

半正定値計画問題への招待

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ワークショップ「最適化理論の産業・諸科学への応用」

九州大学伊都キャンパス

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4. 一般の SDP
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6. 双対性
7. 数値解法
8. センサーネットワークの位置推定問題への応用
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SDP は LP の対称行列空間への拡張

$$\begin{aligned} \text{LP: minimize} \quad & -X_{11} - 2X_{12} - 5X_{22} \\ \text{subject to} \quad & 2X_{11} + 3X_{12} + X_{22} = 7, \quad X_{11} + X_{12} \geq 1, \\ & X_{11} \geq 0, \quad X_{12} \geq 0, \quad X_{22} \geq 0. \end{aligned}$$

$$\begin{aligned} \text{SDP: minimize} \quad & -X_{11} - 2X_{12} - 5X_{22} \\ \text{subject to} \quad & 2X_{11} + 3X_{12} + X_{22} = 7, \quad X_{11} + X_{12} \geq 1, \\ & X_{11} \geq 0, \quad X_{12} \geq 0, \quad X_{22} \geq 0, \\ & \begin{pmatrix} X_{11} & X_{12} \\ X_{12} & X_{22} \end{pmatrix} \succeq \mathbf{O} \text{ (半正定値)}. \end{aligned}$$

- 共通点： X_{11} , X_{12} , X_{22} に関する線形目的関数
- 共通点： X_{11} , X_{12} , X_{22} に関する線形等式，線形不等式条件

SDP は LP の対称行列空間への拡張

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- SDP: 対称行列変数 $\begin{pmatrix} X_{11} & X_{12} \\ X_{12} & X_{22} \end{pmatrix}$ に対する半正定値条件 \Leftrightarrow
 $X_{11} \geq 0, X_{22} \geq 0, X_{11}X_{22} - X_{12}^2 \geq 0$ — 非線形

SDP は LP の対称行列空間への拡張

$$\begin{aligned} \text{LP: minimize} \quad & -X_{11} - 2X_{12} - 5X_{22} \\ \text{subject to} \quad & 2X_{11} + 3X_{12} + X_{22} = 7, \quad X_{11} + X_{12} \geq 1, \\ & X_{11} \geq 0, \quad X_{12} \geq 0, \quad X_{22} \geq 0. \end{aligned}$$

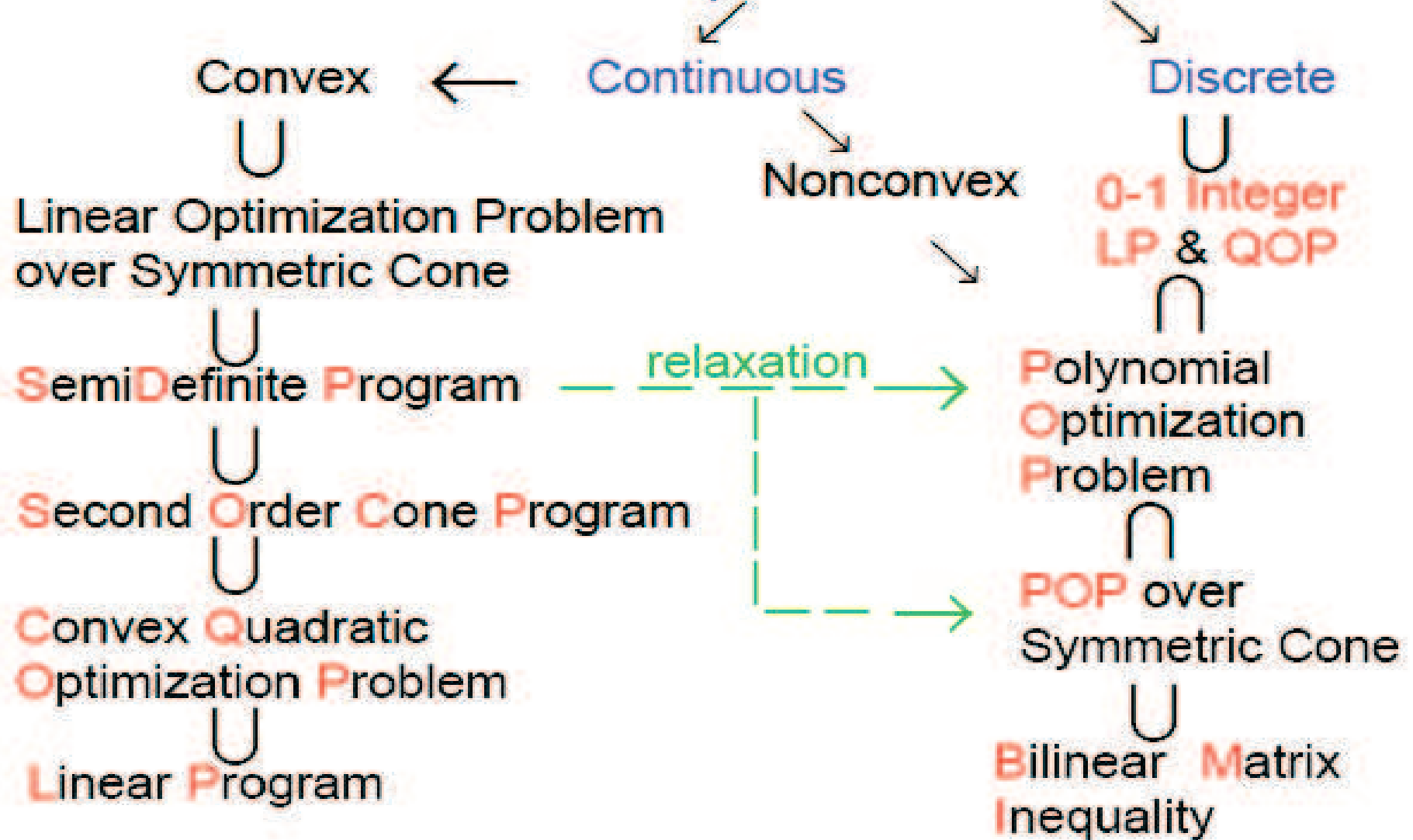
$$\begin{aligned} \text{SDP: minimize} \quad & -X_{11} - 2X_{12} - 5X_{22} \\ \text{subject to} \quad & 2X_{11} + 3X_{12} + X_{22} = 7, \quad X_{11} + X_{12} \geq 1, \\ & X_{11} \geq 0, \quad X_{12} \geq 0, \quad X_{22} \geq 0, \\ & \begin{pmatrix} X_{11} & X_{12} \\ X_{12} & X_{22} \end{pmatrix} \succeq \mathbf{O} \text{ (半正定値)}. \end{aligned}$$

