Connected Components of Regular Fibers of Differentiable Maps

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1. Introduction

Quotient space
Stein factorization
Example
Triangulation of a map
Today’s topic

2. Triangulation of Stein Factorization

3. Application

1. Introduction
$M, N : \text{smooth (}= C^\infty) \text{ manifolds}$

$f : M \to N \quad \text{a smooth map}$
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$M, N : \text{smooth (} = C^\infty\text{) manifolds}$

$f : M \to N$ a smooth map

For $x, x' \in M$, define $x \sim x'$ if

(i) $f(x) = f(x')(= y)$, and

(ii) $x$ and $x'$ belong to the same connected component of $f^{-1}(y)$. 

We denote by $W_f=M=\text{the quotient space}$, which can be regarded as the space of connected components of fibers of $f$.

We denote by $q_f:M\to W_f$ the quotient map.
\[ M, N : \text{smooth (}= C^\infty) \text{ manifolds} \]
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We denote by \( W_f = M/\sim \) the **quotient space**, which can be regarded as the space of connected components of fibers of \( f \).
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\( W_f \) is often called the quotient space or the Reeb space (or the Reeb complex) of \( f \).
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\( W_f \) is often called the quotient space or the Reeb space (or the Reeb complex) of \( f \).

We denote by \( q_f : M \to W_f \) the quotient map.
There exists a unique continuous map $\bar{f} : W_f \to N$ that makes the following diagram commutative:

$$
\begin{array}{ccc}
M & \xrightarrow{f} & N \\
& q_f \swarrow & \searrow \bar{f} \\
& W_f & 
\end{array}
$$
There exists a unique continuous map $\bar{f} : W_f \rightarrow N$ that makes the following diagram commutative:

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\begin{array}{ccc}
M & \xrightarrow{f} & N \\
\downarrow q_f & & \uparrow \bar{f} \\
W_f & & \\
\end{array}
$$

The above diagram is called the **Stein factorization** of $f$. 
There exists a unique continuous map $\tilde{f} : W_f \rightarrow N$ that makes the following diagram commutative:

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\begin{array}{ccc}
M & \xrightarrow{f} & N \\
\downarrow{q_f} & & \downarrow{\tilde{f}} \\
W_f & & &
\end{array}
$$

The above diagram is called the **Stein factorization** of $f$.

Note that $W_f$ is merely a topological space at this moment.
There exists a unique continuous map $\tilde{f} : W_f \to N$ that makes the following diagram commutative:

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M & \xrightarrow{f} & N \\
\downarrow{q_f} & & \nearrow{\tilde{f}} \\
W_f & & \\
\end{array}
$$

The above diagram is called the **Stein factorization** of $f$.

Note that $W_f$ is merely a topological space at this moment.

Note also that each fiber of $q_f$ corresponds to a connected component of a fiber of $f$. 
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Figure 1: Stein factorization
Let \( g : X \rightarrow Y \) be a continuous map between topological spaces.
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Let \( g : X \to Y \) be a continuous map between topological spaces. Then, \( g \) is said to be **triangulable** if there exist **simplicial complexes** \( K \) and \( L \), a **simplicial map** \( s : K \to L \), and **homeomorphisms** \( \lambda : |K| \to X \) and \( \mu : |L| \to Y \) such that the following diagram is commutative:

\[
\begin{array}{ccc}
X & \xrightarrow{g} & Y \\
\uparrow{\lambda} & & \uparrow{\mu} \\
|K| & \xrightarrow{|s|} & |L|,
\end{array}
\]

where \( |K| \) and \( |L| \) are polyhedrons associated with \( K \) and \( L \), respectively, and \( |s| \) is the continuous map associated with \( s \).
Remark 1.1  The notion of the Stein factorization can be similarly defined for any continuous map \( g : X \rightarrow Y \).
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Today’s first topic: If $g$ is triangulable, then so is its Stein factorization?
Remark 1.1 The notion of the Stein factorization can be similarly defined for any continuous map \( g : X \to Y \).

Then, again the quotient space \( W_g \) is merely a topological space.

*Today’s first topic:* If \( g \) is triangulable, then so is its Stein factorization?

