A Lie theoretic proposal on algorithms for the spherical harmonic lighting

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Abstract The spherical harmonics are the angular portion of the solution to the Laplace equation in spherical coordinates and provide a frequency-basis for representing functions on the sphere. The spherical harmonic lighting, as defined by Robin Green at Sony Computer Entertainment in 2003, is a family of real-time rendering techniques that may produce certain realistic shading and shadowing with relatively small overhead lighting. All such spherical harmonic lighting techniques involve replacing parts of standard lighting equations with spherical functions that have been projected into a frequency space using the spherical harmonics as a basis (or a weight space of irreducible finite dimensional representation of the rotation group). In this paper, using a group theoretical background of spherical harmonics and rather simple realization of the space of functions on the two dimensional sphere in the frame work of representation theory, we propose a possible geometry preserving algebraic/efficient computing, which might accelerate the (numerical and exact) computations slightly for spherical harmonic lighting.

1 Introduction

The spherical harmonics are the angular portion of the solution to the Laplace equation in spherical coordinates. The spherical harmonic lighting, as defined by Robin Green at Sony Computer Entertainment in 2003, is a family of real-time rendering techniques that may produce certain realistic shading and shadowing with relatively small overhead lighting. All such spherical harmonic lighting techniques involve replacing parts of standard lighting equations with spherical functions that have been projected into a frequency space (or, in our terminology, a weight space of

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irreducible finite dimensional representation of rotation group) using the spherical harmonics as a basis.

In this paper, we first recall the basic notion/definition for representation theory briefly and provide a group theoretical background of spherical harmonics, and using this, we propose a possible geometry preserving algebraic/efficient computing, which might accelerate the (numerical and exact) computations slightly for spherical harmonic lighting [1, 3, 4, 5, 7] (see also [6, 9]) in some context. A mathematical idea presented here, if it actually works in the rendering process in computer graphics, would not be necessarily limited to the study of computer graphics, whence could be applicable to other fields. Our proposal is based on Lie theory or Representation theory of Lie groups. One of the general ideas or reason why this theory can work for such applications is, practically, to providing simultaneous block-diagonalization of matrices.

Spherical harmonics are orthogonal functions and span rotation invariant spaces on the two dimensional sphere S^2, allowing for efficient, alias-free least squares projection and reconstruction of spherical functions (= functions on the sphere). These properties lead to a number of efficient operations for computing rotations, convolution, and double product integrals [4, 7]. As is well-known, spherical harmonics are used extensively in various fields. They are a basis of the space $L^2(S^2)$ of the square integrable functions on S^2, as the name would suggest. They have been used to solve problems in physics, such as in heat equations, the gravitational and electric fields. They have also been used in quantum chemistry and physics to model the electron configuration in atoms. For the spherical harmonic lighting, in place of the Fourier series expansion on the Euclidean space, one uses the expansion by the spherical harmonics, in other words, replaces exponential functions by the associated Legendre functions (or Legendre spherical functions). From our current point of view, the (usual) Fourier analysis is considered to be based on very simple representation theory of abelian groups $\mathbb{R}^n$, whereas the spherical harmonics is on the representation theory of a non-commutative group $SO(3)$, $SO(n)$ being the rotation group of order n.

To be more explicit, we shall describe certain algebraic treatment for the computation of harmonic expansions, which turns to be a part of the technique at the spherical harmonic lighting, by the framework of harmonic analysis on the sphere $S^2 \cong SO(2)\backslash SO(3)$. More precisely, one considers the irreducible decomposition of the natural action defined by the right translation of $SO(3)$ on $L^2(S^2)$ (i.e. a part of the theory of spherical harmonics) and translate/reformulate the problem into the different Hilbert space using another but unitarily equivalent realization of irreducible representations on the space of polynomials with complex coefficients by the language of the special unitary group $SU(2)$ of degree two (see e.g. [2, 8, 10]). Here the word "unitarily equivalent" means the isometry between two Hilbert spaces with equivalent group actions.

2 Basic notion for representation theory

One recalls here some of fundamental definitions and facts for representation theory of topological groups or Lie groups (see e.g. [2, 8]). The readers may assume that groups considerd in this paper are given by matrices groups such as $SO(n)$, the orthogonal group $O(n)$, the special linear group $SL_n(\mathbb{C})$, etc.