- LP の許容解集合（条件を満たす X_{11}, X_{12}, X_{22} の集合）は多面体集合 \Rightarrow 有限個の頂点のいずれかで最小値が達成 \Rightarrow シンプレックス法
- SDP の許容解集合は凸集合であるが多面体集合ではない

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Classification of Optimization Problems



多くの分野へのさまざまな応用

- システムと制御 — 線形行列不等式 [6]
- 半正定値行列緩和
 - グラフの最大カット問題 [14], 最大クリーク問題
 - 線形, 2次 0-1 計画問題 [24]
 - 多項式最適化 [22, 35]
- ロバスト最適化 [4]
- 量子化学 [51]
- モーメント問題 (応用確率論) [5, 23]
- ...

サーベイ論文 — Todd [39], Vandenberghe-Boyd [45]

ハンドブック — Wolkowicz-Saigal-Vandenberghe [46]

ウェブサイト — Helmberg[15], Wolkowicz [47]

SDP を支える基礎理論

- Self-concordant 理論 [33]
- Euclidean Jordan 代数 [10, 36]
- 主双対内点法 [1, 17, 20, 27, 34]

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$$\begin{aligned} \text{(LP) minimize } & \mathbf{a}_0 \cdot \mathbf{x} \\ \text{subject to } & \mathbf{a}_p \cdot \mathbf{x} = b_p \quad (1 \leq p \leq m), \quad \mathbb{R}^n \ni \mathbf{x} \geq \mathbf{0}. \end{aligned}$$

\mathbb{R} : 実数の集合,

\mathbb{R}^n : n dim. ベクトルの線形空間,

$\mathbf{a}_p \in \mathbb{R}^n$: 定数, n dim. ベクトル ($1 \leq p \leq m$),

$b_p \in \mathbb{R}$: 定数, 実数 ($1 \leq p \leq m$),

$\mathbf{x} \in \mathbb{R}^n$: 変数, n dim. ベクトル,

$\mathbf{a}_p \cdot \mathbf{x} = \sum_{i=1}^n [\mathbf{a}_p]_i x_i$ (\mathbf{a}_p and \mathbf{x} の内積).

(LP) minimize $\mathbf{a}_0 \cdot \mathbf{x}$
 subject to $\mathbf{a}_p \cdot \mathbf{x} = b_p \ (1 \leq p \leq m), \ \mathbb{R}^n \ni \mathbf{x} \geq \mathbf{0}.$

(SDP) minimize $\mathbf{A}_0 \bullet \mathbf{X}$
 subject to $\mathbf{A}_p \bullet \mathbf{X} = b_p \ (1 \leq p \leq m), \ \mathbb{S}^n \ni \mathbf{X} \succeq \mathbf{O}.$

\mathbb{S}^n : $n \times n$ 対称行列の線形空間,

$\mathbf{A}_p \in \mathbb{S}^n$: 定数, $n \times n$ 対称行列 ($0 \leq p \leq m$),

$b_p \in \mathbb{R}$: 定数, 実数 ($1 \leq p \leq m$),

$\mathbf{X} \in \mathbb{S}^n$: $n \times n$ 変数, $n \times n$ 対称行列;

$$\mathbf{X} = (X_{ij}) = \begin{pmatrix} X_{11} & X_{12} & \dots & X_{1n} \\ X_{21} & X_{22} & \dots & X_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ X_{n1} & X_{n2} & \dots & X_{nn} \end{pmatrix} \in \mathbb{S}^n,$$

$$X_{ij} = X_{ji} \in \mathbb{R} \ (1 \leq i \leq j \leq n),$$

$$\begin{aligned}
 \text{(LP)} \quad & \text{minimize} \quad \mathbf{a}_0 \cdot \mathbf{x} \\
 & \text{subject to} \quad \mathbf{a}_p \cdot \mathbf{x} = b_p \quad (1 \leq p \leq m), \quad \mathbb{R}^n \ni \mathbf{x} \geq \mathbf{0}.
 \end{aligned}$$

