MAGMAを利用したフュージョンスキームの探索

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1.Association Scheme

Let X be a finite set, $\{R_i\}_{0\leq i\leq d}$ a collection of binary relations of X. R_i is a subset of $X\times X$. Assume that $\{R_i\}_{0\leq i\leq d}$ is a partition of $X\times X$. We say that $\mathfrak{X}=(X,\{R_i\}_{0\leq i\leq d})$ is an *association scheme, if the following conditions are satisfied.*

1. *(i)* $R_0:=\{(x,x)|x\in X\}$,
2. *(ii)* $X\times X=R_0\cup R_1\cup\cdots\cup R_d$ *and* $R_i\cap R_j=\phi$ $(i\neq j)$,
3. *(iii) For* $i\in\{0,1,\ldots,d\}$, *there exists* $i'\in\{0,1,\ldots,d\}$ *such that* ${}^tR_i=R_{i'}$ *where* ${}^tR_i=\{(y,x)|(x,y)\in R_i\}$,
4. *(iv) For* $f,g,h\in\{0,1,\ldots,d\}$, $\exists p_{f,g}^h\in\mathbb{Z}_{\geq 0}$ *such that* $p_{f,g}^h=|\{z\in X|(x,z)\in R_f,(z,y)\in R_g\}|$ *(intersection number) whenever* $(x,y)\in R_h$.

Example. Johnson Scheme

Let M be a finite set $|M| = m$. For n $(0 < n \leq m/2)$, define a set $\binom{M}{n} = \{N \subset M \mid |N| = n\}$. For any $N_1, N_2 \in \binom{M}{n}$, we can define a Johnson distance $\rho(N_1, N_2) = n - |N_1 \cap N_2|$. Relations are defined $R_i = \{(N_1, N_2) \in \binom{M}{n}^2 \mid \rho(N_1, N_2) = i\}$. $\mathfrak{X} = (\binom{M}{n}, \{R_i\}_{0 \leq i \leq n})$ is an association scheme. We write $J(m, n)$.

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Adjacency matrices

For $i \in \{0, 1, \ldots, d\}$, we define the *adjacency matrix* A_i *of the* i*-th relation as follows.*

$$(A_i)_{x,y} = \begin{cases} 1 & (x, y) \in R_i, \\ 0 & \textit{other wise.} \end{cases}$$

1. *(i),* A_0 *is the identity matrix,*
2. *(ii),* $\sum_{i=0..d} A_i = J$*,*
3. *(iii),* $A_{i'} =^t A_i \in \{A_i\}_{0 \le i \le d}$*,*
4. *(iv), For* $f, g, h \in \{0, 1, \ldots, d\}$*,* $\exists p^h_{f,g} \in \mathbb{Z}_{\ge 0}$ *such that* $A_f A_g = \sum_{0 \le h \le d} p^h_{f,g} A_h$.

$A_f A_g = A_g A_f$ *for any* $f, g\ in \{R_i\}_{0 \le i \le d}$*,* $\mathfrak{X}$ *is called a commutative association scheme.*

Relation matrix

We use relation matrices for inputting association schemes.

$$R = \sum_{i=0}^{d} iA_i$$

example: Johnson scheme

$$(R)_{x,y} = \rho(x,y), x, y \in \binom{M}{n}.$$

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Character Table

Example: Johnson Scheme $J(11, 4)$

$$\begin{pmatrix} 1 & 28 & 126 & 140 & 35 \\ 1 & 17 & 27 & -25 & -20 \\ 1 & 8 & -9 & -10 & 10 \\ 1 & 1 & -9 & 11 & -4 \\ 1 & -4 & 6 & -4 & 1 \end{pmatrix}.$$

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Fusion schemes (subscheme)

Let $\mathfrak{X} := (X, \{R_i\})$, $\mathfrak{Y} := (X, \{S_j\})$ be two association schemes. The scheme $\mathfrak{Y}$ is called a fusion scheme of $\mathfrak{X}$, iff each relations of $\mathfrak{Y}$ is a union of some relations from $\mathfrak{X}$.

Example : Fusion scheme of J(11,4)

Character Table of $J(11,4)$

$$\begin{pmatrix} 1 & 28 & 126 & 140 & 35 \\ 1 & 17 & 27 & -25 & -20 \\ 1 & 8 & -9 & -10 & 10 \\ 1 & 1 & -9 & 11 & -4 \\ 1 & -4 & 6 & -4 & 1 \end{pmatrix}.$$

Fusion scheme of $J(11,4)$ is
$< A_0, A_1 + A_4, A_2 + A_3 >=< E_0, E_1 + E_3 + E_4, E_2 >$.

Known Results

Example

1. $J(2n, n)$: class 2 and class $\frac{n}{2}+1$(n: even), $\frac{n+1}{2}$(n: odd),
2. $J(2n+1, n)$: class $\frac{n}{2}$(n:even), $\frac{n+1}{2}$(n:odd),
3. sporadic: $J(10, 3)$, $J(11, 4)$, $J(12, 4)$,$J(13, 6)$ class 2.
4. $J(14, 7)$ class 3(new?).

Theorem.[Muzychuk]
If $m \geq 3n+4 \geq 13$, then $J(m, n)$ has no non-trivial fusion scheme.

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Improved program

Fact1. Johnson Scheme is P-polynomial, so $R_1 \in \{S_j\}_{0 \le j \le d'}$ implies $\mathfrak{Y} = \mathfrak{X}$.

Fact2. If a subscheme $\mathfrak{Y} = (X, \{S_j\}_{0 \le j \le d'})$ of $\mathfrak{X} := (X, \{R_i\})$ exists, for any S_{j_1},S_{j_2},$S_{j_3} \in \{S_j\}$,

$$\sum_{i \in R_i \subset S_{j_1}, j \in R_j \subset S_{j_2}} p_{i,j}^{k_1} = \sum_{i \in R_i \subset S_{j_1}, j \in R_i \subset S_{j_2}} p_{i,j}^{k_2},$$

$\forall k_1, k_2 \in S_{j_3}$.