We will show that the answer is “Yes” under certain mild conditions.
Remark 1.1  The notion of the Stein factorization can be similarly defined for any continuous map $g : X \rightarrow Y$.

Then, again the quotient space $W_g$ is merely a topological space.

Today’s first topic: If $g$ is triangulable, then so is its Stein factorization?

We will show that the answer is “Yes” under certain mild conditions.

In the second part, we will apply the result for studying components of regular fibers of generic smooth maps.
§2. Triangulation of Stein Factorization

Barycentric subdivision
Triangulation of a Stein factorization
Why barycentric subdivision?
Case of generic maps

§3. Application
Lemma 2.1  Let $s : K \rightarrow L$ be a simplicial map. We denote by $L'$ the barycentric subdivision of $L$. Then, there exists a subdivision $K'$ of $K$ and a simplicial map $s' : K' \rightarrow L'$ such that $|s| : |K| \rightarrow |L|$ coincides with $|s'| : |K'| \rightarrow |L'|$. 
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Theorem 2.2
Suppose $X$ is locally compact and $g$ is proper.
If $g : X \to Y$ is triangulable, then so is its Stein factorization.
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Suppose $X$ is locally compact and $g$ is proper. If $g : X \to Y$ is triangulable, then so is its Stein factorization. That is, we have the commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{g} & Y \\
|K'| & \xrightarrow{|s'|} & |L'| \\
\end{array}
$$

for some simplicial complex $V$, simplicial maps $\varphi : K' \to V$, $\psi : V \to L'$, and a homeomorphism $\Theta$, where $K'$, $L'$, $s'$, etc. are as before.
Why barycentric subdivision?

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   - Triangulation of a Stein factorization
   - Why barycentric subdivision?
   - Case of generic maps

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No Good!
Why barycentric subdivision?

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\[ K' \quad \rightarrow \quad L' \]

OK!
Theorem 2.3 (Shiota, 2000)

*Proper Thom maps* between smooth manifolds are always triangulable.
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In particular, topologically stable proper maps are triangulable.
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In particular, topologically stable proper maps are triangulable.

Corollary 2.4

For smooth manifolds $M$ and $N$, the set of smooth maps $M \to N$ whose Stein factorization is triangulable contains an open and dense subset of the set of all proper smooth maps $C^\infty(M, N)_\text{prop}$ endowed with the Whitney $C^\infty$-topology.
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\(M_0, M_1\) : closed oriented manifolds with \(\dim M_0 = \dim M_1 = m\).
We say that \(M_0\) and \(M_1\) are **oriented cobordant**
if \(\exists\) compact oriented \((m + 1)\)-dimensional manifold \(W\)
such that \(\partial W = (-M_0) \cup M_1\),
where \(-M_0\) denotes the manifold \(M_0\) with the orientation reversed.
$M_0, M_1$ : closed oriented manifolds with $\dim M_0 = \dim M_1 = m$. We say that $M_0$ and $M_1$ are **oriented cobordant** if there exists a compact oriented $(m + 1)$-dimensional manifold $W$ such that $\partial W = (-M_0) \cup M_1$, where $-M_0$ denotes the manifold $M_0$ with the orientation reversed.
The relation “oriented cobordant” defines an equivalence relation. The equivalence class of a manifold $M$ will be denoted by $[M]$. 
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We can define \([M] + [M'] = [M \cup M']\), so that

\[
\Omega_m = \{ [M] \mid M \text{ is a closed oriented } m\text{-dim. manifold} \}
\]

forms an additive group. This is called the \( m\)-dim. oriented cobordism group.
If we ignore the orientations, then we get the \( m \)-dim. (unoriented) cobordism group, denoted by \( \mathcal{N}_m \).
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The groups $\Omega_m$ and $\mathcal{N}_m$ have been extensively studied and their structures have been completely determined.

