Survey on knots associated with complex hypersurface singularities

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§1. An Example

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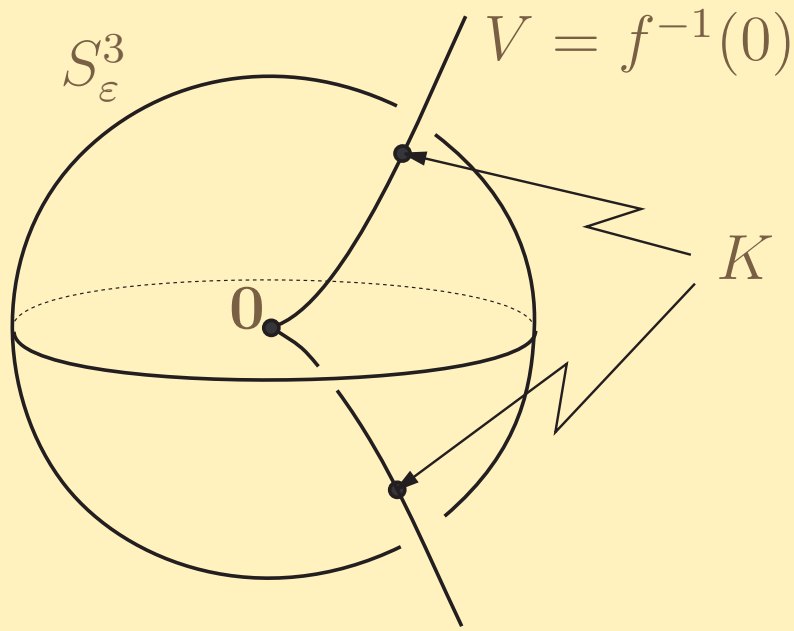
K is a knot in S^3 !



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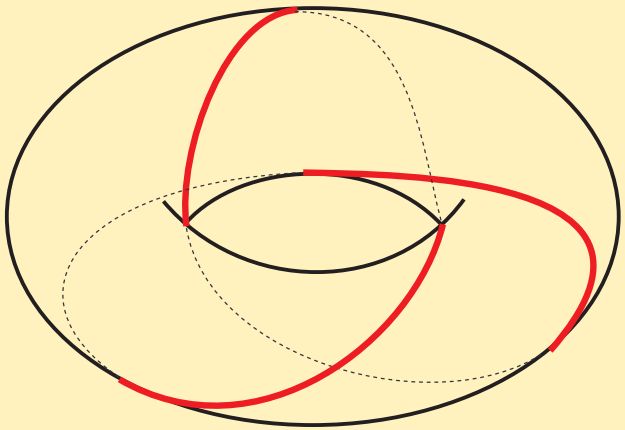

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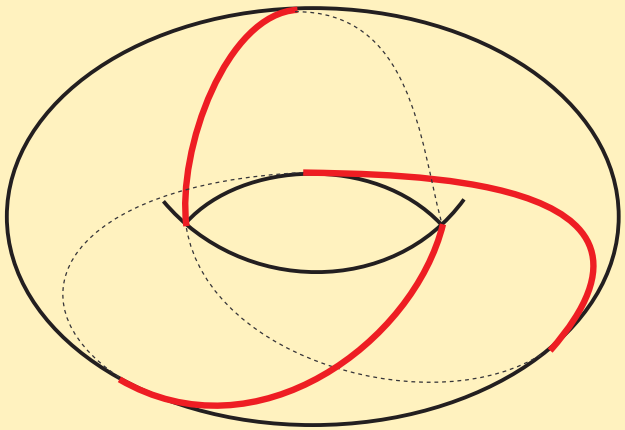

Trefoil knot is a fibered knot.

$f/|f| : S^3_\varepsilon \setminus K \to S^1 \subset \mathbf{C}$ locally trivial fibration

§2. General Results

Complex hypersurface

$f = f(z_1, z_2, \ldots, z_{n+1})$ complex polynomial with $f(\mathbf{0}) = 0$ s.t. $\mathbf{0}$ is an isolated critical point of f, i.e.,

$$\frac{\partial f}{\partial z_1}(z) = \cdots = \frac{\partial f}{\partial z_{n+1}}(z) = 0 \Longleftrightarrow z = \mathbf{0}$$

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In the following, we always assume that $\varepsilon > 0$ is sufficiently small.

