Remarks on quantum interaction models by Lie theory and modular forms via non-commutative harmonic oscillators

Masato Wakayama

Abstract As typically the *quantum Rabi model*, particular attention has been paid recently to studying the spectrum of self-adjoint operators with non-commutative coefficients, not only in mathematics but also in theoretical/experimental physics, e.g. aiming at an application to quantum information processing. The *non-commuta -tive harmonic oscillator* (NcHO) is a self-adjoint operator, which is a generalization of the harmonic oscillator, having an interaction term. The Rabi model is shown to be obtained by a second order element of the universal enveloping algebra of the Lie algebra $\mathfrak{sl}_2$, which is arising from NcHO through the oscillator representation. Precisely, an equivalent picture of the model is obtained as a confluent Heun equation derived from the Heun operator defined by that element via another representation. Though the spectrum of NcHO is not fully known, it has a rich structure. In fact, one finds interesting arithmetics/geometry described by e.g. elliptic curves, modular forms in the study of the spectral zeta function of NcHO. In this article, we draw this picture, which may give a better understanding of interacting quantum models.

Keywords Eichler integral, Heun ODE, non-commutative harmonic oscillator, oscillator representation, Rabi model, spectral zeta function, universal enveloping algebra, zeta regularization.

1 Introduction

The *non-commutative harmonic oscillator* Q (NcHO) is a parity-preserving (or possessing $\mathbb{Z}_2$ symmetry) differential operator introduced in [29, 30] as

$$Q := A\Big(-\frac{1}{2}\frac{d^2}{dx^2}+\frac{1}{2}x^2\Big)+B\Big(x\frac{d}{dx}+\frac{1}{2}\Big),$$

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where A is positive definite symmetric and B is skew-symmetric ($A, B \in \mathrm{Mat}_2(\mathbb{R})$). We assume the Hermitian matrix $A + iB$ is positive definite, i.e. $\det(A) > \mathrm{pf(B)}^2$. The former requirement arises from the formal self-adjointness of Q relative to the inner product on $\mathbb{C}^2 \otimes L^2(\mathbb{R})$. The latter condition guarantees that the eigenvalues of Q are all positive and form a discrete set with finite multiplicity. As a normalized form we may take $A = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ (The assumption is $\alpha\beta > 1$). It should be noted that, when $\alpha = \beta$, Q is unitarily equivalent to a couple of quantum harmonic oscillators, whence the eigenvalues are easily calculated as $\left\{\sqrt{\alpha^2-1}\left(n+\frac{1}{2}\right) \,|\, n \in \mathbb{Z}_{\geq 0}\right\}$ having multiplicity 2 ([30], I). Actually, when $\alpha = \beta$, behind Q, there exists a structure corresponding to the tensor product of the 2-dimensional trivial representation and the oscillator representation (e.g. [6]) of the Lie algebra $\mathfrak{sl}_2$. The clarification of the spectrum in the general $\alpha \neq \beta$ case is, however, considered to be highly non-trivial. Indeed, while the spectrum is well described theoretically by using certain continued fractions [29, 30] and also by Heun's ordinary differential equation (the second order Fuchsian differential equation with four regular singular points in a complex domain, [23, 35]), and in particular there are some results related to the estimate of upper bound of the lowest eigenvalue and distribution of eigenvalues [22, 13, 26, 9], only very little information is available in reality when $\alpha \neq \beta$ (see [27] and references therein. Figure1 represents a numerical graph for the spectrum for the ratio β/α. Note that the eigenvalue curves are continuous w.r.t. the parameter β/α [22]). In fact, only quite recently, it was proved that the multiplicity of each eigenvalue is always less than or equal to 2 by the monodromy representation of Heun's equations [35], and the ground state is simple (and even) [10] using the criterion given in [34] (see also [9]). Therefore, in spite of many studies, the spectral description of the NcHO is still incomplete (see [28] for an overview of the recent progress). One of the difficulties to obtain the eigenfunctions and eigenvalues is, representation theoretically, the apparent lack of an operator which commute with Q (second conserved quantity) besides the Casimir operator, the image of the generator of the center $\mathscr{Z}\mathscr{U}(\mathfrak{sl}_2)$ of the universal enveloping algebra of the Lie algebra $\mathfrak{sl}_2$. (Moreover, it has been shown that there is no annihilation/creation operators associated to NcHO when $\alpha \neq \beta$ [28].)

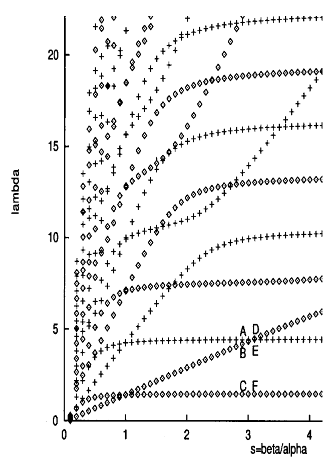

Fig. 1 Approximate N-th eigenvalues $\hat{\lambda}_N$ of Q [22].

Recently, however, particular attention has been paid to studying the spectrum of self-adjoint operators with non-commutative coefficients, in other words, interacting quantum systems, like the *quantum Rabi model* [31, 20, 3, 21, 38], the *Jaynes-Cumming (JC) model* etc., not only in mathematics (e.g. [8]) but also in theoretical physics and experimental physics founded e.g. in the book by Haroche & Raimond [5] (also [39]). For instance, the quantum Rabi model (1937) is known to be the simplest model used in quantum optics to describe interaction of light and matter beyond the harmonic oscillator, and the JC model is the widely studied rotating-wave approximation of the Rabi model (see e.g. [4]).