- $\Omega_m$ is a finitely generated abelian group.
- $\mathcal{N}_m$ is a finitely generated $\mathbb{Z}_2$-module.
- $\Omega_m$ is a finite group unless $m$ is a multiple of four.
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<tbody>
<tr>
<td>$\Omega_*$</td>
<td>$\mathbb{Z}$</td>
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<td>0</td>
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<td>$\mathbb{Z}$</td>
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<td>$\mathcal{N}_*$</td>
<td>$\mathbb{Z}_2$</td>
<td>0</td>
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A closed manifold \( M \) with \( [M] = 0 \) is said to be \textit{(oriented) null-cobordant}. 
M: closed manifold (compact and $\partial M = \emptyset$)

$f : M \to N$ a smooth map with $m = \dim M \geq \dim N = n$.

Assume that $f$ is **triangulable** (e.g. a topologically stable proper map).

---

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\(\Rightarrow W_f\) is an \(n\)-dim. polyhedron.
Cobordism classes of regular fiber components

\[ M \text{: closed manifold (compact and } \partial M = \emptyset) \]
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Assume that \( f \) is \textit{triangulable} (e.g. a topologically stable proper map).
\[ \implies W_f \text{ is an } n\text{-dim. polyhedron.} \]

**Theorem 3.1**

\(1\) \textbf{If a component of a regular fiber of } f \textbf{ is not null-cobordant,}
then \( H_n(W_f; \mathbb{Z}_2) \neq 0. \)
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$M$: closed manifold (compact and $\partial M = \emptyset$)

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Assume that $f$ is triangulable (e.g. a topologically stable proper map).

$\implies W_f$ is an $n$-dim. polyhedron.

Theorem 3.1

(1) If a component of a regular fiber of $f$ is not null-cobordant, then $H_n(W_f; \mathbb{Z}_2) \neq 0$.

(2) Suppose $f$ is an oriented map (i.e. the regular fibers are consistently oriented). If a component of a regular fiber of $f$ is not oriented null-cobordant, then $H_n(W_f; \Omega_{m-n}) \neq 0$. 
Corollary 3.2

(1) If $H_n(W_f; \mathbb{Z}_2) = 0$, then every component of every regular fiber of $f$ is null-cobordant.
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(1) If $H_n(W_f; \mathbb{Z}_2) = 0$, then every component of every regular fiber of $f$ is null-cobordant.

(2) If $f$ is an oriented map and $H_n(W_f; \Omega_{m-n}) = 0$, then every component of every regular fiber of $f$ is oriented null-cobordant.
Let \( s : K \to L \) be a triangulation of \( f : M \to N \).

By Theorem 2.2, we have a triangulation of the Stein factorization:

\[
\begin{array}{c}
|K'| \quad |s'| \quad |L'| \\
\downarrow \varphi \quad \uparrow \psi \\
|V| \\
\end{array}
\begin{array}{c}
M \quad f \quad N \\
q_f \quad \bar{f} \\
W_f
\end{array}
\]

Proof of Theorem 3.1

An \( n \)-cycle of the quotient space

Proof of Lemma 3.3

Proof of Lemma 3.3

A homology class of \( W_f \)
Proof of Theorem 3.1

Let \( s : K \to L \) be a triangulation of \( f : M \to N \).

By Theorem 2.2, we have a triangulation of the Stein factorization:

\[
\begin{array}{ccc}
|K'| & \to & |L'| \\
\uparrow & & \uparrow \\
|\varphi| & \leftrightarrow & |\psi| \\
\downarrow & & \downarrow \\
|V| & \leftrightarrow & W_f
\end{array}
\]

For each \( n \)-simplex \( \sigma \in V \), define

\[
\omega_\sigma := [|\varphi|^{-1}(b_\sigma)] \in \mathcal{M}_{m-n},
\]

where \( b_\sigma \in \sigma \) is the barycenter of \( \sigma \).
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|V| & \xleftrightarrow{} & W_f
\end{array}
\]

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M & \xrightarrow{f} & N \\
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\end{array}
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For each \( n \)-simplex \( \sigma \in V \), define

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\omega_\sigma := [|\varphi|^{-1}(b_\sigma)] \in \Omega_{m-n},
\]

where \( b_\sigma \in \sigma \) is the barycenter of \( \sigma \).