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 \text{(SDP)} \quad & \text{minimize} \quad \mathbf{A}_0 \bullet \mathbf{X} \\
 & \text{subject to} \quad \mathbf{A}_p \bullet \mathbf{X} = b_p \quad (1 \leq p \leq m), \quad \mathbb{S}^n \ni \mathbf{X} \succeq \mathbf{O}.
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{X} \in \mathbb{S}_+^n & \Leftrightarrow \mathbf{X} \in \mathbb{S}^n : \text{半正定値;} \\
 & \mathbf{X} \text{ の固有値が全て非負,} \\
 & \text{2次形式 } \mathbf{u}^T \mathbf{X} \mathbf{u} \geq 0, \quad \forall \mathbf{u} \in \mathbb{R}^n,
 \end{aligned}$$

$$\mathbf{X} \succeq \mathbf{O} \Leftrightarrow \mathbf{X} \in \mathbb{S}_+^n, \exists n,$$

$$\mathbf{A}_p \bullet \mathbf{X} = \sum_{i=1}^n \sum_{j=1}^n [\mathbf{A}_p]_{ij} \mathbf{X}_{ij} \quad (\mathbf{A}_p \text{ と } \mathbf{X} \text{ の内積}).$$

$$\begin{aligned}
 \text{(LP)} \quad & \text{minimize} \quad \mathbf{a}_0 \cdot \mathbf{x} \\
 & \text{subject to} \quad \mathbf{a}_p \cdot \mathbf{x} = b_p \quad (1 \leq p \leq m), \quad \mathbb{R}^n \ni \mathbf{x} \geq \mathbf{0}.
 \end{aligned}$$

$$\begin{aligned}
 \text{(SDP)} \quad & \text{minimize} \quad \mathbf{A}_0 \bullet \mathbf{X} \\
 & \text{subject to} \quad \mathbf{A}_p \bullet \mathbf{X} = b_p \quad (1 \leq p \leq m), \quad \mathbb{S}^n \ni \mathbf{X} \succeq \mathbf{O}.
 \end{aligned}$$

$$\uparrow \left\{ \begin{array}{l} m = 2, \quad n = 2, \quad b_1 = 7, \quad b_2 = 9, \\ \mathbf{X} = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}, \quad \mathbf{A}_0 = \begin{pmatrix} -1 & -1 \\ -1 & -5 \end{pmatrix}, \\ \mathbf{A}_1 = \begin{pmatrix} 2 & 1.5 \\ 1.5 & 1 \end{pmatrix}, \quad \mathbf{A}_2 = \begin{pmatrix} 2 & 0.5 \\ 0.5 & 3 \end{pmatrix}. \end{array} \right.$$

$$\begin{aligned}
 & \text{minimize} \quad -X_{11} - 2X_{12} - 5X_{22} \\
 & \text{subject to} \quad 2X_{11} + 3X_{12} + X_{22} = 7, \quad 2X_{11} + X_{12} + 3X_{22} = 9, \\
 & \quad \quad \quad \begin{pmatrix} X_{11} & X_{12} \\ X_{12} & X_{22} \end{pmatrix} \succeq \mathbf{O}.
 \end{aligned}$$

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等式標準形 (SDP):

$$\min. \quad A_0 \bullet X$$

$$\text{sub.to} \quad A_p \bullet X = b_p \quad (1 \leq p \leq m), \quad \mathbb{S}^n \ni X \succeq O.$$

↑ ?

システムと制御からの SDP の例 (SDP)':

$$\min \quad \lambda$$

$$\text{sub. to} \quad \begin{pmatrix} XA + A^T X + C^T C & XB + C^T D \\ B^T X + D^T C & D^T D - I \end{pmatrix} \preceq \lambda I,$$
$$X \succeq -\lambda I.$$

$X \in \mathbb{S}^n$, $\lambda \in \mathbb{R}$ 変数, A, B, C, D 定数行列, I : 単位行列

- (SDP)' は等式標準形 (SDP) に変換出来るか?
- 理論的には “Yes” $\iff X \in \mathbb{S}^n$ の任意の実線形関数は $A \bullet X = \sum_{i=1}^n \sum_{j=1}^n A_{ij} X_{ij}$, $\exists A \in \mathbb{S}^n$ と表現できる .
- 汎用ソフトウェアを適用するには自由変数を含む等式標準形または等式条件付き LMI 標準形への変換が必要

一般の SDP:

min. x_1, \dots, x_k と X^q ($1 \leq q \leq t$) の線形関数
sub.to x_1, \dots, x_k と X^q ($1 \leq q \leq t$) の線形等式,
 x_1, \dots, x_k と X^q ($1 \leq q \leq t$) の線形 (行列) 不等式,
 $x_1, \dots, x_k \in \mathbb{R}$ (自由変数),
 $X^q \succeq O$ ($1 \leq q \leq t$) (半正定値行列変数).

線形行列不等式

$$F(x_1, \dots, x_k, X_1, \dots, X_q) - B \succeq O,$$

$F : (x_1, \dots, x_k, X_1, \dots, X_q) \rightarrow Y \in \mathbb{S}^m$: 線形, $B \in \mathbb{S}^m$: 定数

$m = 1$ のときは, 通常 of 線形不等式.

実非負変数は 1×1 半正定値対称行列変数とみなす.

$k \times m$ 行列変数 U 自由変数 U_{ij} の集まりとみなす.

