ON BUTSON-TYPE HADAMARD MATRICES

$H(17, 17)$

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An $n \times n$ matrix $H = (H_{ij})$ is called an $H(p, n)$-matrix if $H_{ij}^p = 1$ and $HH^CT = nl$ where $I$ is the $n \times n$ identity matrix.

$\implies$ An $H(p, n)$-matrix is called a Butson-type Hadamard matrix.

**Basic Properties for $H(p, n)$-matrix $H$**

1. A permutation of the rows (columns) of $H$ is permissible.

2. A multiplication of the elements of a row (column) of $H$ by a fixed $p$th root of unity is permissible.
**DIFFERENCE MATRICES**

\[ H : \text{an } H(p, n) \text{-matrix} \]

\[ \implies \text{A DIFFERENCE MATRIX } \text{Diff}(H) \in \text{Mat}_{n \times n}(\mathbb{Z}_p) \text{ of } H: \]

\[ H = (H_{ij}) = \left( \exp\left(\frac{2\pi \sqrt{-1} E_{ij}}{p}\right) \right) \iff \text{Diff}(H) = (E_{ij}) \]

**EXAMPLE :** \( H \) is an \( H(3, 3) \)-matrix

\[
H = \begin{pmatrix}
1 & 1 & 1 \\
1 & e^{2\pi i/3} & e^{4\pi i/3} \\
1 & e^{4\pi i/3} & e^{2\pi i/3}
\end{pmatrix} \iff \text{Diff}(H) = \begin{pmatrix}
0 & 0 & 0 \\
0 & 1 & 2 \\
0 & 2 & 1
\end{pmatrix}
\]
The Fourier matrix $F$ of order $p$ is a $p \times p$ matrix such that

$$\text{Diff}(F) = (ij)^{p-1}_{i,j=0} = \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 \\
0 & 1 & 2 & \cdots & p-1 \\
0 & 2 & 4 & \cdots & 2(p-1) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & p-1 & 2(p-1) & \cdots & (p-1)^2
\end{pmatrix}$$

$\Rightarrow F$ is indeed an $H(p, p)$-matrix.
We will consider ONLY $H(p, p)$-matrix $H$ where $p$ is prime.

For two distinct rows of $H$

\[(\omega_1, \cdots, \omega_p) \perp (\eta_1, \cdots, \eta_p)\]

\[\omega_1 \cdot \eta_1 + \cdots + \omega_p \cdot \eta_p = 0\]

For two distinct rows of $\text{Diff}(H)$

\[(E_{i1}, \cdots, E_{ip}) \perp (E_{j1}, \cdots, E_{jp})\]

\[\{E_{i1} - E_{j1}, \cdots, E_{ip} - E_{jp}\} = \mathbb{Z}_p\]
Fix $i$ and $j$. For $0 \leq a < i$ and $0 \leq b < j$,

$$\text{Diff}(H) = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & E_{ab} & \cdot & E_{aj} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & E_{ib} & \cdot & E_{ij} \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

$$\implies E_{ib} - E_{ab} \neq E_{ij} - E_{aj}.$$
Some Redundancy for Calculation

Our initial part for $\text{Diff}(H)$ is symmetric:

$$\text{Diff}(H) = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 1 & 2 & 3 & 4 & \cdots \\
0 & 2 & E_{22} & E_{23} & E_{24} & \cdots \\
0 & 3 & E_{32} & E_{33} & E_{34} & \cdots \\
0 & 4 & E_{42} & E_{43} & E_{44} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}$$

Symmetric Redundancy

We may assume that $E_{23} \leq E_{32}$.
For $p = 17$ we use a parallel algorithm on super computer with MPI (Massage Passing Interfaces) over TATARA FUJITSU PRIMERGY CX400 (in Kyushu Univ).

**COMPUTATIONAL RESULT**

There is a unique $H(17, 17)$-matrix, namely, the Fourier matrix of order 17.

- This calculation is done by super computer with 64 threads over computational complexity = 68 hours.