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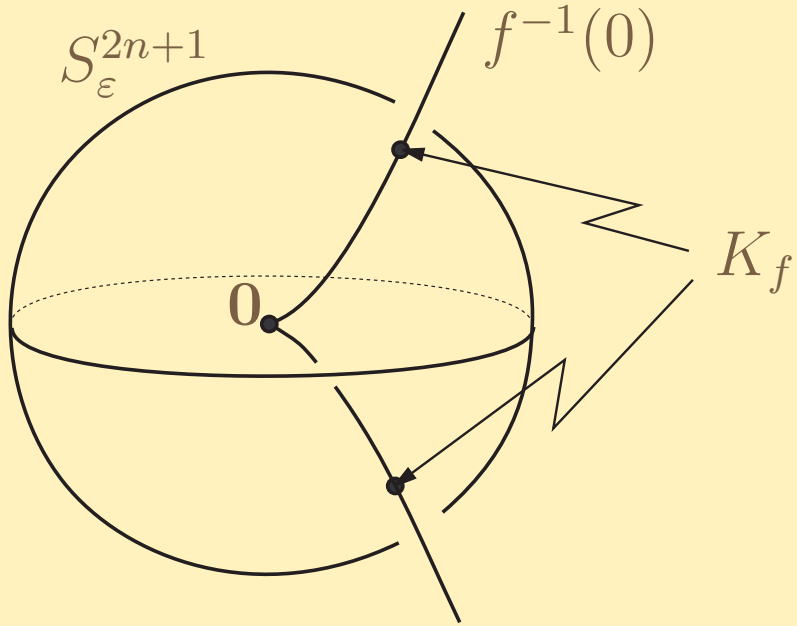

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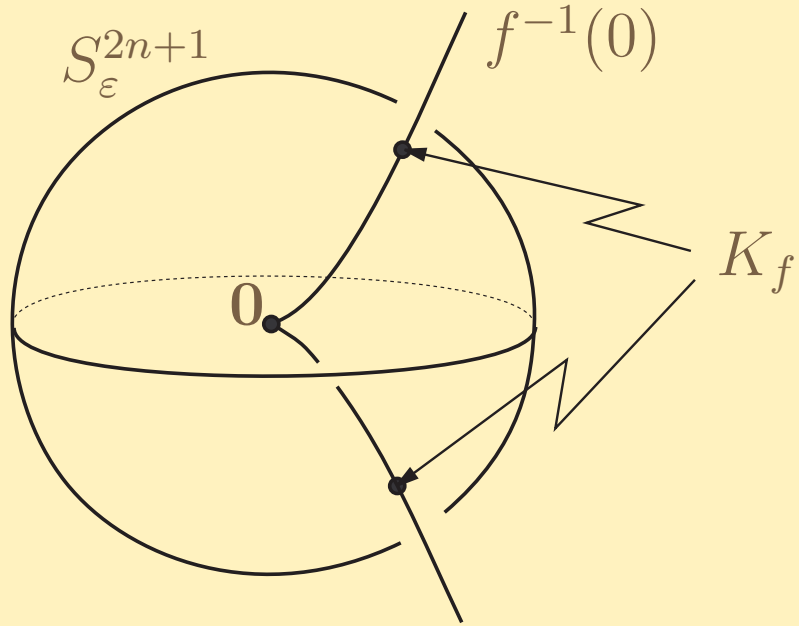