\( \omega_\sigma \) : cobordism class of the regular fiber component corresponding to \( \sigma \subset |V| = W_f \).
An $n$-cycle of the quotient space

Set

$$c_f = \sum_{\sigma} \omega_{\sigma} \sigma \in C_n(V; \mathcal{M}_{m-n}),$$
An $n$-cycle of the quotient space

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where $\sigma$ runs over all $n$-simplices of $V$, and
An $n$-cycle of the quotient space

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$$c_f = \sum_{\sigma} \omega_\sigma \sigma \in C_n(V; \mathcal{N}_{m-n}),$$

where $\sigma$ runs over all $n$-simplices of $V$, and $C_n(V; \mathcal{N}_{m-n})$ denotes the $n$-th chain group of $V$ with coefficients in $\mathcal{N}_{m-n}$. 

Proof of Theorem 3.1

Proof of Lemma 3.3

Proof of Lemma 3.3

A homology class of $W_f$

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where $\sigma$ runs over all $n$-simplices of $V$, and $C_n(V; \mathcal{M}_{m-n})$ denotes the $n$-th chain group of $V$ with coefficients in $\mathcal{M}_{m-n}$.

**Lemma 3.3** \( \partial c_f = 0 \), i.e. $c_f$ is an $n$-cycle.
Proof of Lemma 3.3.

Let $\tau$ be an arbitrary $(n-1)$-simplex of $V$, and let $\sigma_1, \sigma_2, \ldots, \sigma_r$ be the $n$-simplices of $V$ containing $\tau$ as a face. We have only to show

$$\sum_{j=1}^{r} \omega_{\sigma_j} = 0.$$  

(The coefficient of $\tau$ in $\partial C_f$.)
Then, $|s'|^{-1}(\alpha)$ is an $(m - n + 1)$-dim. compact manifold and

$$\partial(|s'|^{-1}(\alpha)) = |s'|^{-1}(b_{\bar{\sigma}_1}) \cup |s'|^{-1}(b_{\bar{\sigma}_2}) = \bigcup_{j=1}^{r} |\varphi|^{-1}(b_{\sigma_j}).$$

Therefore, we have

$$\sum_{j=1}^{r} \omega_{\sigma_j} = \sum_{j=1}^{r} \left[|\varphi|^{-1}(b_{\sigma_j})\right] = 0$$

in $\mathcal{N}_{m-n}$. \hfill \square
Thus, $c_f$ defines a homology class $\gamma_f \in H_n(W_f; \mathcal{M}_{m-n})$. 
Thus, $c_f$ defines a homology class $\gamma_f \in H_n(W_f; \mathcal{M}_{m-n})$.

Since $\dim W_f = n$, we have

$$\gamma_f \neq 0 \iff c_f \neq 0$$
Thus, $c_f$ defines a homology class $\gamma_f \in H_n(W_f; \mathcal{M}_{m-n})$.

Since $\dim W_f = n$, we have

$$\gamma_f \neq 0 \iff c_f \neq 0$$

Furthermore, $c_f \neq 0$ iff there exists a component of a regular fiber which is not null-cobordant.
Thus, $c_f$ defines a homology class $\gamma_f \in H_n(W_f; \mathcal{M}_{m-n})$.

Since $\dim W_f = n$, we have $\gamma_f \neq 0 \iff c_f \neq 0$.

Furthermore, $c_f \neq 0$ iff there exists a component of a regular fiber which is not null-cobordant.

Therefore, if such a regular fiber component exists, we have $H_n(W_f; \mathbb{Z}_2) \neq 0$, since $\mathcal{M}_{m-n} \cong \mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2$.

The case of an oriented map can be treated similarly. $\square$
(1) Let us consider a tree $T$. 

Example 1
(1) Let us consider a tree $T$.
Then, since $H_1(T) = 0$, there exists no Morse function $f_1 : M^5_1 \to \mathbb{R}$ whose quotient space is homeomorphic to $T$ and which has $CP^2$ as a component of a regular fiber.
(2) \exists Morse function \( f_2 : M_2^5 \to \mathbb{R} \) whose quotient space is:

\[
\begin{array}{c}
S^4 \\
\downarrow 2 \\
\downarrow 4 \\
\downarrow 1 \\
\downarrow 3 \\
\downarrow 5 \\
\hline
CP^2 \oplus CP^2 \\
CP^2 \\
\end{array}
\]

The integer at each vertex denotes the index of the corresponding critical point, and the 4-manifold attached to each edge denotes the corresponding regular fiber component.
(2) \exists \text{Morse function } f_2 : M_2^5 \rightarrow \mathbb{R} \text{ whose quotient space is:}

\[ \text{a diagram showing } \mathbb{C}P^2 \# \mathbb{C}P^2, \mathbb{C}P^2, \text{ and } S^4 \text{ attached by edges.} \]

The integer at each vertex denotes the index of the corresponding critical point, and the 4-manifold attached to each edge denotes the corresponding regular fiber component.

