3 FERMION PROCESS AND FREDHOLM DETERMINANT

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Abstract
We construct a family of probability measures on a configuration space associated with Fredholm determinants for symmetric integral operators. For these probability measures we show basic limit theorems (law of large numbers, central limit theorem and large deviation.) Also we deal with shift dynamical systems naturally corresponding to these probability measures on $\mathbf{Z}^1$ and investigate their ergodic properties.

1. Introduction

Let $\mathcal{H}_N$ be the space of $N \times N$-Hermitian matrices, and $P(dX)$ the Gaussian probability measure on $\mathcal{H}_N$ given by

$$P(dX) = Z_N^{-1} \exp(-\mathrm{Tr}(X^2))dX, \tag{1.1}$$

where Z_N^{-1} is the normalized constant and dX is the Lebesgue measure on $\mathcal{H}_N$ identified with $\mathbf{R}^{N^2}$. The pair $(\mathcal{H}_N, P(dX))$ is called Gaussian Unitary Ensemble (GUE). GUE has been investigated by many authors. One of the main concerns about GUE is to study the joint distributions of

H.G.W. Begehr et al. (eds.), Proceedings of the Second ISAAC Congress, Vol. 1, 15–23.

N eigenvalues and its limiting behavior as $N \to \infty$. The joint eigenvalue distribution is well known [8] and given by

$$\mu_N(dx_1, dx_2, \ldots, dx_N) = \tag{1.2}$$
$$C_N^{-1} \prod_{1 \le i < j \le N} (x_i - x_j)^2 \exp\left(-\sum_{i=1}^N x_i^2\right) dx_1, dx_2, \ldots, dx_N.$$

In order to obtain the limiting behavior as $N \to \infty$, we need to take a scaling limit. Here we show two typical scalings. First we consider the empirical measure

$$\xi_N = \frac{1}{N} \sum_{i=1}^N \delta_{x_i/\sqrt{N}}, \tag{1.3}$$

where $\{x_i\}_{i=1}^N$ are the eigenvalues of GUE. Then as $N \to \infty$, one can obtain the so-called Wigner's semi-circle law, that is,

$$\xi_N \to \frac{1}{\pi}\sqrt{2 - x^2}dx \tag{1.4}$$

almost surely.

The second one is the following: set $\lambda_N = \pi/\sqrt{2N}$. Then,

$$\begin{aligned} \lim_{N\to\infty} \lambda_N^n \rho_n^{(N)}(\lambda_N x_1, \ldots, \lambda_N x_n) &= \det\left(K(x_i, x_j)\right)_{i,j=1}^n \\ &:= \rho_n(x_1, \ldots, x_n), \end{aligned} \tag{1.5}$$

where

$$K(x, y) = \frac{\sin \pi(x - y)}{\pi(x - y)} \tag{1.6}$$

and $\rho_n^{(N)}(x_1, x_2, \ldots, x_n)$ is the density correlation function defined by

$$\rho_n^{(N)}(x_1, x_2, \ldots, x_n) = \tag{1.7}$$
$$\frac{N!}{(N-n)!} \int_{\mathbf{R}^{N-n}} \mu_N(x_1, x_2, \ldots, x_N) dx_{n+1} dx_{n+2} \ldots dx_N.$$

In particular, $\rho_1^{(N)}(x)/N$ is the eigenvalue density distribution.

It is natural to ask that does there exist a probability measure on a configuration space which has $\{\rho_n\}_{n=1}^\infty$ as correlation functions? Moreover does there exist a probability measure on a configuration space which has

$$\rho_n(x_1, x_2, \ldots, x_n) = \det\left(K(x_i, x_j)\right)_{i,j=1}^n \tag{1.8}$$

as correlation functions for a kernel $K(x, y)$? The answer is a yes under some conditions for $K(x, y)$.