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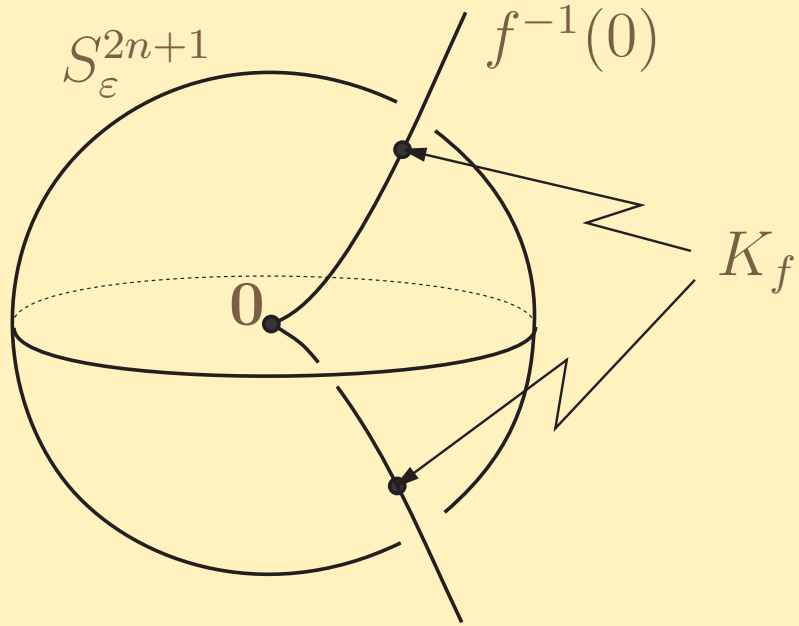

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Furthermore, $(S_\varepsilon^{2n+1}, K_f)$*, or the isotopy class of* K_f *in* S_ε^{2n+1}*, does not depend on* $0 < \varepsilon << 1$.

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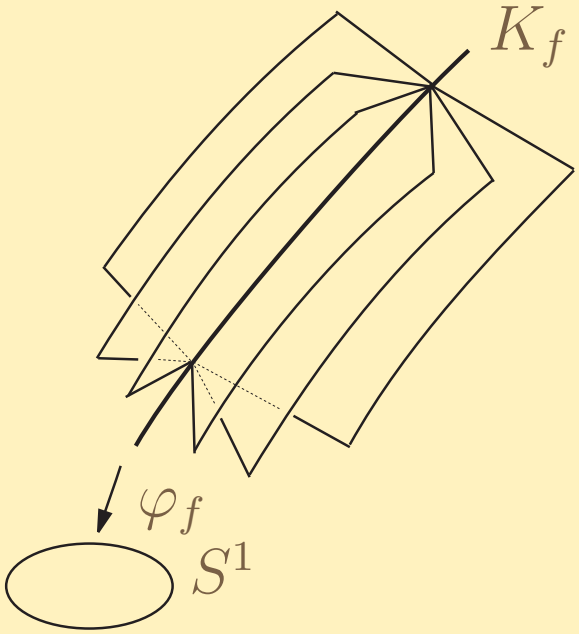

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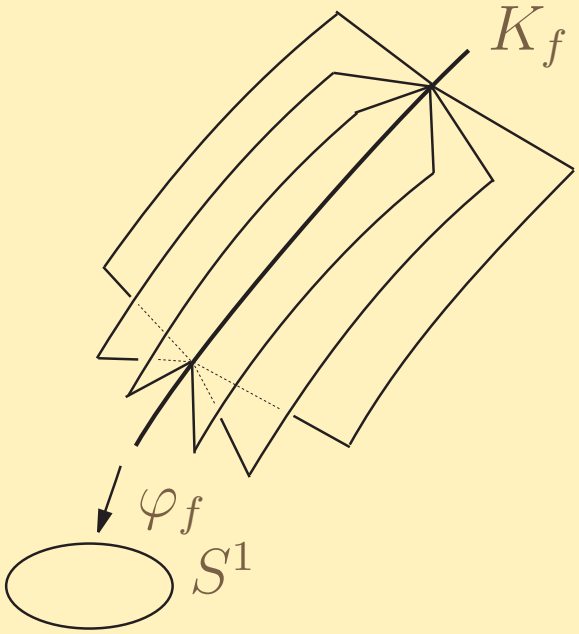

K_f is a fibered knot, but K_f may not be a sphere.

§3. Algebraic Knots

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$n = 1$: fibered link in S^3
$n = 2$: connected 3-manifold in S^5 with simply connected fibers
$n = 3$: simply connected 5-manifold in S^7 with 2-connected fibers
$\vdots$

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However, in higher dimensions, this is no longer true.

Theorem 3.2 (Michel–Weber, 1982 ($n \geq 3$); S, 1987 ($n = 2$))
There exist decomposable algebraic $(2n-1)$-knots.

General two-variable case

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We also have the following

Theorem 3.4 (Yamamoto, 1984)
Two algebraic links are isotopic iff they have the same multi-variable Alexander polynomials.

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Furthermore, K_f is always irreducible as a 3-manifold.
(But, as a 3-knot, it can be decomposable.)

