Some spectral and geometric properties for infinite graphs

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Dedicated to Professor Tokuzo Shiga on his sixtieth birthday

Abstract. For infinite graphs we discuss some topics on spectral and geometric properties, for example, full spectral property, isoperimetric constants, graph-operations, bipartiteness and so on. We also collect some problems around these topics.

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1. Introduction

For finite or infinite graphs, there are many kinds of researches on the relationship between geometric and spectral properties. Some of them clarify the similarities between finite (infinite) graphs and compact (non-compact) manifolds; others clarify the differences between them. The present paper is mainly concerned with spectral and geometric properties for infinite graphs from the latter point of view. There are three main topics, which are discussed in Sections 2, 3 and 4.

In Section 2 we treat laplacians on infinite, abelian covering graphs of finite graphs. The spectrum of our laplacian is a closed subset of the interval $[0, 2]$ for every graph and we say that a graph has “full spectrum property” if the laplacian on it has the whole interval $[0, 2]$ as its spectrum. We give sufficient conditions for an abelian covering graph to have full spectrum property. This property can also

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be considered in the manifold setting if one replaces \([0, 2]\) by \([0, \infty)\), however, our technique used in the proof cannot be directly applied to the manifold case since it heavily depends on graph structure.

In Section 3 we introduce a kind of curvature for planar graphs and give a sufficient condition for both the bottom of the spectrum and the isoperimetric constant to be away from 0. In particular, we determine the isoperimetric constants explicitly for a special class of planar graphs that are called \((d, f)\)-regular planar graphs. We also consider the upper bound of the spectrum. It is unique to graphs simply because a discrete laplacian is a bounded operator while a Laplace-Beltrami operator on a manifold is usually unbounded. We show that the upper bound of the spectrum of a laplacian is closely related to a geometric property, so-called the bipartiteness.

In Section 4 we discuss about three graph-operations: line graphs, subdivisions and para-line graphs. There might be no analogous objects for manifolds unless one considers singular manifolds. We show how the spectra change under those graph-operations, and as an application of these results we determine the spectrum of an infinite Sierpiński lattice graph.

In Section 5, we collect some problems around topics which are discussed in the present paper.

The rest of this section is devoted to the explanation of our setting which is used throughout the present paper. Let \(G = (V(G), E(G))\) be a connected, locally finite graph, where \(V(G)\) is the set of its vertices and \(E(G)\) is the set of its unoriented edges. A graph \(G\) may have self-loops and multiple edges. Considering each edge in \(E(G)\) to have two orientations, we introduce the set of all oriented edges: we denote it by \(A(G)\). For an edge \(e \in A(G)\), the origin vertex and the terminal one of \(e\) are denoted by \(o(e)\) and \(t(e)\), respectively. The inverse edge of \(e\) is denoted by \(\bar{e}\). We sometimes write \([e]\) for the unoriented edge which is obtained from \(e \in A(G)\) (or \(\bar{e}\)) by forgetting its orientation.

Let \(p : A(G) \to (0, 1]\) be a transition probability such that
\[
\sum_{e \in A_x(G)} p(e) = 1,
\]
where \(A_x(G) = \{ e \in A(G) \mid o(e) = x \}\). We assume that \(p\) is reversible, that is, there exists a positive valued function \(m : V(G) \to (0, \infty)\) such that
\[
(1.1) \quad m(o(e))p(e) = m(t(e))p(\bar{e}) \quad (=: m_A(e))
\]
for every oriented edge \(e \in A(G)\). The function \(m\) is called a reversible measure for \(p\) and it is unique, if it exists, up to a multiple constant. We set
\[
C^0(G, C) = \{ f : V(G) \to C \}
\]
as the space of all functions on \(G\) and denote by \(\ell^2(G)\) or \(\ell^2(G, m)\) the space of all square sumvable functions on \(V(G)\) with respect to the inner product \(\langle f_1, f_2 \rangle = \sum_{x \in V(G)} f_1(x) \overline{f_2(x)} m(x)\). Define the discrete laplacian \(\Delta_G : \ell^2(G) \to \ell^2(G)\) by
\[
\Delta_G f(x) = \sum_{e \in A_x(G)} p(e) f(t(e)) - f(x).
\]
It becomes a self-adjoint operator on \(\ell^2(G)\) because of the reversibility (1.1). We set
\[
C^1(G, C) = \{ \omega : A(G) \to C \mid \omega(\bar{e}) = -\omega(e) \text{ for } e \in A(G) \}\]
as the space of all 1-forms on \( G \) and define the coboundary operator \( d : C^0(G, \mathbb{C}) \to C^1(G, \mathbb{C}) \) by
\[
df(e) = f(t(e)) - f(o(e))
\]
for every \( e \in A(G) \). We introduce an inner product on \( C^1(G, \mathbb{C}) \) by
\[
\langle \omega_1, \omega_2 \rangle = \frac{1}{2} \sum_{e \in A(G)} \omega_1(e) \overline{\omega_2(e)} m_A(e).
\]
Then the dual \( \delta \) of \( d \) is defined by the equality \( \langle df, \omega \rangle = \langle f, \delta \omega \rangle \) and expressed by
\[
\delta \omega(x) = -\sum_{e \in A_x} p(e) \omega(e).
\]
It is easy to check that \( \Delta_G = -\delta d \). In Section 3 we use a quadratic form \( \mathcal{E} \) defined by
\[
(1.2) \quad \mathcal{E}(f, f) = \langle -\Delta_G f, f \rangle = \langle df, df \rangle.
\]
In this paper, we often take \( (\deg_G o(e))^{-1} \) as \( p(e) \) and \( \deg_G x \) as \( m(x) \), which are associated with the simple random walk on \( G \). Here \( \deg_G x = \# A_x(G) \) and it is called the degree of \( x \). In this case, they satisfy (1.1) and the laplacian is expressed by
\[
\Delta_G f(x) = (\deg_G x)^{-1} \sum_{e \in A_x(G)} f(t(e)) - f(x).
\]
Our main concern of this paper is to study the relationship between the spectral properties of \( \Delta_G \) and the geometric properties of \( G \).

Finally let us recall the notion of bipartiteness, which plays an important role throughout this paper. We call a graph \( G \) bipartite if \( G \) has no closed path of odd length. In other words, the vertex set \( V(G) \) can be partitioned into two disjoint sets \( V_1 \) and \( V_2 \) in such a way that every edge in \( E(G) \) connects a vertex in \( V_1 \) with a vertex in \( V_2 \).

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2. Full spectrum property for maximal abelian covering graphs

After giving some preliminaries around abelian covering structure of graphs, we discuss some conditions for abelian covering graphs to have full spectrum property in Section 2.2.

2.1. Abelian covering graphs and twisted operators. For a finite connected graph \( M \), we define the chain groups with coefficients in \( \mathbb{Z} \) by
\[
C_0(M, \mathbb{Z}) = \{ \sum_{x \in V(M)} a_x x \mid a_x \in \mathbb{Z} \}, \quad C_1(M, \mathbb{Z}) = \{ \sum_{e \in A(M)} a_e e \mid a_e \in \mathbb{Z} \} / \sim,
\]
where \( \sim \) is the relation \( \tilde{e} = -e \) for \( e \in A(M) \). The boundary map \( \partial : C_1(M, \mathbb{Z}) \to C_0(M, \mathbb{Z}) \) is defined by \( \partial(e) = t(e) - o(e) \). The 1-homology group \( H_1(M, \mathbb{Z}) \) is equal
to ker $\partial$. We also define the cochain groups with coefficients in an abelian group $A$
by
\[ C^0(M, A) = \{ f : V(M) \to A \}, \]
\[ C^1(M, A) = \{ \omega : A(M) \to A \mid \omega(\bar{e}) = -\omega(e) \text{ for } e \in A(M) \}. \]
The coboundary map $d : C^0(M, A) \to C^1(M, A)$ is the same one as was defined in
Section 1 and the 1-cohomology group is defined by $H^1(M, A) = C^1(M, A)/\text{Im}d$. It
is known as the discrete Kodaira-Hodge theorem that $H^1(M, \mathbb{R})$ is identified with
the space $\{ \omega \in C^1(M, \mathbb{R}) \mid \delta\omega = 0 \}$ of harmonic 1-forms. In this identification, the
integral 1-cohomology group $H^1(M, \mathbb{Z})$ corresponds to the lattice subgroup
\[ \{ \omega \in H^1(M, \mathbb{R}) \mid \int_c \omega \in \mathbb{Z} \text{ for every closed path } c \text{ in } M \}, \]
where $\int_c \omega = \sum_{i=1}^n \omega(e_i)$ for $c = (e_1, e_2, \ldots, e_n)$. Here a path $p = (e_1, e_2, \ldots, e_n)$ of
length $n$ is a sequence of oriented edges with $t(e_i) = o(e_{i+1})$ for $i = 1, \ldots, n-1$; a
closed path is a path such that $t(e_n) = o(e_1)$.

We recall that a map $\gamma : V(G) \to V(G)$ is an automorphism if it is a bijection
satisfying the following: two vertices $x_1$ and $x_2$ are adjacent in $G$ if and only if the
vertices $\gamma(x_1)$ and $\gamma(x_2)$ are adjacent in $G$. See also Section 4.3. Now we assume
that an abelian group $\Gamma$ acts freely on $G = (V(G), E(G))$ as an automorphism
group. The quotient $M = \Gamma \backslash G$ naturally has a graph structure and we assume
$M = (V(M), E(M))$ is finite. The canonical morphism $\pi : G \to M$ is a covering
map in the sense that
\begin{enumerate}
\item $\pi$ is surjective as a map between the sets of vertices,
\item the map $\pi|_{A_x(G)} : A_x(G) \to A_{\pi(x)}(M)$ is bijective for every $x \in V(G)$.
\end{enumerate}
Hence $G$ can be regarded as an abelian covering graph of $M$ with covering transformation
map $\Gamma$. In particular, when the transformation group is the 1-homology
group $H_1(M, \mathbb{Z})$ of $M$, the corresponding covering graph of $M$ is said to be the
maximal abelian covering graph of $M$ and is denoted by $M^{ab}$. The covering graph
$M^{ab}$ is maximal in the sense that for any abelian covering graph $G$ of $M$ with covering
transformation group $\Gamma$ there exists a covering map $M^{ab} \to G$ which factorizes
the covering map $M^{ab} \to M$. The covering transformation group $\Gamma$ is isomorphic
to the quotient group $H_1(M, \mathbb{Z})/\Gamma_1$, where $\Gamma_1$ is the covering transformation group
of the covering map $M^{ab} \to G$.

The group $\widehat{H}_1(M, \mathbb{Z})$ of unitary characters of the 1-homology group $H_1(M, \mathbb{Z})$
is identified with the Jacobian torus $J(M) = H^1(M, \mathbb{R})/H^1(M, \mathbb{Z})$ in the following
way. For a given $\omega \in H^1(M, \mathbb{R})$, define the unitary character $\chi_\omega$ by
\[ \chi_\omega(\sigma) = \exp(2\pi \sqrt{-1} \int_c \omega). \]
where $\sigma$ is represented by a closed path $c$, that is a $\mathbb{Z}$-linear combination of closed
paths in $M$. The correspondence $\omega \mapsto \chi_\omega$ gives rise to an isomorphism of the
Jacobian torus onto the group of unitary characters of $H_1(M, \mathbb{Z})$.

We denote by $\widehat{\Gamma}$ the group of unitary characters of $\Gamma$. The surjective homomorphism $\eta : H_1(M, \mathbb{Z}) \to \Gamma$ gives rise to an injective homomorphism $\widehat{\eta} : \widehat{\Gamma} \to \widehat{H}_1(M, \mathbb{Z}) \cong J(M)$. We identify $\widehat{\Gamma}$ with its image in $J(M)$. Then $\chi_\omega \in \widehat{\Gamma}$ if and
only if $\int_c \omega \in \mathbb{Z}$ for every closed path $c$ in $M$ with $\eta(c) = 1$.