例

$$\min x + \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \bullet X \quad \text{sub.to} \quad \begin{pmatrix} X & 2 \\ 21 & x \end{pmatrix} \preceq \mathbf{O}.$$

⇒ LMI 標準形 (等式標準形の双対問題)

$$\begin{aligned} \min & \quad x + 2y_1 + 2y_2 + 3y_3 \\ \text{sub.to} & \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} y_1 + \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} y_2 + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} y_3 \\ & \quad + \begin{pmatrix} \mathbf{O} & 0 \\ \mathbf{O} & 0 \\ 0 & 0 & 1 \end{pmatrix} x + \begin{pmatrix} \mathbf{O} & 2 \\ \mathbf{O} & 1 \\ 2 & 1 & 0 \end{pmatrix} \preceq \mathbf{O}. \end{aligned}$$

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対称行列 A の固有値

$$\begin{aligned}\text{最大固有値} &= \min \{ \lambda : \lambda I \succeq A \} \\ &= \min \{ \lambda : \lambda I - A \succeq O \} . \\ \text{最小固有値} &= \max \{ \lambda : A - \lambda I \succeq O \} .\end{aligned}$$

- A 線形行列不等式 (LMI) $A(\cdot) \succeq O$, ただし $A(\cdot)$ は行列, ベクトルに関する線形写像, は

$$\text{maximize } \lambda \text{ subject to } A(\cdot) - \lambda I \succeq O.$$

と定式化できる．例えば,

$$A(X) = \begin{pmatrix} XA + A^T X + C^T C & XB + C^T D \\ B^T X + D^T C & D^T D - I \end{pmatrix} \succeq O.$$

参考文献

[6] S. Boyd, L. El Ghaoui, E. Feron and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, SIAM, Philadelphia, 1994.

Matrix approximation problem — 1

F_p を $k \times \ell$ 行列とする ($0 \leq p \leq m$). F_0 を F_p ($1 \leq p \leq m$) の線形結合で近似せよ;

$$\text{minimize } \{\|F(\mathbf{x})\| : \mathbf{x} \in \mathbb{R}^m\},$$

$$\text{ただし, } F(\mathbf{x}) = F_0 - \sum_{p=1}^m F_p x_p \text{ for } \forall \mathbf{x} = (x_1, \dots, x_m)^T.$$

- この問題の最適化問題への定式化は norm の取り方に依存する

$$\|A\|_\infty = \max \{|A_{ij}| : 1 \leq i \leq k, 1 \leq j \leq \ell\} \text{ (}\infty \text{ norm)} \Rightarrow \text{LP}$$

$$\|A\|_F = \left(\sum_{i=1}^k \sum_{j=1}^{\ell} A_{ij}^2 \right)^{1/2} \text{ (Frobenius norm)} \Rightarrow \text{convex QP}$$

$$\|A\|_{L_2} = \max_{\|\mathbf{u}\|_2=1} \|A\mathbf{u}\| = (A^T A \text{ の最大固有値})^{1/2}$$

$(L_2 \text{ operator norm}) \Rightarrow \text{SDP}$

Matrix approximation problem — 2

F_p を $k \times \ell$ 行列とする ($0 \leq p \leq m$). F_0 を F_p ($1 \leq p \leq m$) の線形結合で近似せよ;

$$\text{minimize } \{ \|F(\mathbf{x})\| : \mathbf{x} \in \mathbb{R}^m \},$$

$$\text{ただし, } F(\mathbf{x}) = F_0 - \sum_{p=1}^m F_p x_p \text{ for } \forall \mathbf{x} = (x_1, \dots, x_m)^T.$$

$$\|A\|_{L_2} = \max_{\|\mathbf{u}\|_2=1} \|A\mathbf{u}\| = (A^T A \text{ の最大固有値})^{1/2}$$

(L_2 operator norm) \Rightarrow **SDP**

$$\text{minimize } \{ \|F(\mathbf{x})\|_{L_2} : \mathbf{x} \in \mathbb{R}^m \}$$

$$\Downarrow$$
$$\text{minimize " } F(\mathbf{x})^T F(\mathbf{x}) \text{ の最大固有値"}$$

$$\Downarrow$$
$$\text{minimize } \lambda \text{ subject to } \lambda I - F(\mathbf{x})^T F(\mathbf{x}) \succeq O$$

\Downarrow Schur complement

$$\text{minimize } \lambda \text{ subject to } \begin{pmatrix} I & F(\mathbf{x}) \\ F(\mathbf{x})^T & \lambda I \end{pmatrix} \succeq O \text{ (SDP)}$$

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LP の主問題と双対問題

$$(P) \quad \min \quad \mathbf{a}_0 \cdot \mathbf{x} \quad \text{s.t.} \quad \mathbf{a}_p \cdot \mathbf{x} = b_p \quad (\forall p), \quad \mathbf{x} \geq \mathbf{0}.$$

$$(D) \quad \max \quad \sum_{p=1}^m b_p y_p \quad \text{s.t.} \quad \sum_{p=1}^m \mathbf{a}_p y_p + \mathbf{s} = \mathbf{a}_0, \quad \mathbb{R}^n \ni \mathbf{s} \geq \mathbf{0}.$$