Definition 1. A unitary representation of a topological group G is a strongly continuous homomorphism π of G into the group $U(H)$ of unitary operators on a Hilbert space H. Here, a mapping $\pi : G \ni g \mapsto \pi(g) \in U(H)$ is called a homomorphism if it satisfies

$$\pi(gh) = \pi(g)\pi(h) \quad (\forall g, h \in G),$$

and a homomorphism π is called strongly continuous if the mapping $g \mapsto \pi(g)x$ is a continuous mapping of G into H for all $x \in H$. Moreover, one sometimes denotes the representation by a pair of the mapping π and representation space H as (π, H). □

Definition 2. Two unitary representations (π_1, H_1) and (π_2, H_2) are called equivalent if there exists an isometry (i.e. a linear mapping preserving the norm) A of H_1 onto H_2 satisfying

$$A\pi_1(g) = \pi_2(g)A \quad (\forall g \in G).$$

In this case, one writes $(\pi_1, H_1) \cong (\pi_2, H_2)$ (or simply $H_1 \cong H_2$). Obviously, the relation "$\cong$" is an equivalence relation. □

Definition 3. Let (π, H) be a unitary representation of a group G. A closed linear subspace V of H is called invariant under π if one has

$$\pi(g)V \subset H \quad (\forall g \in G).$$

A unitary representation π is called irreducible if $H \neq \{0\}$ and H and $\{0\}$ are the only invariant subspaces of H.

Non-irreducible unitary representations are "decomposed" into irreducible representations. In a sense, the irreducible representations are the "atoms" of unitary representations. □

Definition 4. Let (π_1, H_1) and (π_2, H_2) be two unitary representations of G. A linear operator $T : H_1 \to H_2$ satisfying

$$T\pi_1(g) = \pi_2(g)T \quad (\forall g \in G)$$

is called an intertwiner (or intertwining operator) between H_1 and H_2. □

Proposition 1. *Let (π_1, H_1) and (π_2, H_2) be two finite dimensional unitary representations of G and T be an intertwiner between H_1 and H_2. Then, either $T = 0$ or T is a linear isomorphism.* □

Proposition 2. *Let (π, H) be a unitary representations of G. Then, a closed subspace V of H is invariant under π if and only if the orthogonal projection P_V on V commutes with $\pi(g)$ for all $g \in G$. In this case, the orthogonal complement $V^{\perp}$ is also invariant under π.* □

Let $\mathbf{B}(H)$ be the algebra of bounded linear operators on a Hilbert space H and M be a subset of $\mathbf{B}(H)$. The commutant M' of M is defined by

$$M' = \{L \in \mathbf{B}(H) \,|\, LU = UL \ (\forall U \in M)\}.$$

Theorem 1. *(Schur's Lemma) Let (π, H) be a unitary representations of G and $M = \{\pi(g) \,|\, g \in G\}$. Then, π is irreducible if and only if the commutant M' is equal to the set $\mathbb{C}1$ of scalar operators.* □

Theorem 2. *Any unitary representation π of a compact group G is a (Hilbert space) direct sum of finite-dimensional irreducible unitary representations. In particular, any irreducible representation of a compact group is finite-dimensional.* □

3 Spherical harmonics

The groups treated in the present and subsequent sections such as the special unitary group $SU(2)$ (of degree 2) and special orthogonal groups $SO(2), SO(3)$ are compact. Notice that since one always has an invariant inner product on the representation space by the existence of the Haar measure (invariant measure), any finite-dimensional representation of a compact group is assumed to be unitary [2, 8].

As a matrix group of degree 2, $SU(2)$ acts on the vector space $\mathbb{C}^2$: For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \in SU(2)$ $(|\alpha|^2 + |\beta|^2 = 1, \alpha, \beta \in \mathbb{C})$ the action is defined by

$$\mathbb{C}^2 \ni z = (z_1, z_2) \mapsto zg = (az_1 + cz_2, bz_1 + dz_2) \in \mathbb{C}^2.$$

Note that the action satisfies $z(g_1 g_2) = (zg_1)g_2 \,(\forall g_1, g_2 \in SU(2))$ and $z1 = z$.