Now we assume that the probability $p$ and the reversible measure $m$ on $G$ are
invariant under the $\Gamma$-action. In other words, $p$ and $m$ are defined as the lift of a
transition probability $p_M$ and $m_M$ on $M$, that is, $p(e) = p_M(\pi(e))$ for any $e \in A(G)$ and $m(x) = m_M(\pi(x))$ for any $x \in V(G)$, respectively. For any $\chi \in \widehat{\Gamma}$, we put
\[ \ell^2_{\chi} = \{ f : V(G) \to \mathbb{C} \mid f(\sigma x) = \chi(\sigma)f(x) \text{ for any } \sigma \in \Gamma \} \]
with the inner product $\langle f, g \rangle_{\chi} = \sum_{x \in \mathcal{D}} f(x)\overline{g(x)}m(x)$, where $\mathcal{D}$ is a fundamental set of $V(G)$ for $\Gamma$. Since $\Delta_G$ commutes with the $\Gamma$-action, we can restrict it to $\ell^2_{\chi}$. We call the restriction $\Delta_G|_{\ell^2_{\chi}}$ the twisted laplacian and denote it by $\Delta_{G, \chi}$. It follows from the theory of direct integral that
\[ \text{Spec}(-\Delta_G) = \bigcup_{\chi \in \widehat{\Gamma}} \text{Spec}(-\Delta_{G, \chi}). \tag{2.1} \]
where $\text{Spec}(-\Delta_G)$ is the spectrum set of $-\Delta_G$ on $\ell^2(G)$. For any $\chi \in \widehat{H}_1(M, \mathbb{Z})$, one can find $\theta \in H^1(M, \mathbb{R})$ such that $\chi = \chi_{\theta}$. Moreover, it is known that $\Delta_{G, \chi} : \ell^2_{\chi} \to \ell^2_{\chi}$ is unitarily equivalent to $L_{\theta, M} : \ell^2(M) \to \ell^2(M)$ which is defined by
\[ L_{\theta, M}f(x) = \sum_{e \in A_+(M)} p(e)\exp(2\pi i \theta(e))f(t(e)) - f(x). \tag{2.2} \]
Consequently, we can rewrite (2.1) in terms of $\theta$ as
\[ \text{Spec}(-\Delta_G) = \bigcup_{\theta \in H^1(M, \mathbb{R})} \text{Spec}(-L_{\theta, M}). \tag{2.3} \]
In particular, if $G$ is the maximal abelian covering graph $M^{ab}$ of $M$, we have
\[ \text{Spec}(-\Delta_{M^{ab}}) = \bigcup_{\theta \in H^1(M, \mathbb{R})} \text{Spec}(-L_{\theta, M}). \tag{2.4} \]

2.2. Full spectrum property. By the definition of a discrete laplacian, it holds that $\text{Spec}(-\Delta_G)$ is a closed subset of $[0, 2]$.

**Definition 2.1** (Full Spectrum Property). We say that an infinite graph $G$ has Full Spectrum Property (FSP) if it holds that $\text{Spec}(-\Delta_G) = [0, 2]$ for the laplacian with respect to the simple random walk: for all $e \in A(G)$. $p(e) = (\deg_G o(e))^{-1}$.

Hereafter in Section 2.2, we discuss FSP under the assumption that graphs have abelian covering structure.

The $d$-dimensional integer lattice $\mathbb{Z}^d$ is the maximal abelian covering graph of a $d$-bouquet graph for every natural number $d$, and it is known that $\text{Spec}(-\Delta_{\mathbb{Z}^d}) = [0, 2]$. Then we can say that the infinite graph $\mathbb{Z}^d$ has FSP. On the other hand, the triangular lattice is an abelian covering graph of a 3-bouquet graph but not the maximal one. It does not have FSP since its spectrum set is $[0, 3/2]$. More generally, it follows from Remark 3.12 that a non-bipartite abelian covering graph of a finite graph does not have FSP since it turns out to be “essentially non-bipartite”. It should be noted that the maximal abelian covering graph of a finite graph is always bipartite.

**Proposition 2.2** (cf. [10]). The maximal abelian covering graph $M^{ab}$ of a finite graph $M$ is bipartite. So the spectrum set $\text{Spec}(-\Delta_{M^{ab}})$ is symmetric with respect to 1.

Thus it might be natural to ask whether infinite bipartite covering graphs have FSP. However, the following example shows that it is not always the case.
Example 2.3. Let $M$ be a finite graph such that $V(M) = \{0, 1\}$ and $A(M) = \{e_i, \bar{e}_i, i = 1, 2, 3\}$ with $o(e_i) = 0$ and $t(e_i) = 1$, $i = 1, 2, 3$. The 1-homology group of $M$ is isomorphic to $\mathbb{Z}^2$ and its maximal abelian covering graph is the hexagonal lattice, which has FSP. Take $c_1 = e_1 + \bar{e}_2, c_2 = e_2 + \bar{e}_3$ as a basis of $H_1(M, \mathbb{Z})$ and let $\Gamma_1$ be the subgroup of $H_1(M, \mathbb{Z})$ generated by $c_1$. The quotient graph $M^{ab}$ by $\Gamma_1$ is isomorphic to the abelian covering graph of $M$ with covering transformation group $\Gamma \cong H_1(M, \mathbb{Z})/\Gamma_1$, which is the graph $G$ such that $V(G) = \mathbb{Z}^l$ and $A(G) = \{e_n, \bar{e}_n, e'_m, \bar{e}'_m, n \in \mathbb{Z}, m \in 2\mathbb{Z}\}$, where $o(e_n) = o(e'_n) = n$ and $t(e_n) = t(e'_n) = n + 1$. It is easy to compute the spectrum of $-\Delta_G$ by using (2.3): it is $[0.2/3] \cup [4/3, 2]$, not equal to [0, 2]. Then $G$ does not have FSP. Consequently, $G$ does not necessarily have FSP even if it is an infinite bipartite covering.

In [10], the following problem is raised as a conjecture:

**Conjecture 2.4.** Assume that $G$ is an infinite connected graph such that $G$ is isomorphic to the maximal abelian covering graph $M^{ab}$ of a finite connected graph $M$. Then $G$ has FSP.

We have some partial affirmative answers toward this conjecture:

**Proposition 2.5 ([10]).** Assume $G$ is an infinite connected graph and $G$ is isomorphic to the maximal abelian covering graph $M^{ab}$ of a finite connected graph $M$. Then $G$ has FSP if one of the following conditions holds:

1. $\deg_G x \in 2\mathbb{N}$ for any vertex $x \in V(G)$.
2. $M$ is a $(2d + 1)$-regular graph having a $2\ell$-factor such that $[(d + 1)/2] \leq \ell \leq d$, where $d \geq 1$ and $[x]$ is the minimum integer which is equal or greater than $x$.

In the statement above, the terminology "k-factor" appears. Although this is a well-known notion in graph theory (cf.[1, 2]), here we give the definition:

**Definition 2.6 (k-factor).** Let $G = (V(G), E(G))$ be a finite graph. A graph $H = (V(H), E(H))$ is called a spanning subgraph or a factor of $G$ if $V(H) = V(G)$ and $E(H) \subset E(G)$. In addition, a factor $H$ is called a $k$-factor if $\deg_H x = k$ for every vertex $x \in V(H)$.

Here we give a proof of Proposition 2.5 after recalling a key notion and property which is used in the proof:

**Definition 2.7 (Euler circuit).** A closed path $c = (e_1, e_2, \ldots, e_n)$ in a finite graph $M$ is called an Euler circuit if $n = \#A(M)/2$ and, for each $e \in A(M)$, there exists $i$ ($1 \leq i \leq n$) such that $e_i$ or $\bar{e}_i$ equals to $e$.

**Remark 2.8.** It is well-known as a classical and famous fact in graph theory (cf.[1, 2]) that a finite graph $M$ has an Euler circuit if and only if the degree of each vertex is even.

**Proof of Proposition 2.5.** Firstly we assume the condition (1). Then the quotient graph $M$ also satisfies $\deg_M x \in 2\mathbb{N}$ for any vertex $x \in V(M)$, and $M$ has an Euler circuit $c = (e_1, \ldots, e_n)$. Then we put a 1-form $\theta \in \mathcal{C}^1(M, \mathbb{R})$ as $\theta(e_i) = \alpha \in \mathbb{R}$ and $\theta(\bar{e}_i) = -\alpha$ for every $i = 1, 2, \ldots, n$. It is easy to see that $\{e_i, \bar{e}_i\}_{i=1}^n = A(M)$ and

$$\# \{ e \in A_x(M) \mid \theta(e) = \alpha \} = \# \{ e \in A_x(M) \mid \theta(e) = -\alpha \} = (\deg_M x)/2$$
for every vertex \( x \in V(M) \). For every \( \alpha \), the 1-form \( \theta \) defined as above is harmonic, that is, \( \theta \in H^1(M, \mathbb{R}) \). Denote by \( 1 \) a constant function on \( V(M) \) such that \( 1(x) = 1 \). Then

\[
-L_{\theta,M}1(x) = -(\deg_M x)^{-1} \sum_{e \in A_x(M)} \exp \left( 2\pi \sqrt{-1} \theta(e) \right) 1(t(e)) + 1(x)
\]

\[
= 1 - (\deg_M x)^{-1} \cdot (\deg_M x)/2 \cdot \left( \exp \left( 2\pi \sqrt{-1} \alpha \right) + \exp \left( -2\pi \sqrt{-1} \alpha \right) \right)
\]

\[
= (1 - \cos 2\pi \alpha)1(x).
\]

This implies that the quantity \( 1 - \cos 2\pi \alpha \) is an eigenvalue of \( -L_{\theta,M} \). By (2.4),

\[
\bigcup_{\alpha \in \mathbb{R}} (1 - \cos 2\pi \alpha) \subset \bigcup_{\theta \in H^1(M, \mathbb{R})} \text{Spec}(-L_{\theta,M}) = \text{Spec}(-\Delta_G),
\]

thus we get the conclusion.

Secondly we assume the condition (2). Then each component \( M_j \) of a 2\( \ell \)-factor has an Euler circuit \( c_j = (e_{j1}, e_{j2}, \ldots, e_{j\ell_j}) \) in \( M_j \), and we set a 1-form \( \theta \in C^1(M, \mathbb{R}) \) as \( \theta(e_{ji}) = -\theta(\overline{e_{ji}}) = \alpha \) for every \( i, j \), and \( \theta(\cdot) = 0 \) for other edges. It is easy to see that

\[
\# \{ e \in A_x(M) \mid \theta(e) = \alpha \} = \# \{ e \in A_x(M) \mid \theta(e) = -\alpha \} = \ell
\]

for every vertex \( x \in V(M) \). As was in the proof under the condition (1),

\[
-L_{\theta,M}1(x) = -(2d + 1)^{-1} \sum_{e \in A_x(M)} \exp \left( 2\pi \sqrt{-1} \theta(e) \right) 1(t(e)) + 1(x)
\]

\[
= \frac{2\ell}{2d+1} (1 - \cos 2\pi \alpha)1(x).
\]

Thus we have \([0, 4\ell/(2d+1)] \subset \text{Spec}(-\Delta_G)\), where \(4\ell/(2d+1) > 1\). The conclusion follows from Proposition 2.2 and the proof is completed. \( \square \)

We have obtained some affirmative partial results for this conjecture. However, we also found an example which shows that Conjecture 2.4 needs an additional assumption.

**Example 2.9.** Let \( M_{m,n} \) be a finite graph which looks like a “magnifier glass”:

\[
V(M_{m,n}) = \{ x_0, x_1, \ldots, x_{m-1}, x_m = y_1, y_2, \ldots, y_n \},
\]

\[
E(M_{m,n}) = \{ x_0x_1, x_1x_2, \ldots, x_{m-1}x_m \} \cup \{ y_1y_2, y_2y_3, \ldots, y_{n-1}y_n, y ny_1 \}.
\]

In other words, \( M_{m,n} \) is isomorphic to \( P_m + C_n \), where the symbol \( + \) signifies

![Figure 1. M_{m,n} (m = 4 and n = 8).](image)

that an endpoint of the path \( P_m \) with length \( m \) and a vertex in the cycle \( C_n \) are identified. Let \( G_{m,n} \) be the maximal abelian covering graph \( M_{m,n}^{ab} \) of \( M_{m,n} \). Then
the graph $G_{m,n}$ is an infinite path to which finite paths are attached periodically. Direct calculation shows that the characteristic equation of the twisted operator $L_{\theta, M_{m,n}}$ with $\int_{C_n} \theta = \alpha \in \mathbb{R}$ is equivalent to
\[
T_{m+n}(\lambda) = (2 \cos 2\pi \alpha - T_n(\lambda)) \cdot T_m(\lambda),
\]
where $T_n(\lambda) = \cos(n \arccos \lambda)$ is the Tchebychev polynomial of the first kind. It leads us to the following: let $k$ be the greatest common divisor $\gcd(m,n)$ of $m$ and $n$, and let $N = m + n + 1 - k$. Then the spectrum set of $-\Delta_{G_{m,n}}$ is given by the form
\[
\text{Spec}(-\Delta_{G_{m,n}}) = \bigcup_{i=1}^{N}[a_i, b_i],
\]
where $0 = a_1 < b_1 < a_2 < b_2 < \cdots < a_{N-1} < b_{N-1} < a_N < b_N = 2$. In particular, $G_{m,n}$ does not have FSP. Moreover, when $n/k$ is odd, then $-\Delta_{G_{m,n}}$ has no eigenvalue and $a_i < b_i$ for all $1 \leq i \leq N$; when $n/k$ is even, then $-\Delta_{G_{m,n}}$ has $k$ eigenvalues and there exist $1 < i_1 < i_2 < \cdots < i_k < N$ only for which $a_{i_j} = b_{i_j} (1 \leq j \leq k)$.