弱双対性:

$$\text{LP} \quad : \quad \mathbf{x} \cdot \mathbf{s} = \mathbf{a}_0 \cdot \mathbf{x} - \sum_{j=1}^m b_p y_p \geq 0, \quad \forall \text{許容解 } (\mathbf{x}, \mathbf{y}, \mathbf{s}).$$

$$\text{SDP} \quad : \quad \mathbf{X} \bullet \mathbf{S} = \mathbf{A}_0 \bullet \mathbf{X} - \sum_{j=1}^m b_p y_p \geq 0, \quad \forall \text{許容解 } (\mathbf{X}, \mathbf{y}, \mathbf{S}).$$

SDP の主問題と双対問題

$$(P) \quad \min. \quad \mathbf{A}_0 \bullet \mathbf{X} \quad \text{sub.to} \quad \mathbf{A}_p \bullet \mathbf{X} = b_p \quad (\forall p), \quad \mathbf{X} \succeq \mathbf{O}.$$

$$(D) \quad \max. \quad \sum_{p=1}^m b_p y_p \quad \text{sub.to} \quad \sum_{p=1}^m \mathbf{A}_p y_p + \mathbf{S} = \mathbf{A}_0, \quad \mathbf{S} \succeq \mathbf{O}.$$

LP の主問題と双対問題

$$(P) \quad \min \quad \mathbf{a}_0 \cdot \mathbf{x} \quad \text{s.t.} \quad \mathbf{a}_p \cdot \mathbf{x} = b_p \quad (\forall p), \quad \mathbf{x} \geq \mathbf{0}.$$

$$(D) \quad \max \quad \sum_{p=1}^m b_p y_p \quad \text{s.t.} \quad \sum_{p=1}^m \mathbf{a}_p y_p + \mathbf{s} = \mathbf{a}_0, \quad \mathbb{R}^n \ni \mathbf{s} \geq \mathbf{0}.$$

強双対性:

\exists 許容解 $(\mathbf{x}, \mathbf{y}, \mathbf{s})$ ($\mathbf{x} \geq \mathbf{0}, \mathbf{y} \geq \mathbf{0}$) \Rightarrow

$$\text{LP} : \quad \bar{\mathbf{x}} \cdot \bar{\mathbf{s}} = \mathbf{a}_0 \cdot \bar{\mathbf{x}} - \sum_{j=1}^m b_j \bar{y}_j = 0, \quad \forall \text{最適解 } (\bar{\mathbf{x}}, \bar{\mathbf{y}}, \bar{\mathbf{s}}).$$

\exists 内点許容解 $(\mathbf{X}, \mathbf{y}, \mathbf{S})$ ($\mathbf{X} \succ \mathbf{O}, \mathbf{S} \succ \mathbf{O}$) \Rightarrow

$$\text{SDP} : \quad \bar{\mathbf{X}} \bullet \bar{\mathbf{S}} = \mathbf{A}_0 \bullet \bar{\mathbf{X}} - \sum_{j=1}^m b_j \bar{y}_j = 0, \quad \forall \text{最適解 } (\bar{\mathbf{X}}, \bar{\mathbf{y}}, \bar{\mathbf{S}}).$$

SDP の主問題と双対問題

$$(P) \quad \min. \quad \mathbf{A}_0 \bullet \mathbf{X} \quad \text{sub.to} \quad \mathbf{A}_p \bullet \mathbf{X} = b_p \quad (\forall p), \quad \mathbf{X} \succeq \mathbf{O}.$$

$$(D) \quad \max. \quad \sum_{p=1}^m b_p y_p \quad \text{sub.to} \quad \sum_{p=1}^m \mathbf{A}_p y_p + \mathbf{S} = \mathbf{A}_0, \quad \mathbf{S} \succeq \mathbf{O}.$$

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SDP に対する数値解法

- 主双対内点法 (Primal-Dual Interior-point methods)
CSDP(Borchers[7]),
SDPA(Fujisawa-K-Nakata-Yamashita[49]),
SDPT3(Toh-Todd-Tutuncu[42]), SeDuMi(Sturm[37])
- Dual scaling, DSDP(Benson-Ye-Zhang[3])
- その他の非線形最適化
 - Spectral bundle method(Helmberg-Rendl[17])
 - Gradient-based log-barrier method(Burer-Monteiro[9])
 - PENON(Kocvara [19]) — Augmented Lagrangian
 - Saddle point mirror-prox algorithm
(Lu-Nemirovski-Monteiro[26])

- 中規模 (e.g. $n, m \leq 5000$), 高精度.
- 大規模 (e.g., $n, m \geq 10,000$), 低精度.

● 並列計算:

SDPA \Rightarrow SDPARA(Y-F-K[49]), SDPARA-C(N-Y-F-K[31])

DSDP \Rightarrow PDSDP(Benson[2]), CSDP \Rightarrow Borchers-Young[8]

Spectral bundle method \Rightarrow Nayakkankuppam[32]

目次

1. LP (線形計画) から SDP (半正定値計画) へ
2. SDP の位置づけ, 重要性, 応用
3. 等式標準形
4. 一般の SDP
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6. 双対性
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8. センサーネットワークの位置推定問題への応用
— 半正定値計画緩和の予告編 —

Sensor network localization 問題: $s = 2$ or 3 .

