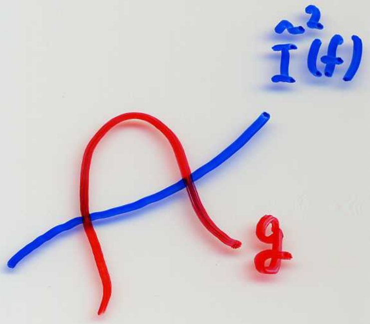
Idea for the Proof

$\forall x \in H_1(N;\mathbb{Z}_2)$

$\exists g : S^1 \cup \cdots \cup S^1 \to N$ C^∞ map

s.t. $g_*\underbrace{[S^1 \cup \cdots \cup S^1]_2}_{\text{fundamental class}} = x$

may assume $g \pitchfork f$



$$\rightsquigarrow \quad \begin{tikzcd} \exists \tilde{M}^2 \arrow[r, "\tilde{g}"] \arrow[d, "\text{stable} \to \exists \tilde{f}"'] \arrow[dr, phantom, "\circlearrowright"] & M^n \arrow[d, "f"] \\ S^1 \cup \cdots \cup S^1 \arrow[r, "g"] & N^{n-1} \end{tikzcd}$$

By applying the prop. to $\hat{f}$, we get

$$\langle \varphi_f \circ s_1^*(\alpha), x \rangle = \langle \varphi_{\tilde{f}} \circ s_1^*(\hat{\alpha}_2'), [S^1 \cup \cdots \cup S^1]_2 \rangle$$

$$= \langle \tilde{f}_! w_2(\tilde{M}^2), [S^1 \cup \cdots \cup S^1]_2 \rangle$$

$$= \langle g^*(f_! w_2(M) + (f_! w_1(M)) \cup w_1(N)), [S^1 \cup \cdots \cup S^1]_2 \rangle$$

$$= \langle f_! w_2(M) + (f_! w_1(M) \cup w_1(N)), x \rangle$$

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