, π):

- $R \subseteq X^3$
- $Rxyz$: 'y more typical than z as an alternative to x'

$$Rxyz \to Rxyy$$
$$(Rxyz \wedge Rxzw) \to Rxyw$$

with

$$x \models \phi \Rightarrow \psi \quad \text{iff} \quad \forall y.(Rxyy \to \exists z.(Rxzy \wedge \forall t.(Rxtz \to M, t \models \psi))).$$

Coalition Logic / Alternating-Time Temporal Logic

- $N = \{1, \ldots, n\}$ set of *agents*
- *Concurrent game structure*: at each state x,
  - sets S_i^x of *moves* ($i \in N$)
  - *outcome function*

$$f_x : (\prod_{i \in N} S_i^x) \to X$$

- Operators $[C]$ 'coalition C can force'

$$x \models [C]\phi \iff \exists \sigma_C \in \prod_{i \in C} S_i^x . \forall \sigma_{N-C} \in \prod_{i \in N-C} S_i^x . f_x\langle \sigma_C, \sigma_{N-C} \rangle \models \phi .$$

Axiomatizing Coalition Logic

$$\neg[C]\bot$$

$$[C]\top$$

$$\neg[\emptyset]\neg\phi \to [N]\phi$$

$$[C](\phi \wedge \psi) \to [C]\phi$$

$$[C]\phi \wedge [D]\psi \to [C \cup D](\phi \wedge \psi) \quad \textit{if } C \cap D \neq \emptyset.$$

Summing up

- Modalities are just logical operators
- Broad range of syntactic and semantic phenomena
- Common features:
  - Models consist of *worlds* / *states*
  - Worlds are connected by *transitions*, broadly construed:
    - relational
    - weighted
    - probabilistic
    - game-based
    - proposition-indexed
    - proximity-based

Tomorrow:

- Everything is coalgebraic :)