Let $\mathbb{C}[z_1, z_2]$ denote the polynomial algebra on $\mathbb{C}^2$. Let $V_m := \mathbb{C}[z_1, z_2]_m$ be the subspace of homogeneous polynomials of degree m of $\mathbb{C}[z_1, z_2]$. Then any polynomial f in V_m can be written uniquely as a linear combination of $m+1$ monomials $z_1^k z_2^{m-k}$ $(0 \le k \le m)$. Hence one defines an $m+1$ dimensional representation (π_m, V_m) of $SU(2)$ by

$$(\pi_m(g)f)(z) := f(zg).$$

The inner product defined by

$$(z_1^k z_2^{m-k}, z_1^\ell z_2^{m-\ell}) = k!(m-k)!\delta_{k,\ell}$$

is invariant under π_n. In other words, equipped with this inner product, (π_m, V_m) turns to be a unitary representation of $SU(2)$. One can prove the following theorem by the representation theory of Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ of $SL_2(\mathbb{C})$.

Theorem 3. *For any non-negative integer m, the unitary representation* (π_m, V_m) *of* $SU(2)$ *is irreducible. Moreover, any irreducible unitary representation of* $SU(2)$ *is equivalent to one of* (π_m, V_m). $\square$

Theorem 4. *Let* $\mathbb{C}[z]_m$ *be the space of polynomials in z of degree less than or equals m. Then one define the action* $\tau_m(g)$ *of* $g = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \in SU(2)$ *on* $\mathbb{C}[z]_m$ *by*

$$(\tau_m(g)p)(z) = (\overline{\beta}z + \alpha)^m p\Big(\frac{\overline{\alpha}z - \overline{\beta}}{\overline{\beta}z + \alpha}\Big) \quad (p \in \mathbb{C}[z]_m).$$

Equipped with the inner product defined by $(z^k, z^\ell)_m = \frac{k!(m-k)!}{(m+1)!}\delta_{k,\ell}$*, the representation* $(\tau_m, \mathbb{C}[z]_m)$ *is unitarily equivalent to* (π_m, V_m) . $\square$

Remark 1. The Hermitian inner product on $\mathbb{C}[z]_m$ defined in the theorem above can be expressed as follows:

$$(p_1, p_2)_m = \frac{m+1}{\pi}\int_{\mathbb{C}} p_1(z)\overline{p_2(z)}(1+|z|^2)^{-m-2}dz.$$

The representation theory of the 3-dimensional rotation group $SO(3)$ can be derived from that of $SU(2)$ described above, because $SU(2)$ is a (double) covering group of $SO(3)$. Actually, the kernel of the adjoint representation Ad of $SU(2)$ on the three dimensional real Lie algebra $\mathfrak{g} = \mathfrak{su}_2(\mathbb{R})$ spanned by

$$X = \frac{1}{2}\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad Y = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad Z = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

(identified to the tangent space of $SU(2)$ at 1 equipped with the Lie bracket $[U, V] := UV - VU$) defined by $\mathrm{Ad}(g)U = gUg^{-1}$ $(U, V \in \mathfrak{g}, g \in SU(2))$ is $\{\pm 1\}$. Then, noting $\{X, Y, Z\}$ is the orthogonal basis with respect to the inner product $\langle A, B\rangle := -2\mathrm{Tr(AB)}$, one observes that the map

$$\mathfrak{g} \ni xX + yY + zZ \mapsto \mathbf{x} = (x, y, z) \in \mathbb{R}^3$$

is an isometry. From this

$$SU(2)/\{\pm 1\} \cong SO(3).$$

This fact actually gives the following.