Let R be a locally compact Hausdorff space with countable basis, $\lambda(dx)$ be a nonnegative Radon measure on R and Q be the space of nonnegative integer-valued Radon measures on R. An element ξ of Q may be expressed as

$$\xi = \sum_i \delta_{x_i} \quad (x_i \in R) \tag{1.9}$$

and then we write

$$\langle \xi, f \rangle = \sum_i f(x_i) \tag{1.10}$$

for a function f on R so far as the right hand side of (1.10) makes sense. It is well known that a measure on Q is characterized by its Laplace transform

$$\int_Q \mu(d\xi) \exp(-\langle \xi, f \rangle). \tag{1.11}$$

Now we define the correlation function ρ_n in this setting. For $\xi \in Q$ and any bounded measurable function f_n on R^n with compact support, denote

$$\langle \xi_n, f_n \rangle = \sum_{x_1, x_2, \ldots, x_n \in \xi}^{*} f_n(x_1, x_2, \ldots, x_n) \tag{1.12}$$

where $\sum^*$ denotes the sum over all mutually distinct points $x_1, x_2, \ldots, x_n$. If $\langle \xi_n, f_n \rangle$ is μ-integrable for any bounded continuous function f_n with compact support on R^n, then the formula

$$\int_Q \langle \xi_n, f_n \rangle \mu(d\xi) = \int_{R^n} f_n(x_1, \ldots, x_n) \lambda_n(dx_1 \ldots dx_n) \tag{1.13}$$

defines a measure λ_n on R^n which is called the n-th correlation measure of μ. Furthermore if λ_n is absolutely continuous with respect to $\lambda^{\otimes n}$, its density $\rho_n(x_1, \ldots, x_n)$ is called the n-th correlation function.

The following is a basic result for the question above.

Theorem 1.1. *Let K be a bounded symmetric integral operator on $L^2(R, \lambda(dx))$ whose spectrum Spec(K) is contained in the unit interval* $[0, 1]$. *Assume that, for each compact subset Λ of R, the restriction $K_\Lambda = 1_\Lambda K 1_\Lambda$ of K to the subspace $L^2(R, \lambda(dx))$ is of trace class(1_Λ being the indicator of the set Λ) and that the spectrum Spec(K_Λ) is contained in the half open interval* $[0, 1)$. *Then there exists a unique probability measure μ on the configuration space Q such that*

$$\int_Q \mu(d\xi) \exp(-\langle \xi, f \rangle) = \det(I - K_\varphi) \tag{1.14}$$

for each nonnegative measurable function f on R with compact support, where K_φ stands for the trace class operator defined as

$$K_\varphi(x,y) = \sqrt{\varphi(x)}K(x,y)\sqrt{\varphi(y)} \tag{1.15}$$

and

$$\varphi(x) = 1 - \exp(-f(x)). \tag{1.16}$$

Furthermore, the correlation functions of the probability measure μ exist and are given by

$$\rho_n(x_1, \dots, x_n) = \det\Big(K(x_i, x_j)\Big)_{i,j=1}^n \quad (n \geq 1) \tag{1.17}$$

and

$$\rho_0 = 1.$$

The Palm measure is the basic concept in point process theory as well as the correlation function (cf. [3]). If μ has mean λ_1, there exists a probability measure μ^x on Q for λ_1-a.e.x satisfying the following equation

$$\int_Q \mu(d\xi) \int_R \xi(dx)u(\xi, x) = \int_R \lambda_1(dx) \int_Q \mu^x(d\xi)u(\xi + \delta_x, x) \tag{1.18}$$

for any bounded measurable function $u(\xi, x)$ on $Q \times R$ with compact support in x. The probability measure μ^x on Q is called the Palm measure.

The point processes associated with Fredholm determinant is closed under the operation to take the Palm measure $\mu \mapsto \mu^x$.