The quantum Rabi model is defined by the Hamiltonian

$$H_{\text{Rabi}}/\hbar = \omega a^{\dagger} a + \Delta \sigma_z + g\sigma_x (a^{\dagger} + a).$$

Here $a = (x + \partial_x)/\sqrt{2}$ (resp. $a^{\dagger} = (x - \partial_x)/\sqrt{2}$) is the annihilation (resp. creation) operator for a bosonic mode of frequency ω, $\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$, $\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ are the Pauli matrices for the two-level system, 2Δ is the energy difference between the two levels, and g denotes the coupling strength between the two-level system and the bosonic mode. The Rabi model considers a two-level atom coupled to a quantized, single-mode harmonic oscillator (in the case of light, this could be a photon in a cavity, as in Figure 2 [33]). Introduced over 70 years ago [31], its applications range from quantum optics, magnetic resonance to solid state and molecular physics. Very recently, the model applies to a variety of physical systems, including cavity quantum electrodynamics, the interaction between light and trapped ions or quantum dots, and the interaction between microwaves and superconducting qubits. Although this model has had an impressive impact on many fields of physics [5],

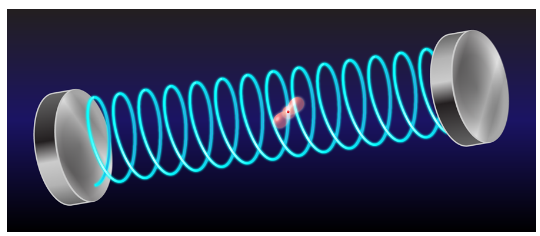

Fig. 2 Courtesy of APS/Alan Stonebraker in E. Solano Physics 4, 68 (2011): The Rabi model describes the simplest interaction between quantum light and matter. The model considers a two-level atom coupled to a quantized, single-mode harmonic oscillator.

only recently (in 2011) could this model be declared solved by D. Braak [3]. It is now pointed out [33] that as physicists gain intuition for Braak's mathematical solution, it is very much expected that the results could have implications for further theoretical and experimental work that explores the interaction between light and matter, from weak to extremely strong interactions.

The NcHO has been similarly expected to provide one of these Hamiltonians describing such quantum interacting systems. In this article, we will observe that the quantum Rabi model is obtained by a second order element $\mathscr{R}$ of the universal enveloping algebra $\mathscr{U}(\mathfrak{sl}_2)$, which is arising from the NcHO through the oscillator representation π' of the Lie algebra $\mathfrak{sl}_2$, and a confluence procedure for Heun's equation under another representation π'_a. Roughly speaking, the quantum Rabi model can be obtained by a confluence process by a "certain rescaling" of the NcHO through their respective Heun's pictures:

$$\begin{array}{ccccc} \text{NcHO} & \xleftarrow{\ \pi'\ } & \underset{\mathscr{U}(\mathfrak{sl}_2)}{\overset{\mathscr{R}}{\cap}} & \xrightarrow[\pi'_a(\cong\varpi_a)]{\mathscr{L}_a} & \text{Heun ODE} \\ & & & & \Big\Downarrow \text{confluence process} \\ & & & & \text{Confluent Heun ODE} \sim \text{Rabi model.} \end{array}$$

2 Number theoretic structure of NcHO

Although the explicit eigenvalues of Q are not known, the spectrum of Q possesses a very rich mathematical structure. Denote the (repeated) eigenvalues of Q by $0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \ldots (\to \infty)$. Define the spectral zeta function of Q by

$$\zeta_Q(s) = \sum_{n=1}^{\infty} \lambda_n^{-s}.$$

This series is absolutely convergent and defines a holomorphic function in s in the region $\mathrm{Re}(s) > 1$. The function $\zeta_Q(s)$ is analytically continued to the whole complex plane $\mathbb{C}$ as a single-valued meromorphic function that is holomorphic, except for a simple pole at $s = 1$ with residue $\frac{\sqrt{\alpha+\beta}}{\sqrt{\alpha\beta(\alpha\beta-1)}}$ [11]. It is notable that $\zeta_Q(s)$ has 'trivial zeros' at $s = 0, -2, -4, \ldots$. When $\alpha = \beta (> 1)$, $\zeta_Q(s)$ is identified (by a elementary holomorphic factor) with the Riemann zeta function $\zeta(s)$.

Similarly to the Apéry numbers which were introduced in 1978 by R. Apéry for proving the irrationality of $\zeta(2)$ and $\zeta(3)$ (see, e.g. [2]), *Apéry-like numbers* have been introduced in [12] for the description of the special values $\zeta_Q(2)$ and $\zeta_Q(3)$. These Apéry-like numbers $J_2(n)$ and $J_3(n)$ share with many of the properties of the original Apéry numbers, e.g. recurrence equations, congruence properties, etc (see [12, 15], also [25]). Actually, the Apéry-like numbers $J_2(n)$ for $\zeta_Q(2)$ obtain a remarkable modular form interpretation, as that shown by F. Beukers [2] in the case of the Apéry numbers. We have shown in [16] that the differential equation satisfied by the generating function $w_2(t)$ of $J_2(n)$ is the Picard-Fuchs equation for the universal family of elliptic curves equipped with rational 4-torsion: $\Omega_{AL} w_2(t) = 0$. The parameter t of this family is regarded as a modular function for the congruence subgroup $\Gamma_0(4) (\cong \Gamma(2)) \subset SL_2(\mathbb{Z})$. Moreover, one observes ([16]) that $w_2(t)$ is considered as a $\Gamma_0(4)$ meromorphic modular form of weight 1 in the variable τ as the classical Legendre modular function $t(\tau) = -\frac{\theta_4(\tau)^2}{\theta_4(\tau)^4}$. We also remark that the modular form $w_2(t)$ can be found at #19 in the list of [37].