$\mathbf{x}^p \in \mathbb{R}^s$: sensor の位置, 座標 (未知) ($p = 1, 2, \dots, m$),

$\mathbf{x}^r = \mathbf{a}^r \in \mathbb{R}^s$: anchor の位置, 座標 (既知) ($r = m + 1, \dots, n$),

$d_{pq}^2 = \|\mathbf{x}^p - \mathbf{x}^q\|^2 + \epsilon_{pq}$ — 距離 (既知) for $(p, q) \in \mathcal{N}$,

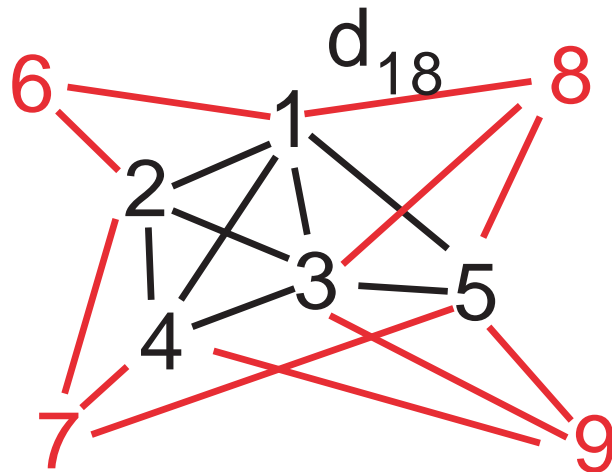
$\mathcal{N} = \{(p, q) : \|\mathbf{x}^p - \mathbf{x}^q\| \leq \rho = \text{a radio range (既知)}\}$

ここで ϵ_{pq} : noise

$m = 5, n = 9$.

1, ..., 5: sensor

6, 7, 8, 9: anchor



anchor の位置は既知

\forall edge に対して距離が既知

sensor の位置を計算せよ

• $\epsilon_{pq} = 0 \Rightarrow$ 多変数 2 次方程式系

$$\|\mathbf{x}^p - \mathbf{x}^q\|^2 = d_{pq}^2 \quad ((p, q) \in \mathcal{N}),$$

$$\mathbf{x}^r = \mathbf{a}^r \quad (r = m + 1, \dots, n)$$

• $\epsilon_{pq} \neq 0 \Rightarrow$ "誤差" の最小化

$$\text{minimize} \quad \sum_{(p,q) \in \mathcal{N}} \left| \|\mathbf{x}^p - \mathbf{x}^q\|^2 - d_{pq}^2 \right|$$

• いずれの場合も半正定値計画緩和 (\approx 近似) が適用出来る

m = sensor の個数, $[0, 1]^2$ 上に random に分布
 4 つの anchor $[0, 1]^2$ の 4 隅に配置
 ρ = radio range = 0.1, noise $\epsilon_{pq} = 0$

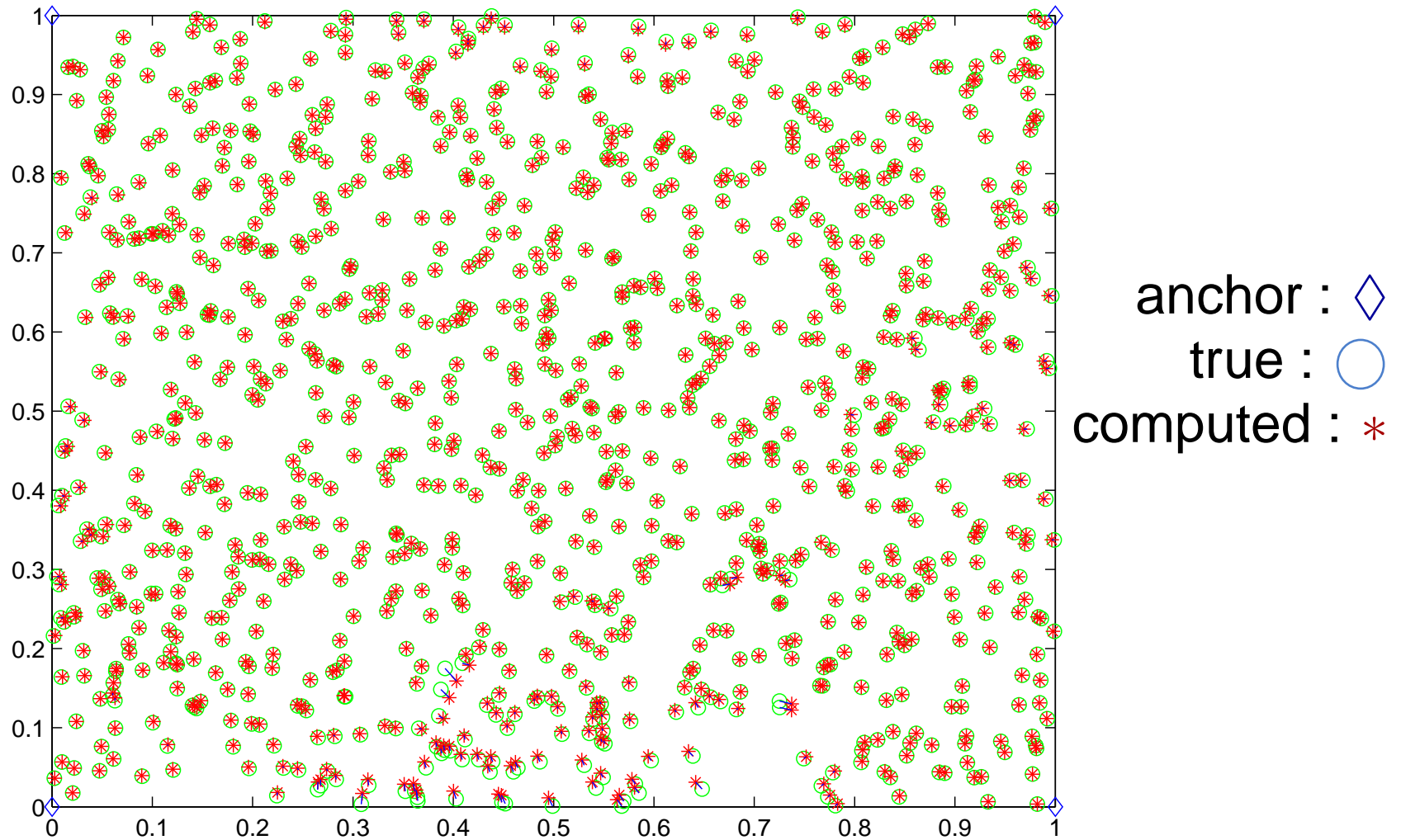
定式化: $\mathcal{N} = \{(p, q) : 1 \leq p < q \leq m, \|\mathbf{x}^p - \mathbf{x}^q\| < \rho\}$,
 $\|\mathbf{x}^p - \mathbf{x}^q\|^2 = d_{pq}^2 \quad (p, q) \in \mathcal{N}$ — 多変数 2 次方程式系,
 $\mathbf{x}^r = \mathbf{a}^r \quad (r = m + 1, \dots, m + 4)$