Theorem 1.2. *The Palm measure μ^x of μ at x is also given by the formula (1.14) with $K(x_1, x_2)$ replaced by*

$$K^x(x_1, x_2) = K(x_1, x_2) - K(x, x_1)K(x, x_2)/K(x, x).$$

The idea of the proof of Theorem 1.1 is simple. Note that $Q(\Lambda) \cong \bigcup_{n=0}^\infty \Lambda^n/\sim$ for each compact set Λ. We define the density function σ_Λ on $\bigcup_{n=0}^\infty \Lambda^n/\sim$ by

$$\sigma_\Lambda(x_1, \dots, x_n) = \det(I - K_\Lambda)\det(J[\Lambda](x_i, x_j))_{i,j=1}^n \text{ on } \Lambda^n, \tag{1.19}$$

where $J[\Lambda] = K_\Lambda(I - K_\Lambda)^{-1}$ and for $n = 0$

$$\sigma_\Lambda(\emptyset) = \det(I - K_\Lambda) \text{ on } \Lambda^0. \tag{1.20}$$

This defines a probability measure μ_Λ on $Q(\Lambda)$. It suffices to check Kolmogorov's consistency condition for the family of probability measures $\{\mu_\Lambda\}_{\Lambda \subset R}$. For the correlation functions we use the principle of inclusion and exclusion.

Remark. Macchi [6, 7] was the first to study such a point process and called it a fermion process. The fermion processes, especially, those associated with the sine kernel and some other special kernels, have been studied in detail by [1, 11, 12]

Example 1.3. Let $R = \mathbf{R}^d$ the real line, dx be the Lebesgue measure and K be the convolution operator with a continuous kernel $k(x)$ defined by

$$k(x) = \left(\frac{1}{2\pi}\right)^d \int_{\mathbf{R}^d} \widehat{k}(t) e^{-itx} dt. \tag{1.21}$$

for an even function $\widehat{k} \in L^1(\mathbf{R}^d)$ taking values in $[0, 1]$. Then one can check the required condition in Theorem 1.1, and so K defines a translation invariant probability measure on the configuration space $Q = Q(\mathbf{R}^d)$. Especially, in the case of $d = 1$, the sine kernel $k(x)$ is the Fourier transform of the indicator function of the interval $[-\pi, \pi]$ so that $\mathrm{Spec}(K) = \{0\} \cup \{1\}$ and it is easy to check that $\mathrm{Spec}(K_\Lambda) \subset (0, 1)$ whenever Λ is a bounded interval.

2. Convolution on $\mathbf{R}^d$

In this section, we always assume the kernel $k(x)$ satisfies the conditions given in above Example 1.3. In this case, the corresponding measure μ is translation invariant as mentioned above.

Lemma 2.1. *Let f be a bounded measurable function with compact support and set $f_N = f(\cdot/N)$. Then, as $N \to \infty$,*

$$\begin{aligned} &\int_Q \langle \xi, f_N \rangle^2 \mu(d\xi) - \left(\int_Q \langle \xi, f_N \rangle \mu(d\xi) \right)^2 \\ &\sim N^d \int_{\mathbf{R}^d} f(x)^2 dx \times \left(\frac{1}{2\pi}\right)^d \int_{\mathbf{R}^d} \widehat{k}(t)(1 - \widehat{k}(t)) dt. \end{aligned} \tag{2.1}$$

Remark. The quantity above for an indicator function f is called the number variance in random matrix theory. Unless $\widehat{k}$ takes values only 0 and 1 almost everywhere, the number variance does not vanish. ¿From this point of view, the point field associated with the sine kernel is a typical degenerate case because the Fourier transform of the sine kernel $\sin \pi x / \pi x$ is an indicator function of the interval $[-\pi, \pi]$

The following are basic limit theorems.

Proposition 2.2. *(Law of large numbers) Let f be a bounded measurable function with compact support. Then*

$$\langle \xi, \frac{f_N}{N^d} \rangle \to \int_{\mathbf{R}^d} f(x)k(0)dx \quad \mu\text{-a.e.}\xi \text{ and in } L^1(Q, \mu), \tag{2.2}$$

where $f_N(\cdot) = f(\cdot/N)$.