The formulas of the special values $\zeta_Q(k)$ for the general cases $k \geq 4$ are much more complicated than those of $k = 2,3$. Thus, we will focus only on the *first anomaly* $R_{k,1}(x)$ (in the terminology by Kimoto) which expresses the 1st order difference (in a suitable sense) of $\zeta_Q(k)$ from $\zeta(k)$ with respect to the parameters α, β [17, 18]. The first anomaly $R_{k,1}(x)$ for $x = 1/\sqrt{\alpha\beta - 1}$ describes the special value $\zeta_Q(k)$ partly. (When $k = 2,3$, $R_{k,1}(x)$ possesses full information of each special value.) The Taylor expansion of $R_{k,1}(x)$ in x yields k-th Apéry-like numbers $J_k(n)$. Then, remarkably, one can show that the generating function $w_k(t)$ of $J_k(n)$ satisfies an inhomogeneous differential equation whose homogeneous part is given by the same Fuchsian differential operator which annihilates $w_2(t)$ as $\Omega_{AL} w_k(t) = w_{k-2}(t)$.

In order to solve this differential equation for $w_4(t)$, it is necessary to "integrate twice" a certain explicitly given modular form. Then one can prove that the generating function $w_4(t)$ can be expressed as a differential of a *residual modular form* multiplied by a modular form (a product and quotient of theta functions) for $\Gamma(2)$. The notion of residual modular forms is a generalization of the Eichler (or automorphic) integral. Note that the Abelian integrals and the Eisenstein series $E_2(\tau)$ of weight 2 for $SL_2(\mathbb{Z})$ are special examples of the Eichler integral. The name "residual" comes from the following two facts.

- Eichler's integral possesses an "integral constant" given by a polynomial in τ, which is known as a period function and computed as residues of the integral when one performs the inverse Mellin transform of L-function of the corresponding modular form.
- To obtain another meaningful expression of such Eichler's integral, we define *differential Eisenstein series* by a derivative of the analytic continuation of generalized Eisenstein series (e.g. [1]) at negative integer points.

We remark that the "residual part" of a differential Eisenstein series is in general given by a rational function in τ, whence it can not be handled in a framework of Eichler integrals. Moreover, one should note that only $w_4(t)$ one can give its explicit expression by a sum of two such differential Eisenstein series [17]. Furthermore, to understand the structure, especially the dimension of a space of residual modular forms, it is important to consider the Eichler cohomology groups [7] associated with several $\Gamma(2)$-modules made by a set of certain functions on the Poincaré upper half plane, such as the space (field) of rational functions $\mathbb{C}(\tau)$, the space of holomorphic/meromorphic functions with some decay condition at the infinity (cusps), etc. In the course of this analysis, we focus on a particular subgroup of the Eichler cohomology group, which we call a periodic cohomology, for the explicit determination of the space of residual modular forms which contains $w_4(t)$. We leave the detailed discussion about this arithmetic study of NcHO to the paper [17] (also [18]).

3 Quantum Rabi and Jaynes-Cummings models

The Hamiltonian of the Rabi model ($\hbar = 1$) reads

$$H_{\text{Rabi}} = \omega a^\dagger a + \Delta \sigma_z + g(\sigma^+ + \sigma^-)(a^\dagger + a),$$

with $\sigma^\pm = (\sigma_x \pm i\sigma_y)/2$. It is regarded as exactly solvable only since the work of [3]. The simpler related model defines the JC Hamiltonian

$$H_{\text{JC}} = \omega a^\dagger a + \Delta \sigma_z + g(\sigma^+ a + \sigma^- a^\dagger).$$

It is known to be integrable, even in the Dicke version with n two-state atoms. Actually, unlike in the Rabi case, the operator

$$\mathscr{I} := a^\dagger a + \frac{1}{2}(\sigma_z + 1)$$

commutes with the Hamiltonian H_{JC} and leads to the solvability of the JC-model. The conservation (i.e. invariance w.r.t. H_{JC}) of $\mathscr{I}$ signifies that the state space decomposes into an infinite sum of two-dimensional invariant subspaces. Each eigenstate of H_{JC} is then labeled by (the eigenstates of $\mathscr{I}$) $0, 1, 2, \ldots$ with a two-valued index, e.g. $+$ and $-$, denoting a basis vector in the two-dimensional subspace which belongs to the eigenspace of $\mathscr{I}$. Representation theoretically, the conserved quantity $\mathscr{I}$ generates a continuous $U(1)$ symmetry of the JC-model which is broken down to $\mathbb{Z}_2$ in the Rabi model due to the presence of the term $(\sigma^+ + \sigma^-)(a^\dagger + a)$. This residual $\mathbb{Z}_2$ symmetry, usually called parity, leads to a decomposition of the state space into just two subspaces $\mathscr{H}_\pm$, each with infinite dimension. Hence the Rabi model shares a similar situation as NcHO.

By [3], one knows that the spectrum of H_{Rabi} consists of two parts, the regular and the exceptional (degenerate) spectrum. Almost all eigenvalues are regular and given by the zeros of the transcendental functions $G_\pm(x)$ in the variable x. The functions $G_\pm(x)$ are defined through the power series in the coupling constant g as

$$G_\pm(x) = \sum_{n=0}^{\infty} K_n(x) \Big[1 \mp \frac{\Delta}{x - n\omega} \Big] \Big(\frac{g}{\omega} \Big)^n,$$

where the coefficients $K_n(x)$ are defined recursively,

$$nK_n(x) = f_{n-1}(x) K_{n-1}(x) - K_{n-2}(x),$$

with the initial condition $K_0 = 1$, $K_1(x) = f_0(x)$, and

$$f_n(x) = \frac{2g}{\omega} + \frac{1}{2g} \Big(n\omega - x + \frac{\Delta^2}{x - n\omega} \Big).$$

The function $G_\pm(x)$ is meromorphic in x having simple poles at $x = 0, \omega, 2\omega, \ldots$ (essentially the eigenvalues of harmonic oscillator). Then the regular energy spectrum of the Rabi model in each invariant subspace $\mathscr{H}_\pm$ with parity $\pm$ is given by the zeros of $G_\pm(x)$: for all zeros $x_n^\pm$ of $G_\pm(x)$, the nth eigenenergy with parity $\pm$ reads $E_n^\pm = x_n^\pm - g^2/\omega$. All exceptional eigenvalues E have the form $E_n^{deg} = n\omega - g^2/\omega$, and the necessary and sufficient condition for the occurrence of the eigenvalue E_n^{deg}

reads $K_n(n\omega) = 0$, which furnishes a condition on the model parameters g and $|\Delta|$. Actually, we have the following interesting result due to Kus [20].