Theorem 5. *For any non-negative integer* ℓ*, there exists an irreducible unitary representation* ρ_ℓ *of* $SO(3)$ *which is given by*

$$\rho_\ell \circ \mathrm{Ad} = \tau_{2\ell} (\cong \pi_{2\ell}).$$

Any irreducible unitary representation of $SO(3)$ *is equivalent to* ρ_ℓ *for some* ℓ*. Moreover, if* $\ell \neq m$*, then* ρ_ℓ *is not equivalent to* ρ_m. $\square$

Since the two dimensional sphere S^2 is realized by a homogeneous space (actually a compact Riemann symmetric space) of $SO(3)$ as $S^2 \cong SO(2)\backslash SO(3) \cong K\backslash SU(2)$, where $K := \{\begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix} \,|\, 0 \leq \theta < 4\pi\}$, one can naturally define a unitary representation of $SO(3)$ on the space of square integrable functions $L^2(S^2)$ on S^2 by right translation. Actually, we will see the irreducible unitary representations of $SO(3)$ are realized on the space of harmonic polynomials or spherical harmonics (thought as a well matched description).

Define a representation T_ℓ of $SO(3)$ on the space $\mathscr{P}_\ell$ of homogeneous polynomials of degree ℓ in three variables $\mathbf{x} = (x,y,z)$ by

$$(T_\ell(g)f)(\mathbf{x}) = f(\mathbf{x}g).$$

Notice that T_ℓ is not irreducible if $\ell \geq 2$, e.g. the space $\mathscr{P}_2$ contains the non-trivial invariant subspace spanned by the quadratic form $x^2+y^2+z^2$.

Let

$$\Delta := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

be the Laplacian on $\mathbb{R}^3$. Define a space of harmonic polynomials of degree ℓ by

$$\mathscr{H}_\ell = \{f \in P_\ell \,|\, \Delta f = 0\}.$$

Note that $\dim \mathscr{H}_\ell = 2\ell + 1$. Moreover, one knows the action of $SO(3)$ (the right translation T_ℓ) commutes with Δ, whence one can define a representation U_ℓ by

$$U_\ell = T_\ell|_{\mathscr{H}_\ell}.$$

Since any element f in $\mathscr{H}_\ell$ is a homogeneous polynomial of degree ℓ, for any $r \geq 0$ one has

$$f(rx,ry,rz) = r^\ell f(x,y,z).$$

Let $\mathscr{K}$ be the restriction of f to S^2. Put

$$\tilde{\mathscr{H}_\ell} := \mathscr{H}_\ell|_{S^2} = \{\mathscr{K}(f) \,|\, f \in \mathscr{H}_\ell\}.$$

Then the map $\mathscr{K}$ is a linear isomorphism of the vector space $\mathscr{H}_\ell$ onto $\tilde{\mathscr{H}_\ell}$. The space $\mathscr{H}_\ell$ is obviously recovered from $\tilde{\mathscr{H}_\ell}$ by the equation above. The elements of $\tilde{\mathscr{H}_\ell}$ are called spherical harmonics of degree ℓ. Since the space $\tilde{\mathscr{H}_\ell}$ is stable under the action $U_\ell(g)\,(g \in SO(3))$, $(U_\ell, \tilde{\mathscr{H}_\ell})$ defines an irreducible unitary representation of $SO(3)$. Actually,

Theorem 6. *As an irreducible unitary representation of* $SO(3)$

$$U_\ell \cong \rho_\ell. \qquad \square$$

The inner product on $\tilde{\mathscr{H}_\ell}$ is naturally given by

$$(f, g)_{S^2} = \int_{S^2} f(\mathbf{x})\overline{g(\mathbf{x})} d\mathbf{x},$$

where $d\mathbf{x}$ is the normalized measure on S^2 given by $d\mathbf{x} = (4\pi)^{-1} \sin\theta \, d\theta d\phi$. Here

$$\mathbf{x} = (x, y, z) = (r \sin\theta \cos\phi, r \sin\theta \sin\phi, r\cos\theta) \quad (r > 0, (\theta, \phi) \in [0, 2\pi] \times [0, \pi])$$

is the polar coordinates of $\mathbb{R}^3$.