We showed that Conjecture 2.4 is true if $M$ is an even-regular graph or an odd-regular graph having some properties. Then this conjecture would be true at least for regular graphs though we do not show at this stage (see also Problem 5.1). On the other hand, by Example 2.9, we now know Conjecture 2.4 itself is not true for all maximal covering graphs. However, we strongly feel that such a case is rare. For example, we believe that Conjecture 2.4 is true if we assume $M$ has no vertex whose degree is 1. We close this section with proposing the following problem.

**Problem 2.10.** Characterize all finite graphs whose maximal abelian covering graphs do not have FSP.

### 3. Spectral and geometric constants

In Section 3.1, we give the isoperimetric constants for a special class of hyperbolic graphs explicitly. In Section 3.2, we introduce a kind of curvature and discuss its properties. In Section 3.3, we show the relationship between non-bipartiteness and spectral/geometric constants.

In Section 3.1 we assume that $p(e) = (\deg_G o(e))^{-1}$ and $m(x) = \deg_G x$, so the quantities, such as $\mathcal{E}(f,f)$ or $\Delta_G$, which appear in this section are correspond to the simple random walk. In Section 3.3 we do not assume it.

#### 3.1. Isoperimetric constant for a $(d,f)$-regular planar graph

In this subsection, we consider the isoperimetric constant for a $(d,f)$-regular planar graph.

The isoperimetric constant is defined as
\[
h(G) = \inf \{ \frac{|E(\partial_x K)|}{\text{Area}(K)} \mid K \text{ is a finite subgraph of } G \}.
\]
where $\text{Area}(K) = \sum_{x \in V(K)} \deg_G x$ and $E(\partial_x K) = \{ xy \in E(G) \mid x \in V(K), y \in V(G) \setminus V(K) \}$. Another equivalent definition is also useful. It is easy to see that
\[
h(G) = \inf \{ \frac{\mathcal{E}(1_K,1_K)}{\|1_K\|^2} \mid K \text{ is a finite subgraph of } G \}.
\]
where $1_K$ is the indicator function of $V(K)$ and $E(f, f)$ was defined in (1.2). We have the following well-known inequality between the isoperimetric constant $h(G)$ and the lower bound $\lambda_0(G)$ of the spectrum of $-\Delta_G$:

$$1 - \sqrt{1 - h^2(G)} \leq \lambda_0(G) \leq h(G) < 1.$$  

(3.2)

The second inequality follows immediately from the variational formulas (3.1) and (3.3).

If $G$ is planar, there is another kind of isoperimetric constant which is a dual notion of $h(G)$: for an embedding $\varphi$ of $G$ in the plane $\mathbb{R}^2$, one can define

$$h^*(G) = h^*(G, \varphi) = \inf \left\{ \frac{|E(\partial_f K)|}{|F(K)|} \mid K \text{ is a finite subgraph of } G \right\},$$

where $F(K)$ is the set of all faces $K$ and $E(\partial_f K) = \{ xy \in E(K) \cap E(F') \mid F' \in F(G) \setminus F(K) \}$.

Both of them are discrete analogues of the isoperimetric constant for a manifold: each denominator corresponds to the “area” of $K$, and each numerator to the “length” of the boundary of $K$. We remark that $h(G)$ is a graph invariant while $h^*(G)$ may depend on the way of embedding in the plane.

Now we introduce a special class of graphs. A graph $G$ is said to be a $(d, f)$-regular planar graph if $G$ is a planar graph satisfying the following:

1. $G$ is $d$-regular in the ordinary sense. That is, $\deg_G x = d$ for every vertex $x \in V(G)$ and $d \geq 3$.
2. $G$ admits an embedding in the plane $\mathbb{R}^2$ in such a way that every face $F$ is an $f$-gon with $f \geq 3$ and the number of vertices contained in each compact subset in $\mathbb{R}^2$ is finite.

Here we call such an embedding a $(d, f)$-regular realization of $G$. For any pair $(d, f)$ with $H(d, f) := 4 - (d - 2)(f - 2) \leq 0$ there exists an infinite $(d, f)$-regular planar graph. For example, the hexagonal lattice, the square lattice $\mathbb{Z}^2$ and the triangular lattice are $(3, 6)$, $(4, 4)$, $(6, 3)$-regular planar graphs, respectively. They correspond to the case $H(d, f) = 0$ and their spectra are explicitly known. As far as we know, the spectrum set of $\Delta_G$ for the other “hyperbolic” cases, $H(d, f) < 0$, has not yet been computed explicitly. Rather we can compute the isoperimetric constant explicitly.

**Theorem 3.1 ([7],[10]).** Let $d \geq 3$ and $f \geq 3$ with $H(d, f) \leq 0$ and $G$ be a $(d, f)$-regular planar graph. Then we have

$$h(G) = \frac{d - 2}{d} \sqrt{\frac{H(d, f)}{H(d, f) - 4}}$$

and for a $(d, f)$-regular realization of $G$, we have

$$h^*(G) = (f - 2) \sqrt{\frac{H(d, f)}{H(d, f) - 4}}.$$

**Remark 3.2.** The case of $f = \infty$ may be understood as a $d$-regular tree $T_d$. In this case, it is well-known that $h(T_d) = (d - 2)/d$ and then the expression of $h(G)$ is still valid by considering its value as the limit as $f$ tends to $\infty$. 
3.2. Combinatorial curvature. In this subsection we always assume $G$ to be a planar graph embedded in the plane. The quantity $H(d, f)$ plays a role like a curvature for a $(d, f)$-regular planar graph $G$ in the previous subsection. For general planar graphs embedded in the plane, one can discuss another kind of curvature which is called in [8] a combinatorial curvature defined as follows: for each vertex $x$,

$$
\kappa_G(x) = 1 - \frac{\deg_G x}{2} + \sum_{i=1}^{\deg_G x} \frac{1}{d(F_i)},
$$

where $F_1, F_2, \ldots, F_{\deg_G x}$ are the faces around a vertex $x$ and $d(F)$ is equal to $f$ if a face $F$ is an $f$-gon.

**Remark 3.3.** For a $(d, f)$-regular planar graph $G$, $\kappa_G(x) = H(d, f)/2f$ for all $x \in V(G)$.

While we can compute the isoperimetric constants for $(d, f)$-regular planar graphs, it is almost impossible to determine the isoperimetric constants for general planar graphs. Here, instead of giving the explicit values, we give a sufficient condition for the positivity of the isoperimetric constant in terms of the combinatorial curvature.

**Theorem 3.4.** If $\kappa_G(x) < 0$ for every vertex $x \in V(G)$, $h^*(G) > 0$.

The combinatorial curvature $\kappa_G$, by definition, takes values in $\mathbb{Q}$. An interesting feature of the combinatorial curvature is that the negative values of $\kappa_G$ do not accumulate to 0 while the positive values may do.

**Theorem 3.5.** If $\kappa_G(x) < 0$, then $\kappa_G(x) \leq -1/1806$.

For details, one can refer to [8].

**Remark 3.6.** A similar type of curvature $\kappa^*_G(\cdot)$ was introduced by M. Gromov [6]. This is a dual notion of the combinatorial curvature discussed above in the sense that it is defined on the set of faces while the above is on the set of vertices. In fact, for a $(d, f)$-regular planar graph $G$, $\kappa^*_G(F) = H(d, f)/2d$ for all $F \in F(G)$, where $F(G)$ is the set of faces of an infinite planar graph $G$. In [8] we also showed that $h(G) > 0$ if $\kappa^*_G(F) < 0$ for every face $F \in F(G)$.

3.3. Non-bipartiteness and Dirichlet form. Now we recall the well-known variational formula for the lower and upper bound for $-\Delta_G$, say $\lambda_0(G)$ and $\lambda_\infty(G)$, respectively:

\begin{equation}
\lambda_0(G) = \inf_{f \in \ell^2(G) \setminus \{0\}} \frac{\mathcal{E}(f, f)}{\|f\|^2}, \quad \lambda_\infty(G) = \sup_{f \in \ell^2(G) \setminus \{0\}} \frac{\mathcal{E}(f, f)}{\|f\|^2}.
\end{equation}

Keeping this formula in mind, we generalize these constants by

\begin{equation}
\alpha_0(G, \mathcal{F}) = \inf_{f \in \mathcal{F} \setminus \{0\}} \frac{\mathcal{E}(f, f)}{\|f\|^2}, \quad \alpha_\infty(G, \mathcal{F}) = \sup_{f \in \mathcal{F} \setminus \{0\}} \frac{\mathcal{E}(f, f)}{\|f\|^2}.
\end{equation}

where $\mathcal{F}$ is a nonempty subset of $\ell^2(G)$ containing at least one element other than 0.

**Example 3.7.** When $\mathcal{F} = \ell^2(G)$, $\alpha_0(G, \mathcal{F})$ and $\alpha_\infty(G, \mathcal{F})$ are, of course, the lower and upper bound of the spectrum of $-\Delta_G$, respectively. Moreover, if we take
the set \( \{ f \in \ell^2(G) : |f(x)| \in \{0, 1\}, \forall x \in V(G) \} \) as \( \mathcal{F} \), then it follows from the variational formula (3.1) that \( \alpha_0(G, \mathcal{F}) \) is nothing but the isoperimetric constant \( h(G) \). Here \( \alpha_\infty(G, \mathcal{F}) \) is the corresponding upper bound, which may be also interesting quantity. See Problem 5.6.

The quantity \( \alpha_\infty(G, \mathcal{F}) \) is closely related to the bipartiteness of a graph \( G \).

**Proposition 3.8.** Suppose \( \mathcal{F} \) is closed under \( U_A \) for any subset \( A \in V(G) \), where \( U_A \) is the unitary operator defined as \( U_A f = f \cdot 1_{A^c} - f \cdot 1_A \). Then

\[
\alpha_0(G, \mathcal{F}) + \alpha_\infty(G, \mathcal{F}) \leq 2.
\]

The equality holds if \( G \) is bipartite.

**Proof.** Here we simply write \( \alpha(G) \) for \( \alpha(G, \mathcal{F}) \) and set \( P = I + \Delta_G \). By (3.4) and the positivity preserving property of \( P \), we obtain, for any \( f \in \mathcal{F} \),

\[
\alpha_0(G) \|f\|^2 + \mathcal{E}(f, f) \leq \mathcal{E}(|f|, |f|) + \mathcal{E}(f, f) = 2[\mathcal{E}(f_+, f_+) + \mathcal{E}(f_-, f_-)] = 2 \left( \langle (I - P)f_+, f_+ \rangle + \langle (I - P)f_-, f_- \rangle \right) \leq 2\|f\|^2,
\]

where \( f_+ = \max(f, 0) \) and \( f_- = \max(-f, 0) \). We can choose a sequence of \( f_n \in \mathcal{F} \) such that \( \mathcal{E}(f_n, f_n)/\|f_n\|^2 \to \alpha_\infty(G) \). Consequently, putting \( f = f_n \) in (3.5) and letting \( n \to \infty \), we obtain

\[
\alpha_0(G) + \alpha_\infty(G) \leq 2.
\]

Suppose \( G \) is bipartite. Then there exists a bipartition \( V_1, V_2 \) such that \( V(G) = V_1 \sqcup V_2 \) as was mentioned in Section 1. Define \( \bar{f} \) by \( \bar{f} = f \cdot 1_{V_1} - f \cdot 1_{V_2} \) for \( f \in \mathcal{F} \). It is obvious from the assumption that \( \bar{f} \in \mathcal{F} \) and it follows easily from the second equality in (3.5) that \( \mathcal{E}(f, \bar{f}) + \mathcal{E}(\bar{f}, \bar{f}) = 2\|f\|^2 \). Then, in the same way as in the above, we obtain \( \alpha_0(G) + \alpha_\infty(G) \geq 2 \), which together with (3.6) implies \( \alpha_0(G) + \alpha_\infty(G) = 2 \). \( \square \)

For finite graphs, it is well-known that the maximal eigenvalue for any reversible transition probability \( p \) is equal to 2 if and only if a graph is bipartite. Also for infinite graphs, one can see from Proposition 3.8 that \( \lambda_0(G) + \lambda_\infty(G) = 2 \) if a graph \( G \) is bipartite. On the other hand, the converse does not hold in general. In fact, let \( \mathbb{Z}^1 = (V(\mathbb{Z}^1), E(\mathbb{Z}^1)) \) be the ordinary one dimensional lattice and \( G = (V(G), E(G)) \) a graph obtained from \( \mathbb{Z}^1 \) by adding an edge. That is, \( V(G) = V(\mathbb{Z}^1) \) and \( E(G) = E(\mathbb{Z}^1) \cup \{[e_0] \} \) with \( o(e_0) = -1 \) and \( t(e_0) = 1 \). Then, although \( G \) becomes non-bipartite due to the odd cycle with length 3 consisting of three vertices \( \{-1, 0, 1\} \), \( \text{Spec}(-\Delta_G) \) is still equal to \([0, 2]\) since the essential spectrum does not change under such a local perturbation.