Note that \( H_1(W_{f2}; \mathbb{Z}) \cong H_1(W_{f2}; \Omega_4) \cong \mathbb{Z} \) is generated by \( \gamma_{f2} \).
Example 3

(3) \exists \text{Morse function } f_3 : M_3^5 \rightarrow \mathbb{R} \text{ whose quotient space is:}

\[
\begin{array}{c}
\text{0} \\
S^4 \\
\text{2} \\
S^2 \times S^2 \\
\text{3} \\
S^2 \times S^2 \\
\text{4} \\
S^2 \times S^2 \\
\text{1} \\
S^2 \times S^2 \\
\text{5} \\
\end{array}
\]

Note that \( W_{f_3} = W_{f_2} \), but \( f_3 = 0 \) in \( H_1(W_{f_3}; \mathbb{Z}) = \mathbb{Z} \), while \( f_2 = 0 \) in \( H_1(W_{f_2}; \mathbb{Z}) \).
Example 3

(3) Morse function \( f_3 : M^5_3 \to \mathbb{R} \) whose quotient space is:

\[
\begin{array}{cccc}
S^2 \times S^2 & S^2 \times S^2 & \vdots & S^4
\end{array}
\]

\[
\begin{array}{c}
S^4 \\
0 \\
2 \\
4 \\
1 \\
3 \\
5
\end{array}
\]

Note that \( W_{f_3} \cong W_{f_2} \), but \( \gamma_{f_3} = 0 \) in \( H_1(W_{f_3}; \mathbb{Z}) \cong \mathbb{Z} \), while \( \gamma_{f_2} \neq 0 \) in \( H_1(W_{f_2}; \mathbb{Z}) \).
Even if every component of every regular fiber is null-cobordant, the source manifold may not be null-cobordant.
Even if every component of every regular fiber is null-cobordant, the source manifold may not be null-cobordant. For example, consider a stable map $f : \mathbb{C}P^2 \to \mathbb{R}^3$. Every component of every regular fiber is diffeomorphic to $S^1$, which is null-cobordant. However, $\mathbb{C}P^2$ is not null-cobordant.
Even if every component of every regular fiber is null-cobordant, the source manifold may not be null-cobordant. For example, consider a stable map \( f : \mathbb{C}P^2 \to \mathbb{R}^3 \). Every component of every regular fiber is diffeomorphic to \( S^1 \), which is null-cobordant. However, \( \mathbb{C}P^2 \) is not null-cobordant.

In fact, for a stable map \( f : M^4 \to \mathbb{R}^3 \), the cobordism class of \( M^4 \) is determined by singular fibers.
By associating an “invariant” of a (regular or singular) fiber component corresponding to certain dimensional simplices of $W_f$, we may be able to define a homology class of $W_f$. 
By associating an “invariant” of a (regular or singular) fiber component corresponding to certain dimensional simplices of $W_f$, we may be able to define a homology class of $W_f$.

**Problem 3.4**

*Study such kind of homology classes and their relations to the geometry and topology of the manifolds and the map.*
§1. Introduction

§2. Triangulation of Stein Factorization

§3. Application

Cobordism of manifolds
Cobordism group
Cobordism groups $\Omega_m$ and $\Omega_\mathbb{R}_m$
Cobordism classes of regular fiber components
Corollary
Proof of Theorem 3.1
An $n$-cycle of the quotient space
Proof of Lemma 3.3
Proof of Lemma 3.3
A homology class of $W_f$
Example 1
Example 2
Example 3
Remark
Problem

Thank you!
Thank you!