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Theorem 3.6 (Brieskorn–Hirzebruch, 1966)
The $(4m-1)$-dimensional manifold K_f is homeomorphic to a sphere. Furthermore, they exhaust all the differentiable structures in bP_{4m}.

Topological sphere

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We denote by $\Delta_f(t)$ the characteristic polynomial of

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When $K_f \approx S^{2n-1}$ (homeo.), the diffeomorphism type of K_f is determined by
(1) the signature of F_f if n is even, and
(2) $\Delta_f(-1) \pmod 8$ if n is odd.

§4. Classification

Seifert form

The **Seifert form** associated with f is the bilinear form

$$L_f : H_n(F_f; \mathbf{Z}) \times H_n(F_f; \mathbf{Z}) \to \mathbf{Z} \quad \text{define by}$$

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- a and b are n-cycles representing $\alpha, \beta \in H_n(F_f; \mathbf{Z})$,
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- a and b are n-cycles representing $\alpha, \beta \in H_n(F_f; \mathbf{Z})$,
- a_+ is obtained by pushing a into the positive normal direction of $F_f \subset S_\varepsilon^{2n+1}$,
- lk is the linking number in S_ε^{2n+1}.

Theorem 4.1 (Durfee, Kato, 1974) *For $n \geq 3$, two algebraic knots K_f and K_g are isotopic $\iff$ the Seifert forms L_f and L_g are isomorphic.*

Simple fibered knots

A $(2n-1)$-dim. fibered knot K in S^{2n+1} is **simple** if
(1) K is $(n-2)$-connected, and (2) the fibers are $(n-1)$-connected.

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For $n = 1, 2$, the above theorem does not hold.

Theorem 4.3 (S, 1999) *For every $k \geq 2$, there exist simple fibered 3-knots $K_1, K_2, \ldots, K_k$ s.t.*
(1) *they are all diffeomorphic as abstract 3-manifolds,*
(2) *their Seifert forms are all isomorphic,*
(3) *K_i and K_j are not isotopic if $i \neq j$.*

$f_t(z_1, z_2, \ldots, z_{n+1})$, $t \in (-\delta, \delta)$.
A family of complex polynomials with isolated critical points at $\mathbf{0}$.

μ-constant deformation

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Theorem 4.4 (Lê–Ramanujam, 1976)
For $n \neq 2$, μ-constant deformation is topologically constant, i.e., K_{f_t} are all isotopic.

For $n = 2$, this is still an open problem.
(Mainly due to the failure of the h-cobordism theorem in low dimensions.)

Brieskorn–Pham type polynomial

For $a_1, a_2, \ldots, a_{n+1} \geq 2$, set

$$f(z_1, z_2, \ldots, z_{n+1}) = z_1^{a_1} + z_2^{a_2} + \cdots + z_{n+1}^{a_{n+1}},$$

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Seifert forms for algebraic knots associated with Brieskorn–Pham type polynomials are known.
In fact, we have the following.

Theorem 4.5 (Yoshinaga–Suzuki, 1978)
For two Brieskorn–Pham type polynomials f and g, the following three are equivanent.

(1) *K_f and K_g are isotopic.*
(2) *f and g have the same set of exponents.*
(3) $\Delta_f(t) = \Delta_g(t)$.

Open problem

However, in general, it is extremely difficult to calculate the Seifert form of a given algebraic knot.

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The following is still an important open problem.

Problem 4.6 *For a given f, compute the Seifert form of the associated algebraic knot K_f.*

Summary

- Complex polynomial $f(z_1, z_2, \ldots, z_{n+1})$ with an isolated critical point at $\mathbf{0}$ gives rise to a fibered knot $K_f \subset S^{2n+1}$, called the **algebraic knot** associated with f.
- For $n = 1$, K_f is an **iterated torus knot**, or a link whose components are iterated torus knots. They are easily classified.
- For $n \geq 3$, K_f is completely determined by its **Seifert form**, but its explicit calculation is unknown in general.
- For $n = 2$, the situation is difficult and it is much harder to understand **algebraic 3-knots** in S^5.
- **Exotic spheres** arise around singularities associated with simple polynomials, called **Brieskorn–Pham type polynomials**. They are sources of interesting examples of high dimensional knots.

Thank you!