We will assume $\omega = 1$ (without loss of generality) in the sequel of this article.

Lemma 1. *Let $P_k^{(n)}(x,y)$ be the polynomial of two variables defined by the following recursion formula:*

$$P_0^{(n)} = 1, \quad P_1^{(n)} = x+y-1,$$
$$P_k^{(n)} = (kx+y-k^2)P_{k-1}^{(n)} - k(k-1)(n-k+1)xP_{k-2}^{(n)}$$

If $P_n^{(n)}((2g)^2, \Delta^2) = 0$, then there exist two linearly independent eigenfucntions $\psi_n^{\pm}$ ("positive and negative parity") of H_{Rabi} corresponding to the eigenvalue $E_n^{deg} = n - g^2$, that is, the multiplicity of E_n^{deg} is 2. $\square$

Remark 1. The eigenfunctions $\psi_n^{\pm}$ are constructed in [20]. Also, if $0 < \Delta < 1$ there exist exactly n distinct positive roots of $P_n^{(n)}(x, \Delta^2)$ as a polynomial in x [20]. Similar polynomials for the NcHO [35] should be well formulated as $P_k^{(n)}$.

The analysis on the Rabi model above have been extensively using the Bargmann representation of bosonic operators which is realized by the following Bargmann transform $\mathscr{B}$ (from real coordinate x to complex variable z).

$$(\mathscr{B}f)(x) = \sqrt{2}\int_{-\infty}^{\infty} f(x)e^{2\pi xz - \pi x^2 - \frac{\pi}{2}z^2}dx.$$

Here the Bargmann space is by definition a Hilbert space of entire functions equipped with the inner product

$$(f|g) = \frac{1}{\pi}\int_{\mathbb{C}} \overline{f(z)}g(z)e^{-|z|^2}d(\mathrm{Re}(z))d(\mathrm{Im}(z)).$$

The main advantage is simply due to the fact; $a^{\dagger} = (x - \partial_x)/\sqrt{2} \to z$ and $a = (x + \partial_x)/\sqrt{2} \to \partial_z$. This makes the Rabi model to be a matrix-valued first order differential operator. The same situation, however, does not appear for NcHO. This explains one of the reasons that the analysis for NcHO is rather difficult and very likely richer.

Now we consider the spectral (Hurwitz type) zeta function of the Rabi model as

$$\zeta_{\mathrm{Rabi}}(s,z) = \sum_{\lambda \in \mathrm{Spec}(H_{\mathrm{Rabi}})} (z-\lambda)^{-s},$$

where the sum runs over all eigenvalues λ of the Rabi model (counted with multiplicity). As in the case of the spectral zeta function $\zeta_Q(s)$ of NcHO, one can easily prove that the sum converges absolutely and uniformly on compacts in the right hall plane $\mathrm{Re(s)} > 1$ so that it defines an analytic function in this region. Then, as in the case of $\zeta_Q(s)$, one can naturally expect that $\zeta_{\mathrm{Rabi}}(s,z)$ has a meromorphic continuation to the whole complex plane $\mathbb{C}$, in particular meromorphic at $s = 0$. If we may assume that $\zeta_{\mathrm{Rabi}}(s,z)$ is holomorphic at $s=0$, we define the zeta regularized product by

$$\prod_{\lambda\in\mathrm{Spec}(H_{\mathrm{Rabi}})}(z-\lambda):=\exp\big(-\frac{d}{ds}\zeta_{\mathrm{Rabi}}(0,z)\big).$$

(Notice that a zeta regularized product is identified with a usual product when the defining series is finite. Moreover, even if $\zeta_{\mathrm{Rabi}}(s,z)$ is not holomorphic at $s=0$, one may still define the zeta regularized product similarly. See [14, 19] and the references therein for zeta regularizations.) It is known that the function $\prod_{\lambda\in\mathrm{Spec}(H_{\mathrm{Rabi}})}(z-\lambda)$ is an entire function whose zeros are exactly given by the λ's. Then, the following claim follows naturally from the results of Braak [3] and Kus [20] above.

Conjecture 1. Let $\prod_{E_n^{deg}\in\mathrm{Spec}(H_{\mathrm{Rabi}})}(z-n)$ be the zeta regularized product defined by the series $\sum_{E_n^{deg}\in\mathrm{Spec}(H_{\mathrm{Rabi}})}(z-E_n^{deg})^{-s}$, where $E_n^{deg}=n-g^2$ denotes the doubly degenerate eigenvalue of the Rabi Hamiltonian. Then, there is a non-zero entire function $C(z)$ such that the following holds.

$$\prod_{\lambda\in\mathrm{Spec}(H_{\mathrm{Rabi}})}(z-\lambda-g^2)=C(z)\Gamma(z)^{-2}G_+(z)G_-(z)\prod_{E_n^{deg}\in\mathrm{Spec}(H_{\mathrm{Rabi}})}(z-n)^2.$$

Remark 2. It is important to study common zeros of the polynomials $P_n^{(n)}(x,\Delta^2)$. Also, the proof should be done by an analytic continuation of the (Hurwitz) spectral zeta function of the Rabi model in an explicit manner.