However, under some reasonable assumption, the converse assertion in the above holds. Before stating the theorem, we introduce a notion of a graph \( G \) being uniformly non-bipartite.

**Definition 3.9.** Let \( N \) be a positive integer. An infinite graph \( G \) is called an \( N \)-non-bipartite graph if the following condition holds: for every vertex \( x \in V(G) \), there exists an odd cycle \( C_{2n+1} \) going through \( x \) with \( n \leq N \).
**Theorem 3.10.** Let \( G \) be an \( N \)-non-bipartite graph. Assume that the reversible transition probability \( p \) is uniformly bounded below by \( p_0 \). Then it holds that
\[
\lambda_0(G) + \lambda_{\infty}(G) \leq 2 - \varepsilon(N, p_0),
\]
where \( \varepsilon(N, p_0) \) depends only on \( N \) and \( p_0 \), and \( 0 < \varepsilon(N, p_0) < 1 \).

**A Sketch of the Proof.** In the discussion below, for simplicity, we restrict ourselves to a special class of \( N \)-non-bipartite graphs: \( G \) is a planar graph embedded in the plane and every face \( F \) of \( G \) is bounded by an odd cycle \( C_{2n+1} \) with \( n \leq N \). A \((d, f)\)-regular planar graph for an odd \( f (\leq 2N + 1) \) is a typical example belonging to this class. The discussion for the general \( N \)-non-bipartite graphs can be seen in [12]. See also Remark 3.12.

A key point is that the estimate of the quadratic form \( \mathcal{E}(f, f) \) is reduced to that of local quadratic forms. In particular, under the above restriction for graphs, \( G \) admits a face decomposition and we only have to estimate the local quadratic forms on faces. Define a local quadratic form \( \mathcal{E}_F(f, f) \) associated with a face \( F \in F(G) \) by
\[
\mathcal{E}_F(f, f) = \frac{1}{4} \sum_{e \in A(F)} |df(e)|^2 m_A(e).
\]
Here we identify a face \( F \) with a cycle of length \( d(F) \) as a finite subgraph of \( G \). If we decompose \( G \) into the faces \( F(G) \), we have
\[
\mathcal{E}(f, f) = \frac{1}{2} \sum_{e \in A(G)} |df(e)|^2 m_A(e) = \sum_{F \in F(G)} \mathcal{E}_F(f, f)
\]
(3.7)
since every edge \( e \in A(G) \) is counted twice in the face decomposition.

Now let \( C_F(x) = \sum_{e \in A(F)} p(e) \) and \( m_F(x) = (1/2)C_F(x)m(x)1_{V(F)}(x) \). Then the operator associated with the form \( \mathcal{E}_F \) on \( \ell^2(G, m_F) \) is given by the laplacian \( -\Delta_F \) (restricted on \( F \)) corresponding to the reversible transition probability (restricted on \( F \))
\[
p_F(e) = C_F(o(e))^{-1}p(e)1_{A(F)}(e).
\]
In other words,
\[
\mathcal{E}_F(f, f) = \langle -\Delta_F f, f \rangle_{m_F},
\]
where \( \Delta_F = I - P_F \). Note that
\[
\|f\|_{m_F}^2 = \frac{1}{2} \sum_{e \in A(F)} |f(o(e))|^2 m_A(e).
\]
and
\[
\|f\|^2 = \sum_{F \in F(G)} \|f\|_{m_F}^2.
\]
(3.8)
The local quadratic form \( \mathcal{E}_F \) is always bounded by \( 2\|f\|_{m_F}^2 \) for any \( F \in F(G) \). Moreover, since every face \( F \) in \( G \) is identified with an odd cycle, we can obtain a stronger (uniform) estimate. Let \( p \) be a reversible transition probability on a cycle \( C_{2n+1} \) which is bounded below by a positive constant \( p_0 \), that is, \( p(e) \geq p_0 \) for any \( e \in A(G) \). We denote the maximal eigenvalues of the laplacian on \( C_{2n+1} \) corresponding to a transition probability \( p \) by \( \lambda_{\infty}(p, n) \). Since \( \lambda_{\infty}(p, n) \) is continuous with respect to \( p \) and the set of all reversible measures bounded below by \( p_0 > 0 \) is closed, the maximum of \( \lambda_{\infty}(p, n) \) attains at some reversible probability
p bounded below. From the remark after Proposition 3.8, \( \lambda_{\infty}(p, n) \) is never equal to 2 since \( C_{2n+1} \) is not bipartite unless \( p \) takes values 0 or 1. Hence \( \lambda_{\infty}(p, n) \) is bounded above uniformly in \( p \) by a constant strictly less than 2 which depends only on \( n \) and \( p_0 \). Hence we obtain the following uniform estimate for the local quadratic forms:

\[
(3.9) \quad \mathcal{E}_F(f, f) \leq (2 - \delta)\|f\|^2_{m_F}, \quad \forall F \in F(G)
\]

for some \( \delta = \delta(N, p_0) > 0 \) depending only on \( N \) and \( p_0 \).

It follows from (3.7), (3.8) and (3.9) that

\[
(3.10) \quad \mathcal{E}(f, f) \leq (2 - \delta)\|f\|^2
\]

for any \( f \in \ell^2(G, m) \). It should be noted that the estimate (3.10) is still valid whenever \( G \) is \( N \)-non-bipartite and the transition probability is uniformly bounded below, although \( \delta \) may vary in \((0, 1)\).

Let \( \lambda_0(G) \) be the bottom of the spectrum of \(-\Delta_G\). Then it is known that there exists a positive function \( h \) such that \(-\Delta_G h = \lambda_0(G) h \) (cf. [4]). Define a new reversible probability \( p_h \) by harmonic transform of \( p \) as

\[
p_h(e) = \frac{1}{1 - \lambda_0(G)} \frac{h(t(e))p(e)}{h(o(e))}
\]

with a reversible measure \( m_h = mh^2 \). Here \( 1 - \lambda_0(G) \) is not equal to 0 by the inequality in (3.2).

**Remark 3.11.** If the original transition probability is uniformly bounded below, then so is the harmonic transform \( p_h \) though the bound \( p_0 \) may be changed. It is an easy consequence of a discrete version of the Harnack inequality: for a positive function \( h \) such that \( \Delta_G h \leq 0 \) it holds that

\[
p(e)h(t(e)) \leq h(o(e)) \leq \frac{1}{p(e)} h(t(e))
\]

for every \( e \in A(G) \).

Consider the Dirichlet form

\[
\mathcal{E}_h(f, f) = (-\Delta_h f, f)_{m_h}
\]

on \( \ell^2(G, m_h) \), where \( \Delta_h = I - P_h \). Then it is easy to see that

\[
\mathcal{E}(f, f) = \lambda_0(G)\|Uf\|^2_{m_h} + (1 - \lambda_0(G))\mathcal{E}_h(Uf, Uf)
\]

for any \( f \in \ell^2(G, m) \), where \( U : \ell^2(G, m) \to \ell^2(G, m_h) \) is a unitary operator defined by \( Uf = f/h \). Suppose \( G \) is \( N \)-non-bipartite. By Remark 3.11, one can apply the uniform estimate (3.10) for \( \mathcal{E}_h \) to obtain

\[
\mathcal{E}(f, f) \leq \lambda_0(G)\|Uf\|^2_{m_h} + (1 - \lambda_0(G))(2 - \delta)\|Uf\|^2_{m_h} = \lambda_0(G) + (1 - \lambda_0(G))(2 - \delta)\|f\|^2
\]

for any \( f \in \ell^2(G, m) \). Hence by the variational formula (3.3) we get

\[
\lambda_{\infty}(G) \leq \lambda_0(G) + (1 - \lambda_0(G))(2 - \delta).
\]

Consequently, we obtain

\[
\lambda_0(G) + \lambda_{\infty}(G) \leq 2 - (1 - \lambda_0(G))\delta.
\]
Remark 3.12. A generalization of Theorem 3.10 is obtained in [12]. We call a graph $G$ satisfying the following condition an "essentially non-bipartite graph": there exists a positive integer $N$ such that, for every edge $e \in E(G)$, there exists a magnifier glass $M_{m,2n+1} \cong P_m + C_{2n+1}$ in $G$ with $m + n \leq N$ and $e \in E(M_{m,2n+1})$. (The definition of $M_{m,n}$ was given in Section 2.2.) Also for any essentially non-bipartite graph $G$, it holds that $\lambda_0(G) + \lambda_\infty(G) < 2$. Details are omitted here.

4. Line graphs, subdivisions and para-line graphs

Here we focus on the various aspects of para-line graphs, which may be unfamiliar. In Section 4.1, we give the definitions of line graph, subdivision and para-line graph, and then discuss the relationship among them. We also show the homological structure of para-line graphs. In Section 4.2, we show the spectral property of para-line graphs. In Section 4.3, we show the injectiveness of the above graph-operations as maps among graphs. Finally, as an application, we determine the spectrum of infinite Sierpiński lattice through the structure of para-line graphs.

In this section, we always assume that a transition probability $p$ is associated with the simple random walk on $G$, that is, $p(e) = (d_{G,O}(e))^{-1}$.

4.1. Covering structure for line graphs and para-line graphs. Line graph is a well-known notion in graph theory while para-line graph, which was first introduced in [9], is not. Now we give the definitions of line graphs and para-line graphs. The definition of line graphs is usually given for simple graphs. Here we give the definitions for general graphs.

The line graph $L(G)$ of $G$ is defined by

$$V(L(G)) = E(G) = \{ e \mid e \in A(G) \},$$
$$A(L(G)) = \{ (e_1, e_2) \in A(G) \times A(G) \mid e_1 \neq e_2, o(e_1) = o(e_2) \},$$

where the incidence of an edge is given as $o((e_1, e_2)) = [e_1]$, $t((e_1, e_2)) = [e_2]$, and $(e_1, e_2) = (e_2, e_1)$.

The para-line graph $P(G)$ of $G$ is defined by

$$V(P(G)) = A(G),$$
$$A(P(G)) = \{ (e_1, e_2) \in A(G) \times A(G) \mid o(e_1) = o(e_2), e_1 \neq e_2 \} \cup \{ (e, e) \mid e \in A(G) \},$$

where $o((e_1, e_2)) = e_1$, $t((e_1, e_2)) = e_2$ and $(e_1, e_2) = (e_2, e_1)$, and $o((e, e)_0) = e$, $t((e, e)_0) = e$ and $((e, e)_0) = (e, e)_0$. The symbol $\cup$ stands for the disjoint union. If there exists a self-loop $e$ in $G$, the oriented edge $(e, e)$ and $(e, e)_0$ appears in the definition of $A(P(G))$. In such a case we regard them as double edges.

In order to mention the relationship between line graphs and para-line graphs, we introduce subdivision graphs that are also well-known in graph theory. The subdivision graph $S(G)$ of $G$ is defined by

$$V(S(G)) = V(G) \cup E(G),$$
$$A(S(G)) = \{ (o(e), e), (e, o(e)) \mid e \in A(G) \},$$

where $o((o(e), e)) = o(e)$, $t((o(e), e)) = [e]$ and $(o(e), e) = (e, o(e))$.