⇒ **SDP** に緩和 (\approx 近似)

m	1000	2000	4000
計算時間	11.7 秒	25.3 秒	41.9 秒
rmsd	4.8e-05	1.5e-05	2.6e-05

- root mean square distance = $\left(\frac{1}{m} \sum_{i=1}^m \|\mathbf{x}^p - \bar{\mathbf{x}}^p(\text{真値})\|^2 \right)^{1/2}$
- **SDP** は SDPA を用いて解いた . CPU = 2.7GHz Intel Core i7

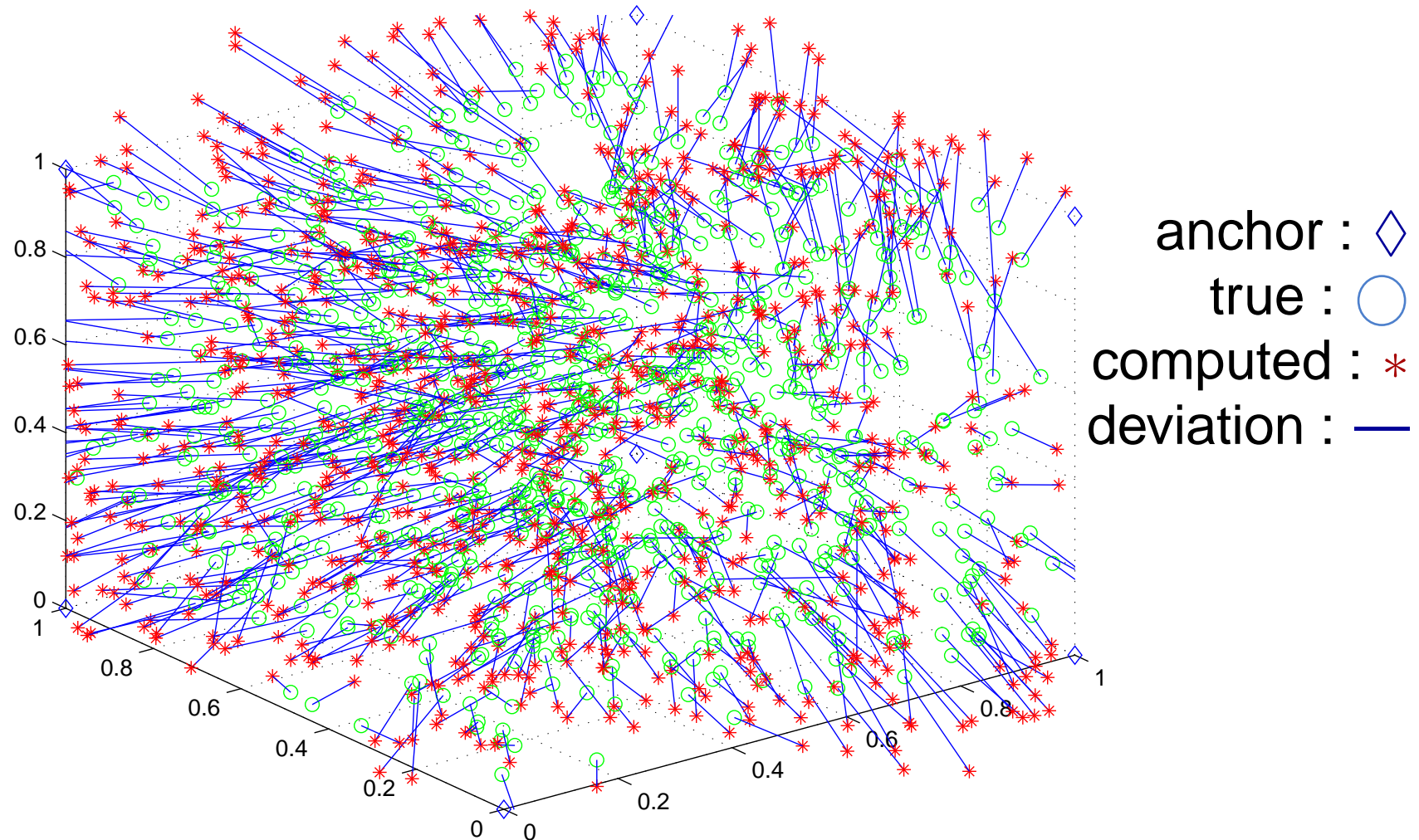
$m = 1000$ sensors, noise $\epsilon_{pq} = 0$