4 Lie algebraic description

To draw the picture more precisely we recall the representation theoretic setting. Let H,E and F be the standard generators of $\mathfrak{sl}_2$ defined by

$$H=\begin{bmatrix}1&0\\0&-1\end{bmatrix},\quad E=\begin{bmatrix}0&1\\0&0\end{bmatrix},\quad F=\begin{bmatrix}0&0\\1&0\end{bmatrix}.$$

They satisfy the commutation relations

$$[H,E]=2E,\ [H,F]=-2F,\ [E,F]=H.$$

For the triplet $(\kappa,\varepsilon,\nu)\in\mathbb{R}^3_{>0}$, define a second order element $\mathscr{R}$ of the universal enveloping algebra $\mathscr{U}(\mathfrak{sl}_2)$ of $\mathfrak{sl}_2$ by

$$\mathscr{R}:=\frac{2}{\sinh 2\kappa}\Big\{\Big[(\sinh 2\kappa)(E-F)-(\cosh 2\kappa)H+\nu\Big](H-\nu)+(\varepsilon\nu)^2\Big\}.$$

Let us consider the representation $(\pi',\mathbb{C}[y])$ of $\mathfrak{sl}_2$ given by

$$\pi'(H) = y\partial_y + 1/2,\ \pi'(E) = y^2/2,\ \pi'(F) = -\partial_y^2/2.$$

Define an inner product on $\mathbb{C}[y]$ by $(f,g)_F = \sqrt{\pi}(f(\partial_y)\bar{g}(y))|_{y=0}\,(f,g \in \mathbb{C}[y])$. Then $(y^m, y^n)_F = \delta_{m,n}\sqrt{\pi}n!$. If we denote by $\overline{\mathbb{C}[y]}$ the completion of $\mathbb{C}[y]$ w.r.t. this inner product, then it is shown that the representation $(\pi', \overline{\mathbb{C}[y]})$ is unitarily equivalent to the oscillator representation of $\mathfrak{sl}_2$ realized on the Hilbert spaces $L^2(\mathbb{R})$.

The following lemma follows immediately from [23] (Corollary 9 with Lemma 8), which translates the eigenvalue problem of Q into a single differential equation.

Lemma 2. *Assume $\alpha \neq \beta$ ($\alpha\beta > 1$). Determine the triplet $(\kappa, \varepsilon, \nu) \in \mathbb{R}^3_{>0}$ by the formulas*

$$\cosh\kappa = \sqrt{\frac{\alpha\beta}{\alpha\beta-1}},\ \ \sinh\kappa = \frac{1}{\sqrt{\alpha\beta-1}},\ \ \varepsilon = \Big|\frac{\alpha-\beta}{\alpha+\beta}\Big|,\ \ \nu = \frac{\alpha+\beta}{2\sqrt{\alpha\beta(\alpha\beta-1)}}\lambda.$$

Then the eigenvalue problem $Q\varphi = \lambda\varphi$ ($\varphi \in L^2(\mathbb{R},\mathbb{C}^2)$) is equivalent to the equation $\pi'(\mathscr{R})u = 0$ ($u \in \overline{\mathbb{C}[y]}$).

Remark 3. Notice that $\pi'(\mathscr{R})$ is a third order differential operator. The correspondence $\varphi \leftrightarrow u$ in the lemma above can be given explicitly. Remark also that the recurrence equation (or its corresponding continued fraction) in [30] is equivalent to this third order differential equation.

4.1 Intertwiners arising from Laplace transforms

In order to obtain a complex analytic picture of the equation $\pi'(\mathscr{R})u = 0$ in Lemma 2 and observe a connection between NcHO and the Rabi model through Heun ODE, we introduce two representations of $\mathfrak{sl}_2$.

Let $a \in \mathbb{N}$. Define first the operator T_a acting on the space of Laurent polynomials $\mathbb{C}[y, y^{-1}]$ (or $y^2\mathbb{C}[y]$) by

$$T_a := -\frac{1}{2}\partial_y^2 + \frac{(a-1)(a-2)}{2}\cdot\frac{1}{y^2}.$$

Define a modified Laplace transform $\mathscr{L}_a$ by

$$(\mathscr{L}_a u)(z) := \int_0^\infty u(yz)e^{-\frac{y^2}{2}}y^{a-1}dy.$$

Then, one finds that

$$(\mathscr{L}_a T_a u)(z) = \big(-\frac{1}{2z}\partial_z + \frac{a-1}{2z^2}\big)(\mathscr{L}_a u)(z) + \frac{1}{2z}u'(0)\delta_{a,1} - \frac{a-1}{2z^2}u(0)\delta_{a,2},$$

where $\delta_{a,k} = 1$ when $k = a$ and 0 otherwise. This can be true whenever $u(0), u'(0)$ and $(\mathscr{L}_a u)(z)$ exist.

We now define a representation π'_a of $\mathfrak{sl}_2$ on $y^{a-1}\mathbb{C}[y]$ by

$$\pi'_a(H) = \pi'(H),\ \pi'_a(E) = \pi'(E),\ \pi'_a(F) = T_a = \pi'(F) + \frac{(a-1)(a-2)}{2}\cdot\frac{1}{y^2}.$$

Moreover, introduce another representation of $\mathfrak{sl}_2$ on $\mathbb{C}[z,z^{-1}]$ by

$$\varpi_a(H) = z\partial_z + \frac{1}{2},\ \varpi_a(E) = \frac{1}{2}z^2(z\partial_z + a),\ \varpi_a(F) = -\frac{1}{2z}\partial_z + \frac{a-1}{2z^2}.$$

Then one easily verifies the following.