Remark 4.1. Let $G$ be a $d$-regular graph. Then $L(G)$ is $(2d-2)$-regular, $P(G)$ is $d$-regular and $S(G)$ is $(d, 2)$-semiregular. Here $G$ is said to be $(d_1, d_2)$-semiregular if $G$ is bipartite with bipartition $V_1 \cup V_2$ and $\deg_G x = d_i$ on $V_i$ ($i = 1, 2$).
The relationship between $G$, $L(G)$, $S(G)$ and $P(G)$ is as follows. See also Figure 2 and Remark 4.3.

**Proposition 4.2.** Let $G$ be a graph. Then the para-line graph $P(G)$ of $G$ is isomorphic to the line graph of the subdivision graph of $G$, that is, $P(G) \cong L(S(G))$.

**Proof.** By the above definitions, it is easy to see that

$$V(L(S(G))) = \{[o(e), e] \mid e \in A(G)\}$$

and

$$A(L(S(G))) = \{(o(e_1), e_1), (o(e_2), e_2) \mid e_1, e_2 \in A(G), o(e_1) = o(e_2), e_1 \neq e_2\}
\cup \{((e, o(e)), (\overline{e}, o(\overline{e}))) \mid e \in A(G)\},$$

where the origin vertices, the terminal ones and the inverse edges are naturally defined. If we identify a vertex $[o(e), e]$ with $e$, an edge $((o(e_1), e_1), (o(e_2), e_2))$ with $(e_1, e_2)$ and an edge $((e, o(e)), (\overline{e}, o(\overline{e})))$ with $(e, \overline{e})$, it is obvious that $L(S(G))$ is isomorphic to $P(G)$. \hfill \Box

**Remark 4.3.** In Figure 2, $A$ and $B$ are the sets of the unoriented edges which are obtained by forgetting the orientation in $\{(e_1, e_2) \in A(G) \times A(G) \mid o(e_1) = o(e_2), e_1 \neq e_2\}$ (solid edges) and $\{((e, o(e)), (\overline{e}, o(\overline{e}))) \mid e \in A(G)\}$ (dashed edges) in $P(G)$, respectively. By the definitions, there exist bijections from $E(L(G))$ to $A$ and from $E(G)$ to $B$. It is easy to see that two graphs $G$ and $L(G)$ are obtained from $P(G)$.
by contracting $A$ and $B$, respectively. Here it is called a contraction of an edge $[e]$ in $G$ to remove $[e] \in E(G)$ and identify $o(e)$ and $t(e)$. We simply say a contraction of $A$ when we contract all edges in a edge set $A$.

When $G$ is a covering graph, $L(G)$ is also a covering graph with the same transformation group. We recall that a covering map $\pi : G \to M$ with covering transformation group $\Gamma$ is normal if $\Gamma$ acts transitively on each fiber $\pi^{-1}(x)$ for $x \in V(M)$.

**Lemma 4.4 ([13]).** Let $G$ be a normal covering graph of a finite graph $M$ with covering transformation group $\Gamma$, then $L(G)$ is a normal covering graph of $L(M)$ with the same covering transformation group $\Gamma$.

The same assertion holds for para-line graphs as a corollary of the above lemma.

**Lemma 4.5.** Let $G$ be a normal covering graph of a finite graph $M$ with covering transformation group $\Gamma$. then $P(G)$ is a normal covering graph of $P(M)$ with the same covering transformation group $\Gamma$.

**Proof.** The action of $\Gamma$ on $G$ induces actions on $S(G)$ and $P(G)$ in a natural manner. The assertion of the lemma for subdivision graphs are almost trivial, and hence Lemma 4.5 follows immediately from Proposition 4.2 and Lemma 4.4. \hfill \square

Now let $\phi_i : C_i(P(M), \mathbb{Z}) \to C_i(M, \mathbb{Z})$ for $i = 0, 1$ be the homomorphism defined as

\[
\begin{align*}
\phi_0(e) &= o(e), \\
\phi_1((e_1, e_2)) &= 0, \\
\phi_1((e, \bar{e})_o) &= e.
\end{align*}
\]

Then we have the following:

**Lemma 4.6.** $(\phi_0, \phi_1)$ is a chain map, that is, $\phi_0 \circ \partial = \partial \circ \phi_1$.

**Proof.** It follows from

\[
\phi_0(\partial(e_1, e_2)) = \phi_0(e_2 - e_1) = o(e_2) - o(e_1) = 0 = \partial(\phi_1((e_1, e_2)))
\]

and

\[
\phi_0(\partial(e, \bar{e})_o) = \phi_0(\bar{e} - e) = o(\bar{e}) - o(e) = t(e) - o(e) = \partial(\phi_1((e, \bar{e})_o)).
\]

\hfill \square

**Lemma 4.7.** Let $\phi_1 : H_1(P(M), \mathbb{Z}) \to H_1(M, \mathbb{Z})$ be the induced homomorphism. Then it is surjective.

**Proof.** An element $\sigma \in H_1(M, \mathbb{Z})$ is represented by a closed path $c = (e_1, \ldots, e_n)$ in $M$ such that $\bar{e}_i \neq e_{i+1}$ for $i = 1, 2, \ldots, n$. Define a closed path $c'$ in $P(M)$ by

\[
c' = ((e_1, \bar{e})_1, (e_2, \bar{e}_2), (e_3, \bar{e}_3), \ldots, (e_{n-1}, \bar{e}_{n-1}, (e_n, \bar{e})_n, (\bar{e}_n, e_1)).
\]

It is obvious that $\phi_1(c') = c$. \hfill \square

**Proposition 4.8.** Let $G$ be the abelian covering graph of a finite graph $M$ with transformation group $\Gamma$. Let $\eta$ and $\eta^p$ be the surjective homomorphisms associated with the covering maps $\pi : G \to M$ and $\pi^p : P(G) \to P(M)$, respectively. Then the homomorphism $\eta^p : H_1(P(M), \mathbb{Z}) \to \Gamma$ coincides with the composition $\eta \circ \phi_1$:

\[
H_1(P(M), \mathbb{Z}) \xrightarrow{\phi_1} H_1(M, \mathbb{Z}) \xrightarrow{\eta} \Gamma.
\]
PROOF. Let \( c' \) be a closed path in \( P(M) \) and its lift to \( P(G) \) be \( \tilde{c}' \). and write

\[
\tilde{c}' = (\tilde{e}_1, \tilde{e}_2, \ldots, \tilde{e}_n).
\]

where \( \tilde{e}_i = (\tilde{e}_i, \tilde{e}_{i+1}) \text{ or } (\tilde{e}_i, \tilde{e}_{i+1})_0 \in A(P(M)). \) Replacing \( \tilde{e}_i \) with two edges \( \tilde{e}_i, \tilde{e}_i \) if \( \tilde{e}_i = (\tilde{e}_i, \tilde{e}_{i+1}) \) and with one edge \( \tilde{e}_i \) if \( \tilde{e}_i = (\tilde{e}_i, \tilde{e}_{i+1})_0 \) in the expression of \( \tilde{c}' \) for every \( i = 1, 2, \ldots, n \), we obtain a new sequence of edges of \( G \) and denote it by \( \tilde{c} \). It is easy to see that \( \tilde{c} \) is a path in \( G \) such that the projection \( c = \pi(\tilde{c}) \) is a closed path in \( M \) with \( \phi_1(c') = c \). Now we recall that \( \eta : H_1(M, \mathbb{Z}) \to \Gamma \) and \( \eta^P : H_1(P(M), \mathbb{Z}) \to \Gamma \) are given by \( \eta(c) o(\tilde{c}) = t(\tilde{c}) \) for a closed path \( c \) in \( M \) with a lift \( \tilde{c} \) in \( G \) and \( \eta^P(c') o(\tilde{c}') = t(\tilde{c}') \) for a closed path \( c' \) in \( P(M) \) with a lift \( \tilde{c}' \) in \( P(G) \). respectively. It is easy to check that \( \eta^P(c') o(\tilde{c}') = t(\tilde{c}) \). Since the \( \Gamma \)-action is free, we obtain \( \eta^P = \eta o \phi_1 \).

\[\square\]

**Remark 4.9.** The surjective homomorphism \( \phi_1 \) induces the injective homomorphism \( \phi_1^* : H^1(M, \mathbb{R}) \to H^1(P(M), \mathbb{R}) \) through

\[\phi_1^*(\theta((e_1, e_2))) = \theta(\phi_1((e_1, e_2))) = 0. \quad \phi_1^*(\theta((e, e)_0)) = \theta(\phi_1((e, e)_0)) = \theta(e).\]

We can identify \( \hat{\Gamma} \) with a subset of \( J(M) \) and \( J(P(M)) \) by the maps \( \hat{\eta} : \hat{\Gamma} \to J(M) \) and \( \hat{\eta}^P : \hat{\Gamma} \to J(P(M)) \). By the above lemma, the relationship between \( \hat{\eta} \) and \( \hat{\eta}^P \) is given by \( \hat{\eta}^P = \phi_1^* o \hat{\eta} \). Hence the cohomology class of \( \phi_1^* \theta \in C^1(P(M), \mathbb{R}) \) is naturally identified with a character \( \chi = \chi_{\theta} \in \hat{\Gamma} \).

### 4.2. Spectrum for line graphs and para-line graphs.

Once we know the spectrum of \( G \), we get the spectra of \( L(G), S(G) \) and \( P(G) \). The proofs of the following three theorems can be found in [18]. Theorems 4.10 and 4.11 are also discussed from the supersymmetric point of view in [17]. Recall a transition probability \( p(e) \) is assumed to be given by \( (\deg_G o(e))^{-1} \).

**Theorem 4.10.** Let \( G \) be an infinite \( d \)-regular graph with \( d \geq 3 \) and \( L(G) \) its line graph of \( G \). Then

\[
\text{Spec}(\Delta_{L(G)}) = \frac{d}{2d-2} \text{Spec}(\Delta_G) \cup \left\{ \frac{d}{d-1} \right\}.
\]

where \( \frac{d}{d-1} \) is an eigenvalue with infinite multiplicity.

**Theorem 4.11.** Let \( G \) be an infinite \( d \)-regular graph with \( d \geq 3 \) and \( S(G) \) its subdivision graph. Let \( \varphi_S(x) = 2x(2-x) \). Then

\[
\text{Spec}(\Delta_{S(G)}) = \varphi_S^{-1}(\text{Spec}(\Delta_G)) \cup \{1\}
\]

where 1 is an eigenvalue with infinite multiplicity.

**Theorem 4.12.** Let \( G \) be an infinite \( d \)-regular graph with \( d \geq 3 \) and \( P(G) \) its para-line graph. and \( \varphi_P(x) = -d \cdot x^2 + (d + 2)x \). Then

\[
\text{Spec}(\Delta_{P(G)}) = \varphi_P^{-1}(\text{Spec}(\Delta_G)) \cup \{1\} \cup \left\{ \frac{d+2}{d} \right\}.
\]

where 1 and \( \frac{d+2}{d} \) are eigenvalues with infinite multiplicity.

The corresponding results for laplacians on line graphs and subdivisions of finite graphs are found in e.g. [3]. Here we give the corresponding result for the relationship between twisted laplacians \( L_{\theta,M} \) on a finite graph \( M \) and \( L_{\theta,P(M)} \) on its para-line graph \( P(M) \).
We recall the definition of $L_{\theta,M}$ (see also (2.2)). Since we treat only regular graphs, $L_{\theta,M} = A_{\theta,M}/d - I$ and $L_{\theta,P(M)} = A_{\theta,P(M)}/d - I$, where $A_{\theta,M}$ and $A_{\theta,P,P(M)}$ are the twisted adjacency operators defined by

$$A_{\theta,M}f(x) = \sum_{e \in A_{\theta,M}(M)} \exp(2\pi \sqrt{-1}\theta(e)f(t(e))$$

and

$$A_{\theta,P,P(M)}F(e) = \sum_{\substack{e' \in A(M) \\ \phi(e') = \phi(e), e' \neq e}} \exp(2\pi \sqrt{-1}\theta_P(e,e'))F(e')$$

$$+ \exp(2\pi \sqrt{-1}\theta_P((e,\bar{e})_o))F(\bar{e}).$$

For our theorem, we consider $\phi_1^* \theta \in C^1(P(M),R)$ as was defined in Remark 4.9 through $\theta \in C^1(M,R)$ by

$$\phi_1^* \theta((e_1, e_2)) = 0, \quad \phi_1^* \theta((e, \bar{e})_o) = \theta(e).$$

Then we have the following theorem.