3 dim, 1000 sensors, radio range = 0.3, noise $\epsilon_{pq} \leftarrow N(0, 0.1)$;

(誤差を含んだ距離) $\hat{d}_{pq} = (1 + \epsilon_{pq})d_{pq}$ (真の距離)

定式化：多変数の2次等式条件の下での線形目的関数の最小化
⇒ **SDP** に緩和 (\approx 近似) 計算時間 82.9 秒

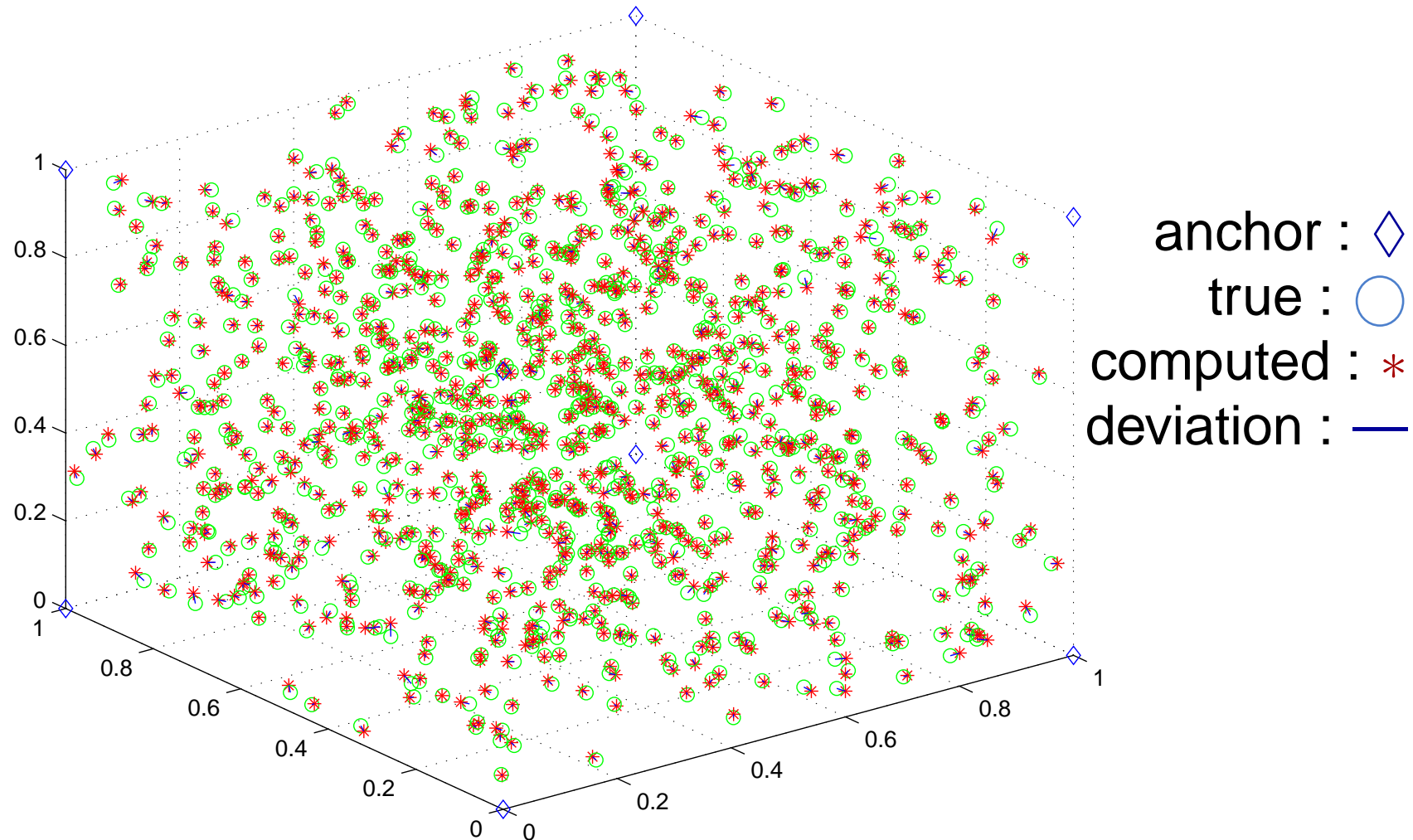


O : Sensor true locations vs * : the ones computed by SFSDP

3 dim, 1000 sensors, radio range = 0.3, noise $\epsilon_{pq} \leftarrow N(0, 0.1)$;

(誤差を含んだ距離) $\hat{d}_{pq} = (1 + \epsilon_{pq})d_{pq}$ (真の距離)

定式化：多変数の2次等式条件の下での線形目的関数の最小化
⇒ **SDP** に緩和 (\approx 近似) + **最急降下法** 計算時間 112.0 秒



O : Sensor true locations vs * : the ones computed by SFSDP

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