Lemma 3. *Let $a \neq 1,2$. Then one has*

$$\mathscr{L}_a\pi'_a(X) = \varpi_a(X)\mathscr{L}_a\ (X \in \mathfrak{sl}_2).$$

Furthermore, when $a = 1$ (resp. $a = 2$) the restriction of $\mathscr{L}_1$ (resp. $\mathscr{L}_2$) to the space of even (resp. odd) functions turns out to be an intertwiner between two representations $\pi'(= \pi'_1)$ (resp. $= \pi'_2$)) and ϖ_1 (resp. ϖ_2).

Remark 4. Observe that $\mathscr{L}_a$ defines an isometry. For instance, assume $a = 1$. If $u(y) = \sum_{n=0}^{N} u_n y^n \in \mathbb{C}[y]$ then $(\mathscr{L}_1 u)(z) = \frac{1}{\sqrt{2}}\sum_{n=0}^{N} u_n \Gamma(\frac{n+1}{2})(\sqrt{2}z)^n$. Moreover, if one defines the inner product in z-space such that $\{z^n \,|\, n \in \mathbb{N}\}$ forms an orthogonal basis and $(z^n, z^n)_1 = \frac{2\Gamma(\frac{n}{2}+1)}{\Gamma(\frac{n+1}{2})}$, then $\mathscr{L}_1$ is an isometry. The others are similar.

Since $\varpi_a(E)z^{-a} = 0$, one has the following second equivalence: the representation $(\pi'_a, y^{2-a}\mathbb{C}[y^2])$ can be considered as the Langlans quotient of the representations $(\varpi_a, \mathbb{C}[z^2, z^{-2}])$ or $(\varpi_a, z\mathbb{C}[z^2, z^{-2}])$ depending on the parity of a.

Lemma 4. *The operator $\mathscr{L}_a$ gives the equivalence of irreducible modules of $\mathfrak{sl}_2$:*

$$(\pi'_a, y^{a-1}\mathbb{C}[y^2]) \cong (\varpi_a, z^{a-1}\mathbb{C}[z^2]),$$
$$(\pi'_a, y^{2-a}\mathbb{C}[y^2]) \cong (\varpi_a, z^{a}\mathbb{C}[z^2, z^{-2}]/z^{-a}\mathbb{C}[z^{-2}]).$$

Moreover, the Casimir operator $Z_C := 4EF + H^2 - 2H \in \mathscr{Z}\mathscr{U}(\mathfrak{sl}_2)$ takes the value $(a-1)(a-2) - \frac{3}{4}$ in both representations $(\pi'_a, y^{a-1}\mathbb{C}[y^2])$ and $(\pi'_a, y^{2-a}\mathbb{C}[y^2])$.

Remark 5. There is a symmetry $a \leftrightarrow 3-a$ for π'_a. Actually, when $a \notin \mathbb{Z}$, there is a equivalence between two representations π'_a and π'_{3-a} in a suitable setting.

4.2 Heun differential operators

In this section, we follow the results from [35]. Recall the operator $\mathscr{R} \in \mathscr{U}(\mathfrak{sl}_2)$. Then, one observes

$$\varpi_a(\mathscr{R}) = \Big\{(z^2+z^{-2}-2\coth 2\kappa)(\theta_z+\frac{1}{2}) + (a-\frac{1}{2})(z^2-z^{-2}) + \frac{2\nu}{\sinh 2\kappa}\Big\}(\theta_z+\frac{1}{2}-\nu) + \frac{2(\varepsilon\nu)^2}{\sinh 2\kappa},$$

where $\theta_z = z\partial_z$. Hence, conjugating by z^{a-1} one obtains the following lemma.

Lemma 5. *For each integer a one has*

$$z^{-a+1}\varpi_a(\mathscr{R})z^{a-1} = \Big\{(z^2+z^{-2}-2\coth 2\kappa)(\theta_z+a-\frac{1}{2}) + (a-\frac{1}{2})(z^2-z^{-2}) + \frac{2\nu}{\sinh 2\kappa}\Big\}(\theta_z+a-\frac{1}{2}-\nu) + \frac{2(\varepsilon\nu)^2}{\sinh 2\kappa}. \square$$

Furthermore, notice that the operators $\varpi_a(H), \varpi_a(E)$ and $\varpi_a(F)$ are invariant under the symmetry $z \to -z$. This implies that the $\varpi_a(\mathscr{R})$ can be expressed in terms of the variable z^2. We therefore put $w := z^2\coth\kappa$. Using $z\partial_z = 2w\partial_w$ and the relations

$$\begin{aligned} &z^2+z^{-2}-2\coth 2\kappa = (\tanh\kappa)w^{-1}(w-1)(w-\coth^2\kappa),\\ &z^2-z^{-2} = (\tanh\kappa)w^{-1}(w^2-\coth^2\kappa),\\ &2/\sinh 2\kappa = (\tanh\kappa)(\coth^2\kappa-1),\end{aligned}$$

factoring out the leading coefficient of $\varpi_a(\mathscr{R})$ in its expression one obtains

Proposition 1. *The following relation holds:*

$$z^{-a+1}\varpi_a(\mathscr{R})z^{a-1} = 4(\tanh\kappa)\, w(w-1)(w-\coth^2\kappa)H^a(w,\partial_w),$$

where $H^a(w,\partial_w)$ is the Heun differential operator given as follows:

$$H^a(w,\partial_w) = \frac{d^2}{dw^2} + \left(\frac{3-2\nu+2a}{4w} + \frac{-1-2\nu+2a}{4(w-1)} + \frac{-1+2\nu+2a}{4(w-\coth^2\kappa)}\right)\frac{d}{dw} + \frac{\frac{1}{2}(a-\frac{1}{2})(a-\frac{1}{2}-\nu)w - q_a}{w(w-1)(w-\coth^2\kappa)}.$$

Here the accessory parameter q_a is given by

$$q_a = \Big\{-(a-\frac{1}{2}-\nu)^2 + (\varepsilon\nu)^2\Big\}(\coth^2\kappa-1) - 2(a-\frac{1}{2})(a-\frac{1}{2}-\nu).$$

5 Heun operators' description for NcHO

The equivalence between the spectral problem of Q and the existence/non-existence of holomorphic solutions of Heun ODE's in a certain complex domain is described in [23] for odd parity and in [35] for even parity. The proof follows from the following quasi-intertwining property of the operator $\mathscr{L}_j$ resulted from Lemma 3 and the realization of the representation ϖ_j.