**Theorem 4.13.** Let $M$ be a finite connected $d$-regular graph with $d \geq 2$ and $P(M)$ its para-line graph, and $\varphi_P(x) = -d \cdot x^2 + (d+2)x$. Let $\lambda = |V(M)|$ and $0 \leq \mu_1 \leq \mu_2 \leq \cdots \leq \mu_N \leq 2$ be the eigenvalues of $-L_{\theta,M}$. Then the characteristic polynomial of $-L_{\phi_1^* \theta,P(M)}$ is given by

$$(\lambda - 1)^{K-1}(\lambda - \frac{d+2}{d})^{K-1} \prod_{i=1}^{N} (\varphi_P(\lambda) - \mu_i)$$

where $K = \text{rank} H_1(M,Z) = \frac{d-2}{2} \lambda + 1$.

**Remark 4.14.** In the case where $G$ is an abelian covering graph of a finite graph $M$, Theorem 4.13 recovers Theorem 4.12 since the spectrum of $\Delta_G$ is the union of the eigenvalues of $L_{\theta,M}$ (see (2.3)).

The multiplicities of the eigenvalues 1 and $\frac{d+2}{d}$ may be $K$ since $\varphi_P^{-1}\{0\} = \{\frac{d+2}{d}, 0\}$ and $\varphi_P^{-1}\{2\} = \{1, \frac{2}{d}\}$. They depend on the bipartiteness of $M$ and the integrals of $\theta$ over cycles:

i) $M$ is bipartite and $\int_c \theta \in Z$ for any closed path $c$.

ii) $M$ is non-bipartite and $\int_c \theta \in Z$ for any closed path $c$.

iii) $M$ is non-bipartite and $\int_c \theta \in Z$ for any closed path $c$ of even length and $\int_c \theta \in Z + 1/2$ for any closed path $c$ of odd length.

iv) Otherwise.

The exact multiplicities of 1 and $\frac{d+2}{d}$ are determined according to the fact that $-L_{\theta,M}$ has the simple eigenvalue 0 in the cases i) and ii); the simple eigenvalue 2 in the cases i) and iii).

Here we give a sketch of the proof in the rest of this subsection. Let $\Phi_i : \ell^2(M) \rightarrow \ell^2(P(M))$ for $i = 0, 1$ be defined by

$$\Phi_0f(e) = f(o(e)), \quad \Phi_1f(e) = \exp(2\pi \theta(e))f(t(e)).$$

It is easy to see that both $\Phi_0$ and $\Phi_1$ are injective and their dual operators are given by

$$\Phi_0^*F(x) = \sum_{e \in A_x} F(e), \quad \Phi_1^*F(x) = \sum_{e \in A_x} \exp(2\pi \theta(e))F(\bar{e}).$$
Define two subspaces $E$ and $E_0$ of $\ell^2(P(M))$ by

$$E = \Phi_0(\ell^2(M)) + \Phi_1(\ell^2(M)), \quad E_0 = \Phi_0(\ell^2(M)) \cap \Phi_1(\ell^2(M)).$$

If $F \in E_0$, there exist $f, g \in \ell^2(M)$ such that $F = \Phi_0 f + \Phi_1 g$, and then

$$f(t(e_2)) = \exp(-2\pi \sqrt{-1}(\theta(e_1) + \theta(e_2))) f(o(e_1))$$

for any path $(e_1, e_2)$ of length $2$. From this equality, it is easy to see that $E_0 = \{0\}$ in the case iv). We fix a vertex $x_0$ and define the function $s_{x_0}$ by

$$s_{x_0}(x) = \exp(-2\pi \sqrt{-1} \int_p \theta) s_{x_0}(x_0)$$

if there exists a path $p$ of even length joining $x_0$ to $x$; otherwise $s_{x_0}(x) = 0$. In the cases i), ii) and iii), it is well-defined and satisfies $\Phi_0 s_{x_0} = \Phi_1 s_{x_0}$.

In the case i), let $V_0$ and $V_1$ be a bipartition, and take $x_0 \in V_0$ and $x_1 \in V_1$. Then we can define two linearly independent such functions $s_{x_0}$ and $s_{x_1}$ by (4.2) and so $E_0 = \text{Span}\{\Phi_0 s_{x_0}, \Phi_0 s_{x_1}\}$. In the cases ii) and iii), for any $x \in V(M)$ there exists a path $p$ of even length joining to $x_0$ to $x$. Then $E_0 = \text{Span}\{\Phi_0 s_{x_0}\}$. We summarize as follows:

**Lemma 4.15.** Let $s_{x_0}$ and $s_{x_1}$ be defined as above. Then

$$E_0 = \begin{cases} \{0\}, & \text{case iv).} \\ \text{Span}\{\Phi_0 s_{x_0}\}, & \text{cases ii), iii).} \\ \text{Span}\{\Phi_0 s_{x_0}, \Phi_0 s_{x_1}\}, & \text{case i).} \end{cases}$$

In particular, $\dim E = 2|V(M)| - 2$ in the case i); $\dim E = 2|V(M)| - 1$ in the cases ii), iii); $\dim E = 2|V(M)|$ in the case iv).

Theorem 4.13 follows from Lemmas 4.16 and 4.17 together with Lemma 4.15.

**Lemma 4.16.** Let $A_{\theta,M}$ and $A_{\theta^*,P(M)}$ be adjacency operators on $\ell^2(M)$ and $\ell^2(P(M))$, respectively. Then

$$\begin{cases} A_{\theta^*,P(M)} \Phi_0 = (d - 1) \Phi_0 + \Phi_1, \\ A_{\theta^*,P(M)} \Phi_1 = \Phi_0 (A_{\theta,M} + I) - \Phi_1. \end{cases}$$

In particular, $A_{\theta^*,P(M)}$ leaves the subspace $E$ invariant.

**Proof.** It follows from a direct computation. \hfill \Box

Now we define a self-adjoint operator $J_\theta : \ell^2(P(M)) \to \ell^2(P(M))$ by

$$J_\theta F(e) = \exp(2\pi \sqrt{-1} \theta(e)) F(\bar{e}).$$

It is easy to see that $J_\theta^2 = I$ and $J_\theta \Phi_0 = \Phi_1$. Hence $J_\theta$ can be restricted on $E_1^\perp$, the orthogonal complement of $E$ in $\ell^2(P(M))$, and its eigenvalues on $E_1^\perp$ are either 1 or $-1$. We consider the two subspaces of $E_1^\perp$:

$$E_1^\perp_S = \{ F \in E_1^\perp \mid J_\theta F = F \},$$

$$E_1^\perp_A = \{ F \in E_1^\perp \mid J_\theta F = -F \}.$$

It is easy to see that $E_1^\perp = E_1^\perp_S \oplus E_1^\perp_A$. \hfill \Box
Lemma 4.17. Let $E^+_{\mathcal{S}}$ and $E^+_{\mathcal{A}}$ be as above. Then $E^+_{\mathcal{S}}$ and $E^+_{\mathcal{A}}$ are the eigenspaces of the eigenvalues 0 and $-2$ of $A_{\theta'} P(M)$, respectively, and
\[
\dim E^+_{\mathcal{S}} = \begin{cases} K, & \text{cases i), iii)}, \\ K - 1, & \text{cases ii), iv)}. \end{cases} \quad \text{and} \quad \dim E^+_{\mathcal{A}} = \begin{cases} K, & \text{cases i), iii)}, \\ K - 1. & \text{cases iii), iv)}. \end{cases}
\]
where $K = \text{rank} H_1(M, \mathbb{Z}) = \frac{d-2}{2} N + 1$.

Proof. The first assertion immediately follows from the formula
\[
A_{\theta'} P(M) F(e) = \Phi_0^* F(o(e)) + (J_\theta - I) F(e).
\]
Let $\{\sigma_i\}_{i=1}^K$ be a $\mathbb{Z}$-basis of 1-homology group $H_1(M, \mathbb{Z})$. Every $\sigma_i$ can be represented by a closed path in $M$, say $c_i = (e_1^{(i)}, \ldots, e_{n_i}^{(i)})$, such that the edges $\{e_j^{(i)}\}_{j=1}^{n_i}$ in $c_i$ are all distinct. Now we define functions $F_i, G_i \in \ell^2(P(M))(i = 1, 2, \ldots, K)$ by
\[
F_i(e) = (-1)^{k-1} \exp(-2\pi \sqrt{-1} \int_{c_{1,k}} \theta), \\
G_i(e) = \exp(-2\pi \sqrt{-1} \int_{c_{1,k}} \theta).
\]
if $e = e_k^{(i)}$ for some $k$ ($1 \leq k \leq n_i$), where $c_{1,k} = (e_1^{(i)}, \ldots, e_{k-1}^{(i)})(k \geq 2)$ and $c_{1,1} = 0$; otherwise $F_i(e) = G_i(e) = 0$. Note that the definitions of $F_i$ and $G_i$ depend on the choice of an edge as $e_k^{(i)}$ in the closed path $c_i$. The functions $\{F_i\}_{i=1}^K$ are linearly independent and so are $\{G_i\}_{i=1}^K$. We also define $Q_S$ and $Q_A$ by $Q_S = I + J_\theta$ and $Q_A = I - J_\theta$. It is easy to see the following:

(1) For a closed path $c_i$ of even length, $Q_S F_i$ is an eigenfunction with respect to 0 if and only if $\int_{c_i} \theta \in \mathbb{Z}$: $Q_A G_i$ is an eigenfunction with respect to $-2$ if and only if $\int_{c_i} \theta \in \mathbb{Z}$.

(2) For a closed path $c_i$ of odd length, $Q_A G_i$ is an eigenfunction with respect to $-2$ if and only if $\int_{c_i} \theta \in \mathbb{Z}$.

(3) For a closed path $c_i$ of odd length, $Q_S F_i$ is an eigenfunction with respect to 0 if and only if $\int_{c_i} \theta \in \mathbb{Z} + \frac{1}{2}$.

Suppose that there exists a closed path, say $c_1$, such that $\Phi_0^* Q_A F_1(x) \neq 0$ for some $x \in V(c_1)$, and if it exists, then it must be $o(e_1^{(1)})$ by the definition of $F_1$. We may assume that $x \in V(c_i)$ and $x = o(e_i^{(i)})$ for any $i = 1, 2, \ldots, K$. Indeed, if necessary, we can take a path $P = (e_1, e_2, \ldots, e_{\ell})$ such that $x = o(e_1)$ and $t(e_{\ell}) \in V(c_i)$ in $M$ so that a new closed path $\tilde{c}_i = P c_i P = (e_1, e_2, \ldots, e_{\ell}, e_1^{(i)}, \ldots, e_{n_i}^{(i)}, \bar{e}_{\ell}, \bar{e}_{\ell-1}, \ldots, \bar{e}_1)$ obviously satisfies that $x \in V(\tilde{c}_i)$, $x = o(e_1^{(1)})$ and $\int_{\tilde{c}_i} \theta = \int_{c_i} \theta$. We consider linear combinations of $F_i$'s instead of themselves:
\[
\tilde{F}_i = (\Phi_0^* Q_A F_1(x)) F_i - (\Phi_0^* Q_A F_1(x)) F_1.
\]
It is easy to see that $Q_A \tilde{F}_i$ is an eigenfunction with respect to $-2$ for $i = 2, 3, \ldots, K$.

In the same way as above, suppose that there exists at least one closed path, say $c_1$, such that $\Phi_0^* Q_S G_1(y) \neq 0$ for some $y \in V(c_1)$. If we consider linear combinations of $G_i$'s
\[
\tilde{G}_i = (\Phi_0^* Q_S G_1(y)) G_i - (\Phi_0^* Q_S G_1(y)) G_1,
\]
then $Q_S \tilde{G}_i$ is an eigenfunction with respect to 0 for $i = 2, 3, \ldots, K$.

Through all the discussion above, we get the dimensions of $E^S_{\tilde{G}}$ and $E^A_{\tilde{G}}$ in each case. \qed

### 4.3. Injectiveness of taking line graphs and para-line graphs

In this subsection we assume $G = (V(G), E(G))$ is a connected, simple graph which may be infinite. Here a graph is said to be simple if it has no self-loops and no multiple edges. Also we often use the notation like $xy \in E(G)$ as an unoriented edge whose endvertices are $x$ and $y$.

Here we show the following result, which follows from Proposition 4.21 and Corollaries 4.22 and 4.23 below.

**Theorem 4.18.** Let $G$ be the set of all infinite simple connected graphs. Then the maps $L, S, P : \mathcal{G} \to \mathcal{G}$ are injective up to isomorphisms.