Proposition 2.3. *(Central limit theorem) Let f be a bounded measurable function with compact support and assume $\int_{\mathbf{R}^d} f(x)dx = 0$. Then*

$$\lim_{N\to\infty} \int_Q \mu(d\xi) \exp\left(i\langle \xi, \frac{f_N}{N^{d/2}} \rangle \right) = \exp\left(-\frac{1}{2}\sigma^2 \|f\|_2^2 \right), \tag{2.3}$$

where

$$\sigma^2 = \left(\frac{1}{2\pi}\right)^d \int_{\mathbf{R}^d} \widehat{k}(t)(1 - \widehat{k}(t))dt \tag{2.4}$$

and $f_N(\cdot) = f(\cdot/N)$.

Proposition 2.4. *(Large deviation) Let f be a nonnegative function with compact support and set $\varphi = 1 - \exp(-f(x))$. Then,*

$$\lim_{N\to\infty} \frac{1}{N^d} \log \int_Q \mu(d\xi) \exp\left(-\langle \xi, f_N \rangle\right) = \tag{2.5}$$

$$\left(\frac{1}{2\pi}\right)^d \int_{\mathbf{R}^d} dt \int_{\mathbf{R}^d} dx \log\left(1 - \widehat{k}(t)\varphi(x)\right),$$

where $f_N(\cdot) = f(\cdot/N)$.

3. Shift

The following is the discrete version of Theorem 1.1.

Theorem 3.1. *Let R be a countable set, λ be a measure on R and K be a bounded operator on $\ell^2(R, \lambda)$ with $Spec(K) \subset [0,1]$. Then there exists a unique probability measure μ on the configuration space $Q = Q(R)$ which satisfies the following property: for any disjoint finite subsets Λ_0 and Λ_1 with union $\Lambda = \Lambda_0 \cup \Lambda_1$,*

$$\mu(0^{\Lambda_0} 1^{\Lambda_1}) = \det(P_{\Lambda_0}(I_\Lambda - K_\Lambda) + P_{\Lambda_1} K_\Lambda), \tag{3.1}$$

where P_{Λ_i} are the projection operators onto $\ell^2(\Lambda_i, \lambda)$. Here we abbreviated notation as

$$\mu(0^{\Lambda_0} 1^{\Lambda_1}) = \mu\left(\xi \in Q \ ; \ \xi\{i\} = 0 \ (i \in \Lambda_0), \ \xi\{j\} = 1 \ (j \in \Lambda_1)\right). \tag{3.2}$$

Remark. In the continuous case, Poisson point process is not contained in our class, however, in the discrete case, Bernoulli measure is the fermion point process associated with the operator αI for $0 < \alpha < 1$. The difference comes from the fact that the identity operator I restricted on a compact set on a discrete space is a compact operator while that on a continuum space is not.

In the discrete case, we can check Kolmogorov's consistency condition directly. The following simple identity is crucial.

Lemma 3.2. *Let A be an $N \times N$-matrix and P_Λ an $N \times N$-diagonal matrix such that*

$$P_\Lambda(i,i) = \begin{cases} 1 & \text{if } i \in \Lambda^c \\ 0 & \text{if } i \in \Lambda, \end{cases} \tag{3.3}$$

for each subset $\Lambda \subset \{1,2,\ldots,n\}$. Then

$$\sum_{\Lambda \subset \{1,2,\ldots,n\}} (-1)^{|\Lambda|} \det(A - P_\Lambda) = 1. \tag{3.4}$$

Moreover, if A is a symmetric matrix such that $0 \le A \le I$, then

$$(-1)^{|\Lambda|} \det(A - P_\Lambda) \ge 0. \tag{3.5}$$

Example 3.3. Let $R = \mathbf{Z}^d$, λ be the Haar measure on $\mathbf{Z}^d$ such that $\lambda(\{x\}) = 1$ for any $x \in \mathbf{Z}^d$, and K be a convolution operator with kernel k defined by

$$k(n) = \left(\frac{1}{2\pi}\right)^d \int_{\mathbf{T}^d} \widehat{k}(\theta) e^{in\theta} d\theta, \tag{3.6}$$

for an even function $\widehat{k} \in L^1(\mathbf{T}^d)$ taking values in $[0,1]$. Then K satisfies the conditions required in Theorem 3.1 and the corresponding probability measure μ is translation invariant.