Proposition 2. *The element $\mathscr{R} \in \mathscr{U}(\mathfrak{sl}_2)$ satisfies the following equations:*

$$(\mathscr{L}_1\pi'(\mathscr{R})u)(z) = \varpi_1(\mathscr{R})(\mathscr{L}_1 u)(z) + (\nu - \tfrac{3}{2})u'(0)z^{-1},$$
$$(\mathscr{L}_2\pi'(\mathscr{R})u)(z) = \varpi_2(\mathscr{R})(\mathscr{L}_2 u)(z) - (\nu - \tfrac{1}{2})u(0)z^{-2}.$$

In particular, the eigenvalue problem $Q\varphi = \lambda\varphi$ for the even and odd case is respectively equivalent to the equation

$\varpi_1(\mathscr{R})(\mathscr{L}_1 u)(z) = 0$ *(the even case)* and $\varpi_2(\mathscr{R})(\mathscr{L}_2 u)(z) = 0$ *(the odd case).* □

Noting that, for instance the even case,

$$\varpi_1(\mathscr{R}) = 4(\tanh\kappa)\, w(w-1)(w-\alpha\beta)H_\lambda^+(w,\partial_w),$$

one has the following by Proposition 2 ([35]). The odd case was obtained in [23].

Theorem 1. *There exist linear bijections:*

$$\text{Even}: \{\varphi \in L^2(\mathbb{R},\mathbb{C}^2) \,|\, Q\varphi = \lambda\varphi,\, \varphi(-x) = \varphi(x)\} \xrightarrow{\sim} \{f \in \mathscr{O}(\Omega) \,|\, H_\lambda^+ f = 0\},$$
$$\text{Odd}: \{\varphi \in L^2(\mathbb{R},\mathbb{C}^2) \,|\, Q\varphi = \lambda\varphi,\, \varphi(-x) = -\varphi(x)\} \xrightarrow{\sim} \{f \in \mathscr{O}(\Omega) \,|\, H_\lambda^- f = 0\},$$

where Ω is a simply-connected domain in $\mathbb{C}$ (w-space) such that $0,1 \in \Omega$ while $\alpha\beta \notin \Omega$, $\mathscr{O}(\Omega)$ denotes the set of holomorphic functions on Ω, and $H_\lambda^\pm = H_\lambda^+(w,\partial_w)$ are the Heun ordinary differential operators given respectively by

$$H_\lambda^+(w,\partial_w) := \frac{d^2}{dw^2} + \left(\frac{\frac{1}{2}-p}{w} + \frac{-\frac{1}{2}-p}{w-1} + \frac{p+1}{w-\alpha\beta}\right)\frac{d}{dw} + \frac{-\frac{1}{2}\left(p+\frac{1}{2}\right)w - q^+}{w(w-1)(w-\alpha\beta)},$$
$$H_\lambda^-(w,\partial_w) := \frac{d^2}{dw^2} + \left(\frac{1-p}{w} + \frac{-p}{w-1} + \frac{p+\frac{3}{2}}{w-\alpha\beta}\right)\frac{d}{dw} + \frac{-\frac{3}{2}pw - q^-}{w(w-1)(w-\alpha\beta)}.$$

Here $p = \frac{2\nu-3}{4}$ with $\nu = \frac{\alpha+\beta}{2\sqrt{\alpha\beta(\alpha\beta-1)}}\lambda$. The accessory parameters $q^\pm = q^\pm(\lambda,\alpha,\beta)$ can be explicitly expressed by the parameters α,β and eigenvalue λ [35]. □

Remark 6. The modified Laplace transform $\hat{u}(=\mathscr{L}_2 u)$ in [23] defines the intertwiner when restricting to the space of odd functions but does not for the even case.

6 Capturing the Rabi model by $\mathscr{R}$

In this section, employing the standard confluence process of Heun equations, we observe that the Rabi model can be obtained from $\mathscr{R} \in \mathscr{U}(\mathfrak{sl}_2)$ by a suitable choice of a triple $(\kappa,\varepsilon,\nu) \in \mathbb{R}^3$. In the sequel, we assume $a \in \mathbb{R}$, not necessarily an integer.

6.1 Confluent Heun's equation derived from Rabi's model

The Schrödinger equation $H_{\rm Rabi}\varphi = E\varphi$ of the quantum Rabi model is reduced to the following second order differential equation (see e.g. [38, 35]):

$$\frac{d^2 f}{dz^2} + p(z)\frac{df}{dz} + q(z)f = 0,$$

where

$$p(z) = \frac{(1-2E-2g^2)z-g}{z^2-g^2}, \quad q(z) = \frac{-g^2z^2+gz+E^2-g^2-\Delta^2}{z^2-g^2}.$$

Write $f(z) = e^{-gz}\phi(x)$, where $x = (g+z)/2g$. Substituting f into the equation above, one finds that the function ϕ satisfies the following confluent Heun equation (by a similar calculation in [38]). Actually, one has $H_1^{\rm Rabi}\phi = 0$, where

$$H_1^{\rm Rabi} := \frac{d^2}{dx^2} + \left\{-4g^2 + \frac{1-(E+g^2)}{x} + \frac{1-(E+g^2+1)}{x-1}\right\}\frac{d}{dx} + \frac{4g^2(E+g^2)x+\mu}{x(x-1)}$$

with the accessory parameter $\mu = (E+g^2)^2 - 4g^2(E+g^2) - \Delta^2$.