Let us recall graph-isomorphisms. Two graphs $G$ and $G'$ are isomorphic ($G \cong G'$) if there exists a bijection $\varphi$ from $V(G)$ to $V(G')$ satisfying the following: two vertices $x_1$ and $x_2$ are adjacent in $G$ if and only if the vertices $\varphi(x_1)$ and $\varphi(x_2)$ are adjacent in $G'$. Such a map $\phi$ is called an isomorphism. In particular, if $G = G'$, it is called an automorphism.

Now we introduce the notion of edge-isomorphism. Further topics about edge-isomorphisms can be found in, e.g., [2]. We call $G$ and $G'$ are edge-isomorphic if there exists a bijection $\psi$ from $E(G)$ to $E(G')$ satisfying the following: two edges $[e_1]$ and $[e_2]$ in $G$ are adjacent (that is, $[e_1]$ and $[e_2]$ have a common vertex in their endvertices) if and only if $\psi([e_1])$ and $\psi([e_2])$ in $G'$ are adjacent.

**Remark 4.19.** $G$ and $G'$ are edge-isomorphic if and only if $L(G)$ and $L(G')$ are isomorphic.

If $G$ and $G'$ are isomorphic, then they are also edge-isomorphic, equivalently. $L(G)$ and $L(G')$ are isomorphic. In fact, if $\varphi$ is an isomorphism from $G$ to $G'$, setting $\psi([e]) = \varphi(x)\varphi(y)$ for any edge $[e] = xy \in E(G)$, one can see that $\psi$ is an edge-isomorphism. In this case, this $\psi$ is said to be the induced edge-isomorphism (by the isomorphism $\varphi$). However, the converse direction is not true. Indeed, two graphs $X_1 = K_3$ and $X_2 = K_{1,3}$ are edge-isomorphic but not isomorphic. In addition, the following $X_i (i = 3, 4, 5)$ has an edge-automorphism (that is, an edge-isomorphism from $X_i$ to itself) which is not induced by any automorphism: $X_3 = K_4$, $X_4 = K_4 - \text{"one edge}"$ and $X_5 = K_4 - \text{"P}_2"$. Fortunately, for simple and finite graphs, it is known that “bad” graphs are only five graphs $X_i (i = 1, 2, 3, 4, 5)$ mentioned above, which is proved by H. Whitney [22]. It can be generalized for infinite graphs.

**Proposition 4.20.** Let $G_1$ and $G_2$ be connected and simple graphs which may be infinite, and we assume that $G_1$ is not isomorphic to any $X_i (i = 1, 2, 3, 4, 5)$. If $\psi$ is an edge-isomorphism from $G_1$ to $G_2$, then $\psi$ is induced by an isomorphism from $G_1$ to $G_2$.

**Proof.** If graphs $G_1$ and $G_2$ are finite, Proposition 4.20 is obtained by Whitney ([22]). Thus we may assume $G_1$ and $G_2$ are infinite. From the assumption, there exists an edge-isomorphism $\psi$ from $G_1$ to $G_2$. Let $S_0$ be a connected finite subgraph of $G_1$. which is induced by $V(S_0)$. with $|V(S_0)| \geq 5$ and set $T_0 = \langle \psi(E(S_0)) \rangle_{G_2}$. 

which is the subgraph of $G_2$ induced by the edge set $\psi(E(S_0))$. Denote the restriction of $\psi$ to $E(S_0)$ by $\psi_0 = \psi|_{E(S_0)}$. Then it is obvious that $\psi_0$ is an edge-isomorphism from $S_0$ to $T_0$, and hence $\psi_0$ is induced by an isomorphism $\varphi_0$ from $S_0$ to $T_0$ since $S_0$ is finite and not isomorphic to any $X_i$ ($i = 1, 2, 3, 4, 5$).

For any subgraph $K$ in $G$, we define a neighborhood $N_G(K)$ of $K$ by the set $N_G(K) = \{ x \in V(G) \setminus V(K) \mid xy \in E(G) \text{ for some } y \in V(K) \}$. Now $G_1$ is an infinite, connected graph, so there exists a vertex $w \in N_{G_1}(S_0)$ and one can choose two edges $uv \in E(S_0)$ and $vw \notin E(S_0)$. Since $uv \notin E(S_0)$ and $\psi$ is an edge-isomorphism, $\psi(uv) \notin E(T_0)$ and is adjacent to $\psi(uv) = \varphi_0(u)\varphi_0(v)$. Then the edge $\psi(uv)$ in $G_2$ must be $\varphi_0(u)\bar{w}$ or $\varphi_0(v)\bar{w}$ for some $\bar{w} \in V(G_2) \setminus V(T_0)$. Since $S_0$ is connected and $|V(S_0)| \geq 5$, there exists a vertex $a(\neq u, v)$ in $S_0$ such that $au \in E(S_0)$ or $av \in E(S_0)$. If $au \in E(S_0)$, since $au$ and $uv$ is not adjacent, $\psi(au) = \varphi_0(a)\varphi_0(u)$ and $\psi(uv)$ do not have a common vertex, and hence $\psi(uv)$ is $\varphi_0(v)\bar{w}$; if $av \in E(S_0)$, since $av$ and $uv$ is adjacent, $\psi(au) = \varphi_0(a)\varphi_0(v)$ and $\psi(uv)$ have a common vertex, and hence $\psi(uv)$ is $\varphi_0(u)\bar{w}$. In both cases, $\psi(uv)$ must be $\varphi_0(v)\bar{w}$. Similarly, for any edge joining $b \in V(S_0)$ and $w$, it holds that $\psi(bw) = \varphi_0(b)\bar{w}'$ for some $\bar{w}' \notin V(S_0)$. Since $\psi(aw)$ and $\psi(bw)$ are adjacent in $G_2$, $\bar{w}$ and $\bar{w}'$ must coincide. Hence $\bar{w}$ is determined uniquely by $w$. Therefore, setting

$$\varphi_1(x) = \begin{cases} \varphi_0(x), & \text{if } x \in V(S_0), \\ \bar{w}, & \text{if } x = w, \end{cases}$$

one can easily check that $\varphi_1$ is an isomorphism from $S_1 = (V(S_0) \cup \{ w \})_{G_1}$ to $T_1 = (V(T_0) \cup \{ \bar{w} \})_{G_2}$; it induces the edge-isomorphism $\psi|_{E(S_1)}$, the restriction of $\psi$ to $E(S_1)$.

Continuing this procedure, we can define inductively an isomorphism $\varphi_k$ from $S_k$ to $T_k$ so that $\{ S_k \}_{k=0}^{\infty}$ and $\{ T_k \}_{k=0}^{\infty}$ are increasing sequences of finite induced subgraphs of $G_1$ and $G_2$ with $V(G_1) = \bigcup_{i=0}^{\infty} V(S_i)$ and $V(G_2) = \bigcup_{i=0}^{\infty} V(T_i)$, respectively, and $\varphi_k|_{S_\ell} = \varphi_\ell$ if $k \geq \ell \geq 0$. We define a map $\varphi : V(G_1) \rightarrow V(G_2)$ by

$$\varphi(x) = \varphi_k(x) \quad \text{if } x \in S_k.$$  

It is well-defined by the consistency of $\varphi_k$'s. Then it is easy to see that $\varphi$ is an isomorphism from $G_1$ to $G_2$ and induces the edge-isomorphism $\psi$, that is, $\psi(xy) = \varphi(x)\varphi(y) \in E(G_2)$ for any edge $xy \in E(G_1)$. This completes the proof.  

Remark 4.19 and Proposition 4.20 shows that the following proposition.

**Proposition 4.21.** Let $G_1$ and $G_2$ be two graphs such that $G_1$ is not isomorphic to $X_1 = K_3$ or $X_2 = K_{1,3}$, and $G_2$ is not isomorphic to $X_2 = K_{1,3}$ or $X_1 = K_3$, respectively. Then $L(G_1)$ is isomorphic to $L(G_2)$ if and only if $G_1$ is isomorphic to $G_2$.

**Proof.** By the definition of line graphs. there exists a bijection from $E(G)$ to $V(L(G))$ such that any two vertices in $L(G)$ are adjacent if and only if two of the corresponding edges in $G$ are adjacent. As mentioned in Remark 4.19, $L(G_1)$ is isomorphic to $L(G_2)$ if $G_1$ is isomorphic to $G_2$.

Assume that $L(G_1)$ is isomorphic to $L(G_2)$ and that both of $G_1$ and $G_2$ are not isomorphic to $K_3$ and $K_{1,3}$. By Remark 4.19, there exists an edge-isomorphism $\psi$ from $G_1$ to $G_2$. If $G_1$ is not isomorphic to any $X_i$ ($i = 3, 4, 5$), it follows from Proposition 4.20 that $\psi$ is induced by an isomorphism from $G_1$ to $G_2$. Thus
$G_1 \cong G_2$. It is easy to see that $G$ is the one and only one graph which has $L(G)$ as its line graph if $G$ is isomorphic to $X_i$ ($i = 3, 4, 5$).

As corollaries, we can obtain the following.

**Corollary 4.22.** Let $G_1$ be a subdivision graph of a graph. If there exists an edge-isomorphism $\psi$ from $G_1$ to a graph $G_2$, then $\psi$ is induced by an isomorphism from $G_1$ to $G_2$.

**Proof.** Since a subdivision graph $G_1$ is bipartite and each vertex in one of the partite sets has degree 2, $G_1$ is not isomorphic to any $X_i$ ($i = 1, 2, 3, 4, 5$). Thus the conclusion holds.

**Corollary 4.23.** Let $G_1$ and $G_2$ be two graphs. Then $P(G_1)$ is isomorphic to $P(G_2)$ if and only if $G_1$ is isomorphic to $G_2$.

**Proof.** Recall that $P(G) \cong L(S(G))$. By the previous results, we only have to see the relationship between $G$ and $S(G)$. It is obvious by definition that $S(G_1) \cong S(G_2)$ if $G_1 \cong G_2$. Thus it is sufficient to show that $G_1 \cong G_2$ if $S(G_1) \cong S(G_2)$. So we suppose $S(G_1) \cong S(G_2)$.

If every vertex in $S(G_1)$ has degree 2, then $S(G_1)$ is isomorphic to $C_{2n}$ ($n = 1, 2, \ldots$) or an infinite path. In such cases, it is obvious that $G_1 \cong G_2$. It also holds for $S(G_2)$ since every vertex in $S(G_2)$ has degree 2 by an isomorphism. Hence $G_1 \cong G_2$.

Assume that $S(G)$ has at least one vertex whose degree is not 2. Since $S(G)$ is bipartite, we may take its bipartition such that one partite set has a vertex whose degree is not 2 and the degree of every vertex in the other partite set is 2. So we can determine which partite set is $V(G)$ (or $E(G)$). By the assumption, there exists a bijection $\varphi$ mapping $V(G_1)$ to $V(G_2)$ and $E(G_1)$ to $E(G_2)$ in such a way that $x$ and $[e]$ are adjacent in $S(G_1)$ if and only if $\varphi(x)$ and $\varphi([e])$ are adjacent in $S(G_2)$. On the other hand, two vertices $x$ and $y$ are adjacent and joined by an edge $e$ in a graph $G$ if and only if $x$ and $y$ are adjacent to $[e]$ in $S(G)$. Hence it is shown that $G_1 \cong G_2$ if $S(G_1) \cong S(G_2)$.

**4.4. Spectrum for infinite Sierpiński lattice.** Spectral analysis for laplacians on fractals has been studied by many authors. Among many works, we focus on the spectrum of the laplacian on an infinite Sierpiński lattice. M. Fukushima and T. Shima computed the eigenvalues and the density of states of the laplacian on an infinite Sierpiński lattice by using Dirichlet form theory and showed that the oscillatory behaviors of the integrated density of states [5]. A. Teplyaev [21] showed that the set of eigenfunctions with compact support for the laplacian on the infinite Sierpiński lattice with Neumann boundary condition is complete and this is also the case where the Sierpiński lattice has no boundary, which implies the absence of continuous part of the spectrum.