Theorem 7. *The space $\tilde{\mathscr{H}_\ell}$ has an orthonormal basis $\{Y_\ell^m\}_{-\ell \le m \le \ell}$ defined by*

$$Y_\ell^m(\theta, \phi) = (-1)^m \sqrt{\frac{(2\ell+1)(\ell+m)!}{(\ell-m)!}} e^{im\phi} P_\ell^{-m}(\cos\theta),$$

where $P_\ell^m(x)$ is the associated Legendre function defined by

$$P_\ell^m(x) = \frac{1}{2^\ell \ell!}(1-x^2)^{m/2} \frac{d^{\ell+m}}{dx^{\ell+m}}(x^2-1)^\ell.$$

These Y_ℓ^m are the matrix elements of the representation $(U_\ell, \tilde{\mathscr{H}_\ell})$ and called Legendre's spherical functions (see Theorem 8, Example 1 below). □

Remark 2. There is a relation:

$$P_\ell^{-m}(x) = (-1)^m \frac{(\ell-m)!}{(\ell+m)!} P_\ell^m(x).$$

Remark that several different normalizations are in common use (in quantum mechanics, seismology, geodesy, magnetics, etc.) for the Laplace spherical harmonic functions $Y_\ell^m(\theta, \phi)$.

Since these representations U_ℓ exhaust all (equivalence classes) of irreducible unitary representations of $SO(3)$ one obtains

$$L^2(S^2) = \Sigma_{\ell=0}^\infty \oplus \tilde{\mathscr{H}_\ell} \quad \text{(a direct sum)}.$$

This shows that a (square integrable, hence in particular, continuous) function on the unit sphere S^2 can be expanded by the spherical harmonics $Y_\ell^m(\theta, \phi)\,(-\ell \le m \le \ell, \ell = 0, 1, 2, \ldots)$. This fact is the consequence of the following Peter-Weyl theorem of compact group:

Theorem 8. *Let $\widehat{G} = \{\pi\}$ be the unitary dual of G, the set of all equivalence classes of irreducible unitary representations of G. Take a representative (π, V) of π (using the same letter). Put $d_\pi = \dim_{\mathbb{C}} V$. Then the family $B_G := \{\sqrt{d_\pi}(\pi(g)v_i, v_j) \mid \pi \in \widehat{G}, 1 \le i, j \le d_\pi\}$ is a complete orthonormal family of $L^2(G)$.* □

Example 1. The unitary dual $\widehat{SO(3)}$ of $SO(3)$ can be parametrized by the set $\mathbb{Z}_{\ge 0}$ of non-negative integers ℓ. Since $S^2 \cong SO(2)\backslash SO(3)$, the Legendre spherical function is given by a matrix element $(\rho_\ell(g)f, f_0)$, where f_0 is a $SO(2)$-fixed vector, i.e. $\rho_\ell(k)f_0 = f_0$ for $k \in SO(2)$.

Remark 3. To illustrate the distributions of colors on the sphere determined by the harmonics basis, usually one replaces the basis $Y_\ell^m(\theta,\phi)$ by the real valued spherical harmonics $y_\ell^m(\theta,\phi)$ as follows.

$$y_\ell^m(\theta,\phi) := \begin{cases} \sqrt{2}\,\mathrm{Re}(\mathrm{Y}_\ell^{\mathrm{m}})\,(\mathrm{m}>0) \\ Y_\ell^0\,(m=0) \\ \sqrt{2}\,\mathrm{Im}(\mathrm{Y}_\ell^{\mathrm{m}})\,(\mathrm{m}<0) \end{cases} .$$