Remark 7. Setting $f(z) = e^{gz}\phi(x)$, where $x = (g-z)/2g$, one obtains another equation $H_2^{\rm Rabi}\phi = 0$. Here

$$H_2^{\rm Rabi} := \frac{d^2}{dx^2} + \left\{-4g^2 + \frac{1-(E+g^2+1)}{x} + \frac{1-(E+g^2)}{x-1}\right\}\frac{d}{dx} + \frac{4g^2(E+g^2-1)x+\mu}{x(x-1)}.$$

6.2 Confluence process of the Heun equation

Put $t = \coth^2\kappa (>1)$. The Heun operator $H^a(w,\partial_w)$ derived from $\varpi_a(\mathscr{R})$ is given by

$$H^a(w,\partial_w) = \frac{d^2}{dw^2} + \left(\frac{3-2\nu+2a}{4w} + \frac{-1-2\nu+2a}{4(w-1)} + \frac{-1+2\nu+2a}{4(w-t)}\right)\frac{d}{dw} + \frac{\frac{1}{2}(a-\frac{1}{2})(a-\frac{1}{2}-\nu)w - q_a}{w(w-1)(w-t)}.$$

The corresponding generalized Riemann scheme ([32]) is expressed as

$$\begin{pmatrix} 1 & 1 & 1 & 1 & \\ 0 & 1 & t & \infty & ; w \;\; q_a \\ 0 & 0 & 0 & a-\frac{1}{2} & \\ \frac{1+2\nu-2a}{4} & \frac{5+2\nu-2a}{4} & \frac{5-2\nu-2a}{4} & \frac{-1-2\nu+2a}{4} & \end{pmatrix}.$$

Here the first line indicates the s-rank of each singularity. Replace a (resp. ν) by $a+p$ (resp. $\nu+p$) in the expression of $H^a(w,\partial_w)$ above. It follows then that

$$A := \frac{1}{4}(-1-2\nu+2a), B := a+p+\frac{1}{2}, C := \frac{1}{4}(3-2\nu+2a) = 1+A, D := A.$$

Write it as

$$w(w-1)(w-t)H^a(w,\partial_w) = w(w-1)(w-t)\partial_w^2$$
$$+\Big[C(w-1)(w-t)+Dw(w-t)+(A+B+1-C-D)w(w-1)\Big]\partial_w + ABw - q_a.$$

Consider a confluence process of the singular points at $w=t$ and ∞ (Table 3.1.2 in [32]). The process is given as $t := \rho^{-1}, B := r\rho^{-1}$ and $\rho \to 0$ (equivalently $p \to \infty$):

$$-\lim_{\rho\to 0} w(w-1)(w-t)\rho H^a(w,\partial_w)$$
$$= w(w-1)\partial_w^2 + \big[C(w-1)+Dw-rw(w-1)\big] - rAw + \lim_{\rho\to 0}\rho q_a.$$

Now we take $\varepsilon = k\rho$ for some constraint k. Then one has a confluent Heun equation.

$$\frac{d^2\phi}{dw^2} + \Big[-r+\frac{1+A}{w}+\frac{A}{w-1}\Big]\frac{d\phi}{dw} + \frac{-rAw-(2A)^2-4A+k^2}{w(w-1)}\phi = 0.$$

Notice that $w=\infty$ is an irregular singularity with s-rank 2 (see e.g. [32]). Compare this with the confluent Heun operator H_1^{Rabi} for the Rabi model. Then, taking $r = 4g^2, A = -(E+g^2)$ with a suitable choice of k in this equation gives the latter.

Remark 8. Let $K = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Similarly to Lemma 2, one can show that the eigenvalue problem $KQK\varphi = \lambda\varphi$ is equivalent to the equation $\pi'(\tilde{\mathscr{R}})u = 0$, where

$$\tilde{\mathscr{R}} := \frac{2}{\sinh 2\kappa}\Big\{(H-\nu)\Big[(\sinh 2\kappa)(E-F)-(\cosh 2\kappa)H+\nu\Big]+(\varepsilon\nu)^2\Big\} \in \mathscr{U}(\mathfrak{sl}_2).$$

Then, a confluence procedure for $\varpi_a(\tilde{\mathscr{R}})$ similarly to that of $\varpi_a(\mathscr{R})$ yields H_2^{Rabi} in Remark 7. Moreover, one can find an element $\mathscr{K}$ (resp. $\tilde{\mathscr{K}}$) $\in \mathscr{U}(\mathfrak{sl}_2)$ of order two such that $\varpi_a(\mathscr{K})$ (resp. $\varpi_a(\tilde{\mathscr{K}})$) essentially provides H_1^{Rabi} (reps. H_2^{Rabi}) [36].

7 Conclusion

So far, even well-developed Lie theory has never contributed the spectral problems of quantum interaction models in a definite way. One of the simplest reasons is obviously the absence of the creation/annihilation operators. Hence the observation in this article might provide a new insight. Also, probably, as many of physicists may think there is an important questions: What are the detailed meaning of the exact solvability of [3], if it really differs from integrability? At the same time, according to [33], no second operator - integral of motion - exists. Therefore we should explore a fundamentally new category of exact solvability. As we have seen, the NcHO can be a "mother" of the Rabi model through the confluence procedure, whence one may expect to obtain an unified understanding of some sort of quantum interaction models using (in general, much higher $\mathbb{R}$-rank) Lie groups/algebras.

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