We define an infinite Sierpiński lattice (see Figure 3). Let $\{e_i\}_{i=1}^d$ be the standard basis of $\mathbb{R}^d$ and $e_0$ be the 0 vector. Define $f_i : \mathbb{R}^d \to \mathbb{R}^d$ by $f_i(x) = \frac{1}{2}(x + e_i)$ ($0 \leq i \leq d$). Furthermore we define $V_n$ inductively as follows:

$$V_0 = \bigcup_{0 \leq i < j \leq d} \{(1-t)e_i + te_j \in \mathbb{R}^d \mid 0 \leq t \leq 1\}$$
and

\[ V_n = f_0^{-1} \left( \bigcup_{i=0}^{d} f_i(V_{n-1}) \right), \quad n \geq 1. \]

We regard \( \tilde{S}_d = \bigcup_{n \geq 0} V_n \) as an infinite graph which is 2d-regular except at the origin and the degree of the origin is \( d \). Here the set of vertices of \( V_0 \) is identified with \( \{e_i\}_{i=0}^{d} \) and \( V(\tilde{S}_d) \) with the set of all vertices defined inductively. We prepare two copies of an infinite graph \( \tilde{S}_d \) and identify the vertices (the origins) of degree \( d \). We call the infinite 2d-regular graph constructed here the \( d \)-dimensional Sierpiński lattice and denote it by \( S_d \).

The theorem is the following.

**Theorem 4.24 ([5], [21]).** Let \( S_d \) be the \( d \)-dimensional infinite Sierpiński lattice with \( d \geq 2 \) and \( \Delta_{S_d} \) the laplacian associated with the simple random walk on \( S_d \). Then

\[ \text{Spec}(\Delta_{S_d}) = \bigcup_{k=0}^{\infty} \left\{ \rho^{-k} \left( \frac{d+1}{2d} \right) \right\} \cup \left\{ \rho^{-k} \left( \frac{d+3}{2d} \right) \right\} \cup \left\{ \frac{d+1}{d} \right\}, \]

where \( \rho(x) = -2dx^2 + (d+3)x \).

Here we give another proof of Theorem 4.24. Our method is based on the relationship between the spectrum of the laplacian for an infinite graph \( G \) and that for a para-line (line) graph of \( G \) as was discussed in Section 4.2 and on the self-similar structure of the Sierpiński lattice.

Now, in the same way as in the above, we define an auxiliary graph, an infinite Sierpiński pre-lattice (see Figure 4). Define \( g_i : \mathbb{R}^d \to \mathbb{R}^d \) by \( g_i(x) = \frac{1}{3}(x + 2e_i) \) \((0 \leq i \leq d)\). where \( \{e_i\}_{i=0}^{d} \) are the same as before. Furthermore we define \( W_n \) inductively as follows:

\[ W_0 = \bigcup_{0 \leq i < j \leq d} \left\{ (1-t)e_i + te_j \in \mathbb{R}^d \mid 0 \leq t \leq 1 \right\} \]

and

\[ W_n = g_0^{-1} \left( \bigcup_{i=0}^{d} g_i(W_{n-1}) \right), \quad n \geq 1. \]
We regard $\tilde{G}_d = \cup_{n \geq 0} W_n$ as an infinite graph which is $(d + 1)$-regular except at the origin and the degree of the origin is $d$. We prepare two copies of an infinite graph $\tilde{G}_d$ and join the vertices (the origins) of degree $d$ by an edge. We call the infinite $(d + 1)$-regular graph constructed here the $d$-dimensional Sierpiński pre-lattice and denote it by $G^*_d$. The next observation is crucial.

**Lemma 4.25.** (1) Let $G_d^*$ be the set of all infinite, simple connected, $d$-regular graphs. Then the $d$-dimensional Sierpiński pre-lattice $G_d^*$ is a fixed point of the map $P : G_d \to G_d$, that is, $P(G_d^*) = G_d^*$.

(2) The $d$-dimensional Sierpiński lattice $S_d$ is the line graph of the graph $G_d^*$, that is, $S_d = L(G_d^*)$.

**Proof.** (1) If we regard $W_0$ as a graph, the corresponding graph is isomorphic to a complete graph $K_d$. Let $K_d^{(1)}$ be the graph which corresponds to $W_1$, and let $L_d$ be the para-line graph of a star graph $K_{1,d}$. The graph $L_d$ consists of a $K_d$ and $d$ edges each of which attaches to each vertex of $K_d$. It is easy to see that the para-line graph of $L_d$ is isomorphic to the graph obtained from $L_d$ by replacing $K_d$ with $K_d^{(1)}$. Thus $P(G_d^*)$ is nothing but the graph obtained from $G_d^*$ by replacing every $K_d$ with $K_d^{(1)}$, and hence $P(G_d^*)$ is isomorphic to $G_d^*$.

(2) By Remark 4.3, $L(G_d^*)$ is obtained from $P(G_d^*) \cong G_d^*$ by contracting all the edges joining $K_d$'s. Thus the resultant graph is obviously isomorphic to $S_d$. □

![Figure 4. Infinite Sierpiński pre-lattice (d = 2).](image)

**Lemma 4.26.** Let $K(\mathbb{R}^d)$ be the set of all compact subsets of $\mathbb{R}^d$ with the Hausdorff metric $d_H$. Let $\Theta : K(\mathbb{R}^d) \to K(\mathbb{R}^d)$ be defined as

$$\Theta(A) = f^{-1}(A) \cup B,$$

where $f$ is a continuous map such that each branch of the inverse $f^{-1}$ is a contraction map on $\mathbb{R}^d$ and $B$ is a compact subset of $\mathbb{R}^d$. Then $\Theta$ is a contraction map on $(K(\mathbb{R}^d), d_H)$ and, in particular, $\Theta$ has the unique fixed point $A^* \in K(\mathbb{R}^d)$.

**Proof.** It is well-known that $(K(\mathbb{R}^d), d_H)$ is a complete metric space. Hence in order to show the existence and uniqueness of the fixed point of the map $\Theta$, it is sufficient to prove that $\Theta$ is a contraction. In fact, it follows from the following: for any contraction map $q$ on $\mathbb{R}^d$ satisfying that $|q(x) - q(y)| \leq C|x - y|$ with $0 < C < 1$, it holds that

$$d_H(q(A), q(B)) \leq Cd_H(A, B),$$

$$d_H(A_1 \cup A_2, B_1 \cup B_2) \leq \max(d_H(A_1, B_1), d_H(A_2, B_2)).$$
Proof of Theorem 4.24. Since \( G_d^* \) is \((d + 1)\)-regular, by Theorem 4.12 and by Lemma 4.25, we obtain
\[
\text{Spec}(-\Delta_{G_d^*}) = \text{Spec}(-\Delta_{P(G_d^*)})
\]
(4.5)
\[
= \varphi_P^{-1}(\text{Spec}(-\Delta_{G_d^*})) \cup \{1\} \cup \left\{\frac{d + 3}{d + 1}\right\},
\]
where \( \varphi_P(x) = -(d + 1)x^2 + (d + 3)x \). Lemma 4.26 shows that the above equation (4.5) has a unique solution, and it is given by
\[
\text{Spec}(-\Delta_{G_d^*}) = \bigcup_{k=0}^{\infty} \left\{\varphi_P^{-k}(1) \cup \varphi_P^{-k}\left(\frac{d + 3}{d + 1}\right)\right\}.
\]
Since \( S_d \) is the line graph of \( G_d^* \) (Lemma 4.25), we obtain
\[
\text{Spec}(-\Delta_{S_d}) = \frac{d + 1}{2d} \left(\bigcup_{k=0}^{\infty} \left\{\varphi_P^{-k}(1) \cup \varphi_P^{-k}\left(\frac{d + 3}{d + 1}\right)\right\} \cup \left\{\frac{d + 1}{d}\right\}\right)
\]
\[
= \bigcup_{k=0}^{\infty} \left\{\rho^{-k}\left(\frac{d + 1}{2d}\right) \cup \rho^{-k}\left(\frac{d + 3}{2d}\right) \cup \left\{\frac{d + 1}{d}\right\}\right\},
\]
where \( \rho(x) = (\tau \circ \varphi_P \circ \tau^{-1})(x) = -2dx^2 + (d + 3)x \) with \( \tau(x) = \frac{(d+1)x}{2d} \). \(\square\)

Remark 4.27. The spectrum set of any fixed point of the map \( P : G_d \to G_d \) should be the same as that of the infinite Sierpiński pre-lattice.

5. Problems

In this section, we collect some problems around the topics in this paper. First three problems are concerned with FSP.

Problem 5.1. Does \( M_{\text{ab}} \) always have FSP when \( M \) is regular? If \( M \) is even-regular, \( M_{\text{ab}} \) has FSP by Proposition 2.5. If \( M \) is odd-regular and bipartite, then, by the classical fact in graph theory (cf. [1], [2]), \( M \) has a 1-factor. As a result, \( M \) has an even factor satisfying the condition (2) in Proposition 2.5, therefore \( M_{\text{ab}} \) has FSP. It is remained for regular graphs to show that \( M_{\text{ab}} \) has FSP even if \( M \) is a non-bipartite odd-regular graph.

Problem 5.2. Full spectrum conjecture has been inspired by several topics in (cf. [16, 20]) for a non-compact manifold \( X \) such that \( X \) has an isometry group \( \Gamma \) acting discontinuously on \( X \) with a compact quotient space \( \Gamma \backslash X \). One might say that an maximal abelian covering manifold of a compact manifold has FSP if the spectrum of the Laplace-Beltrami operator on it is equal to \([0, \infty)\). The method used in our proof of Proposition 2.5 depends on graph structure. Can it be transplanted to the case of manifolds?

Problem 5.3. In Definition 2.1 we defined FSP as a graph property by fixing the simple random walk. When we replace the simple random walk with another reversible random walk, what is a suitable formulation of FSP? For example, we consider the asymmetric reversible random walk on \( \mathbb{Z}^1 \) defined by \( p(e) = p \) and \( p(\bar{e}) = q \) for every \( e \in A(G) \) with \( o(e) = x \) and \( t(e) = x + 1 \). Then the spectrum set is \([1 - 2\sqrt{pq}, 1 + 2\sqrt{pq}]\). Taking account of Theorem 3.10, it might be said to have
FSP in a sense, even if $p$ is not associated with the simple random walk. Likewise, when we consider a non-amenable covering graph, what is a suitable formulation of FSP for such a covering graph? In this case, the spectrum is also away from 0 (cf. [20]).

When graphs are hyperbolic, there are few examples whose spectra are completely determined. Once one finds a graph for which one can compute the spectrum, by using Theorems 4.10, 4.11 and 4.12, one can make many examples.

**Problem 5.4.** We have obtained the isoperimetric constant for a $(d, f)$-regular planar graph in Theorem 3.1. Determine the spectrum for a $(d, f)$-regular planar graph. At least determine $\lambda_0(G)$ for $(d, f)$-regular planar graphs.

The next problem, which is originally raised in [8] as a conjecture, is also related to the result in Section 3:

**Problem 5.5.** Let $G$ be a planar graph. When $\kappa_G(x) > 0$ for every vertex $x$ in $G$, is $G$ always a finite graph? It is a discrete version of the well-known Myers theorem in Riemannian geometry which says that a complete Riemannian manifold with Ricci curvature bounded below by a positive number is compact.

The next two problems are concerned with the results in Section 3.

**Problem 5.6.** Let $\mathcal{F}$ be the set of all the square summable functions on $V(G)$ taking values only in $\{-1, 0, 1\}$. We denote $\alpha_0(G, \mathcal{F})$ and $\alpha_\infty(G, \mathcal{F})$ for the $\mathcal{F}$ by $h(G)$ and $h_\infty(G)$, respectively. As mentioned in Example 3.7, $h(G)$ is nothing but the isoperimetric constant. For a finite graph $G$, both $\lambda_0(G)$ and $h(G)$ are equal to 0 and $\lambda_\infty(G)$, of course, can be directly computed as the maximal eigenvalue. When $G$ is bipartite, $h_\infty(G) = 2$ by Proposition 3.8. For general non-bipartite graph, what is $h_\infty(G)$? Are there any algorithms for computing $h_\infty(G)$?

**Problem 5.7.** By Theorem 3.10 and Remark 3.12 it holds that $\lambda_0(G) + \lambda_\infty(G) < 2$ when $G$ is essentially non-bipartite. Does it also hold that $h(G) + h_\infty(G) < 2$ when $G$ is essentially non-bipartite?

The following is concerned with results in Sections 4.3 and 4.4.

**Problem 5.8.** Do there exist any fixed points of the map $P : \mathcal{G}_d \to \mathcal{G}_d$ other than the infinite Sierpiński pre-lattice (see Remark 4.27)? In the proof of Theorem 4.24 which says the spectrum set is Cantor-like, we gave the set equation for the spectrum by using the fact that the infinite Sierpiński pre-lattice is a fixed point of the map $P$. By emphasizing this fact can one show the absence of continuous singular spectrum to give another proof of some results in [21]?

**References**


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