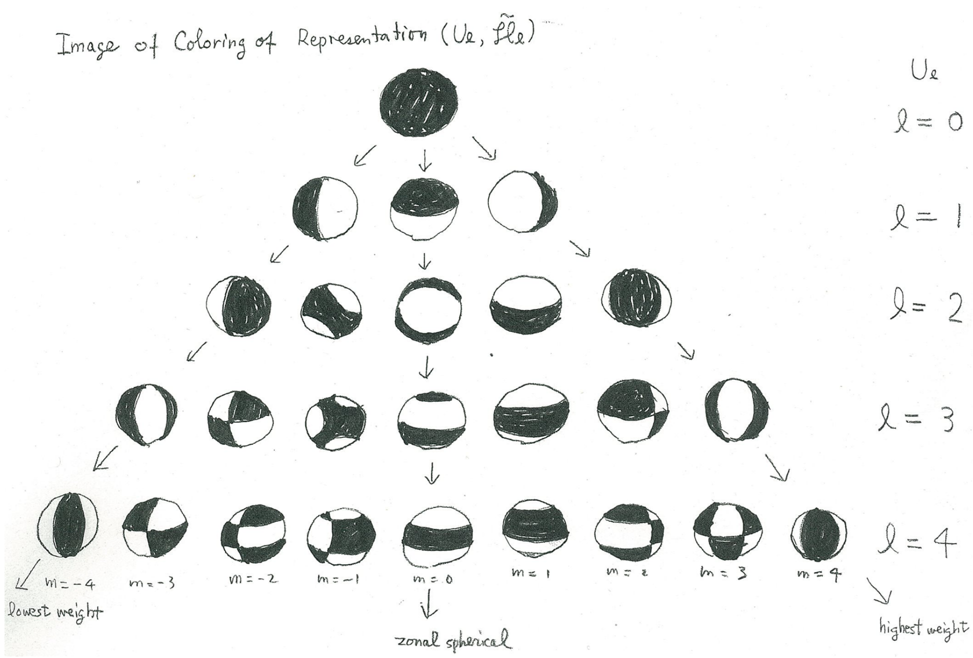

Fig. 1 Number of the figures $2\ell+1$ in each horizontal row is the dimension of irreducible representation $(U_\ell, \tilde{\mathscr{H}}_\ell)$. (The figure is inspired from [4, 7].)

The standard way to visualize the spherical harmonics can be seen in e.g. [4, 7]. For instance, one of those is to distort a unit sphere, by scaling each point radialy by the basilute value of the function and soloing it based on the sign. The figure above illustrates such coloring only on the sphere for the first 5 spherical harmonics corresponding to the irreducible representation $(U_\ell, \tilde{\mathscr{H}}_\ell)\,(\ell = 0,1,2,3,4.)$ (Non-negative integers ℓ are called the bands.) The green parts indicate regions where the function is positive, while white parts represent negative ones.

4 Harmonic expansions by differentiation

Recall now the intertwiner A_ℓ (and its inverse) between $\mathbb{C}[w]_{2\ell}$ and $\mathscr{H}_\ell$. It is given explicitly as (see [2])

$$(A_\ell p)(\mathbf{x}) := \frac{2\ell+1}{\pi}\int_{\mathbb{C}} p(w)\overline{H(\mathbf{x},\, w)}^{\ell}(1+|w|^2)^{-2\ell-2}dw,$$

where $H(\mathbf{x},\, w) := (x+iy)w^2+2zw-(x-iy)$ (cf. Remark 1). Notice that, for w fixed, as a function of $\mathbf{x}=(x,y,z)\in\mathbb{R}^3$, one observes immediately that $H(\mathbf{x},\, w)^\ell\in\mathscr{H}_\ell$.

By this description, one may transform the stage of calculations from the one using spherical harmonics to the one using simple monomials $z^m\in\mathbb{C}[z]_\ell$ by Theorem 5 together with Theorem 4. Namely, in some part of the spherical harmonic lighting technique, one might avoid rather complicated recurrence formulas and/or differential equation of the associated Legendre functions P_ℓ^m.

Recall the facts that in terms of the polar coordinate, Δ can be expressed as

$$\begin{aligned}\Delta =& \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\Delta_{S^2},\\ \Delta_{S^2} :=& \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial^2\phi}.\end{aligned}$$

Notice that the last expression is actually known to be the image of the Casimir element $\mathscr{C}\in\mathscr{Z}U(\mathfrak{so}_3)$, the center of the universal enveloping algebra $U(\mathfrak{so}_3)(\cong U(\mathfrak{su}_2))$, under the (infinitesimal) right action by $SO(3)$. Therefore, since the representation U_ℓ is irreducible, one finds that $Y_\ell(\theta,\phi)$ is the eigenfunction of Δ_{S^2} with eigenvalue $-\ell(\ell+1)$ (by noticing that $\Delta r^\ell=\ell(\ell+1)r^\ell$):

$$\Delta_{S^2}Y_\ell(\theta,\phi) = -\ell(\ell+1)Y_\ell(\theta,\phi).$$

At the harmonic expansion of a spherical function f, one practically considers the approximation $\tilde{f}_N$ truncated by high frequency irreducible components $U_\ell\,(\ell\geq N+1)$:

$$f\approx\tilde{f}_N := \sum_{\ell=0}^{N}\sum_{|m|\leq\ell}a_\ell^m Y_\ell^m,$$

where we put $a_\ell^m=(f,\,Y_\ell^m)_{S^2}$ (but not computing here this integral). Let us consider the situation that one may assume that $f=\tilde{f}_N$ for some large N. (Actually, what one can detect practically is limited by bounded frequency components).