Remark. The discrete versions of Proposition 2.2, 2.3 and 2.4 can be proved in the same way. Especially, the discrete version of Proposition 2.4 is a generalization of Szegö's theorem for Toeplitz operators [10].

¿From now on we consider the case where $R = \mathbf{Z}^1$ and K is a convolution operator. Since μ has no multiple points, we can identify $Q(\mathbf{Z}^1)$ with $\{0,1\}^{\mathbf{Z}^1}$, and so we obtain a shift dynamical system $(\{0,1\}^{\mathbf{Z}^1}, \mu, \sigma)$ where σ stands for the translation to the left or the shift transformation: $(\sigma\xi)\{n\} = \xi\{n+1\}$. This provides us a new family of shift dynamical systems. We show ergodic properties of the shifts.

Theorem 3.4. *Let K be a convolution operator with kernel k given by (3.6). Then the shift dynamical system $(\{0,1\}^{\mathbf{Z}}, \mu, \sigma)$ is mixing. Moreover, if $\|K\| < 1$ and*

$$\sum_{n\in\mathbf{Z}} |n||k(n)|^2 < \infty, \tag{3.7}$$

the shift is weak Bernoulli, i.e., $(\{0,1\}^{\mathbf{Z}}, \mu, \sigma)$ is isomorphic to the Bernoulli system.

Proposition 3.5. *Let K be a convolution operator with kernel k given by (3.6). Then the shift dynamical system $(\{0,1\}^{\mathbf{Z}}, \mu, \sigma)$ has positive entropy:*

$$h_\mu(\sigma) = -\lim_{\Lambda\to\mathbf{Z}^1} \frac{1}{|\Lambda|} \sum_{\Lambda_0\subset\Lambda} \mu(0^{\Lambda_0}1^{\Lambda_1}) \log \mu(0^{\Lambda_0}1^{\Lambda_1}) > 0, \tag{3.8}$$

where $\Lambda = \Lambda_0 \cup \Lambda_1$.

Definition 3.6. *For the potential $\{\Phi_n\}_{n\geq 1}$; $\Phi_n : R^n \to \mathbf{R}$, we define the total energy U by*

$$U(x|\xi) = \Phi_1(x) + \sum_{y\in\xi} \Phi_2(x,y) + \sum_{y,z\in\xi} \Phi_3(x,y,z) + \cdots. \tag{3.9}$$

The probability measure μ is called a Gibbs measure if

$$\int_Q \mu(d\xi)F(\xi) = \int_Q \mu(d\xi)e^{-U(x|\xi_{\Lambda^c})}F(\delta_x + \xi_{\Lambda^c}) \tag{3.10}$$

for any compact Λ and bounded Borel measurable function F, where ξ_{Λ^c} is the restriction of ξ to Λ.

Theorem 3.7. *Let K be the convolution operator on $\ell^2(\mathbf{Z}^d)$ and assume the Fourier transform of the convolution kernel k takes the values in the unit open interval: $0 < \widehat{k}(\xi) < 1$. Let j be the convolution kernel of $J = (I-K)^{-1}K$ and set $J_\xi = (j(x-y))_{x,y\in\xi}$ and $j_\xi = (j(x))_{x\in\xi}$ for $\xi \subset \mathbf{Z}^d$. Then, the measure μ is Gibbs and*

$$U(0|\xi) = -\log(j(0) - \langle J_\xi^{-1} j_\xi, j_\xi\rangle). \tag{3.11}$$

Furthermore, the measure μ is the unique Gibbs measure with potential (3.11).

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