Define the (projection) operator $P_\ell^N:\sum_{j=0}^N\oplus\mathscr{H}_j\to\mathscr{H}_\ell$ by

$$P_\ell^N := \prod_{\ell'=0,\ell'\neq\ell}^{N}\frac{\Delta_{S^2}+\ell'(\ell'+1)}{-\ell(\ell+1)+\ell'(\ell'+1)}.$$

It follows immediately that

$$P_\ell^N \tilde{f}_N = \sum_{|m|\le\ell} a_\ell^m Y_\ell^m.$$

Then one has

$$a_\ell^m = (P_\ell^N \tilde{f}_N, Y_\ell^m)_{S^2} = (A_\ell^{-1} P_\ell^N \tilde{f}_N, z^{\ell-m})_{2\ell}.$$

Now we give an explanation how to obtain the inverse isomorphism A_ℓ^{-1} from the space $\mathscr{H}_\ell$ to $\mathbb{C}[w]_m$. Let $F \in \mathscr{H}_\ell$. Write F as

$$F(x,y,z) = g_0(x,y) + g_1(x,y)z + \cdots + g_\ell(x,y)z^\ell.$$

Note the fact that $g_j(x,y) \in \mathbb{C}[x,y]_{\ell-j}$ (i.e., is a polynomial of homogeneous degree $\ell - j$). Put $\Delta_{x,y} := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. Then it is immediate that

$$\begin{aligned}\Delta_{x,y}F &= \Delta_{x,y}g_0 + \cdots + \Delta_{x,y}g_\ell z^\ell,\\ \frac{\partial^2}{\partial z^2}F &= 2g_2 + \cdots + \ell(\ell-1)g_\ell z^{\ell-2}.\end{aligned}$$

Since g_ℓ (reps. $g_{\ell-1}$) is a constant (resp. a linear function) and $\Delta = \Delta_{x,y} + \frac{\partial^2}{\partial z^2}$, one observes that F is a harmonic polynomial if and only if the following condition are satisfied:

$$g_k = -\frac{1}{k(k-1)}\Delta_{x,y}g_{k-2} \quad (2 \le k \le \ell).$$

Hence one finds that there is a one-to-one correspondence between $\mathscr{H}_\ell$ and $\mathbb{C}[w]_{2\ell}$ as follows:

$$\begin{aligned}\mathscr{H}_\ell \ni F \mapsto (g_0, g_1) \in \mathbb{C}[x,y]_\ell \oplus \mathbb{C}[x,y]_{\ell-1} &\cong \mathbb{C}[w]_\ell \oplus \mathbb{C}[w]_{\ell-1}\\ &\cong \mathbb{C}[w^2]_\ell \oplus w\mathbb{C}[w^2]_{\ell-1} \cong \mathbb{C}[w]_{2\ell}\end{aligned}$$

Notice that the latter three isomorphisms are obviously all algebraic. Since $g_0 = F(x,y,0)$ and $g_1 = \frac{\partial}{\partial z}F(x,y,z)|_{z=0}$, using the reproducing kernel $K_z(w) = K(w, z) := (1+\bar{z}w)^{2\ell}$ (more precisely, the even and odd parts of the reproducing kernels) of the Hilbert space $\mathbb{C}[w]_{2\ell}$, one can essentially construct the inverse of the intertwiner A_ℓ.

This allows us to compute the coefficient a_ℓ^m from $P_\ell^N \tilde{f}_N$ avoiding numerical integration process such as using Monte-Carlo integration by random numbers. (via computing $P_\ell^N \tilde{f}_N(x,y,0)$ and $\frac{\partial}{\partial z}P_\ell^N \tilde{f}_N(x,y,z)|_{z=0}$. In other words, one may compute the inner product of the right hand side in purely algebraic way. Therefore, implementation of the idea provided in this section to computers for spherical harmonic lighting would be desirable.

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