PROBLEMS

OSAMU SAEKI

Problem 1. Let M and N be smooth manifolds and $f : M \to N$ a smooth map. Define the notion of the "most natural map" (or the "simplest map", or the "standard map", or anything similar) among the *generic* smooth maps in the homotopy class of f, and study such maps (existence, uniqueness, their topological properties, etc.).

Problem 2 ([13]). Describe the Euler class e of an oriented S^1-bundle in terms of the space $C^\infty(S^1, \mathbf{R}^2)$. Note that Kazarian [7] has obtained some results in terms of $C^\infty(S^1, \mathbf{R})$.

For example, for an oriented S^1-bundle E, if there exists a map $E \to \mathbf{R}^2$ that is an immersion of rotation number ± 1 on each fiber, then the S^1-bundle is necessarily trivial, i.e. $e(E) = 0$.

Problem 3 ([13]). Characterize those Morse functions $S^1 \to \mathbf{R}$ which can be lifted to an embedding into $\mathbf{R}^2$.

Problem 4. Does there exist a special generic map $f : M^n \to \mathbf{R}^2$ of a closed orientable n-dimensional manifold M^n into the plane that cannot be lifted to an immersion into $\mathbf{R}^{n+1}$?

Problem 5 ([8]). Let G be an arbitrary finite graph without loops or isolated vertices.

(1) Is there an *embedding* $\eta : M^2 \to \mathbf{R}^3$ of a closed orientable surface such that the Reeb graph of the associated height function is homeomorphic to G ?

(2) Is there an *embedding* $\eta : M^2 \to \mathbf{R}^3 \setminus \{0\}$ of a closed orientable surface such that the Reeb graph of the associated distance function from the origin is homeomorphic to G ?

Problem 6. It is known that every graph 3-manifold M^3 admits a simple stable map into $\mathbf{R}^2$ and that the singular set of such a map is a graph link in M^3 [9]. Characterize those graph links which appear as the singular set of a simple stable map.

Problem 7. Let M^3 be a graph 3-manifold. Determine the smallest number of singular set components for simple stable maps $M^3 \to \mathbf{R}^2$.

Problem 8. Let M^3 be a closed 3-manifold. Let $\mathcal{W}(M^3)$ be the set of all compact polyhedrons that appear as the quotient space W_f for a stable map $f : M^3 \to \mathbf{R}^2$. Does $\mathcal{W}(M_1^3) = \mathcal{W}(M_2^3)$ imply that M_1^3 is diffeomorphic to M_2^3 ?

Date: May 25, 2011.

2000 *Mathematics Subject Classification.* Primary 57R45; Secondary 57N10, 57N13, 57Q60, 57R40, 57R42, 58K30, 32S55.

Key words and phrases. Singularity theory.

The author has been supported in part by Grant-in-Aid for Scientific Research (No. 23654028), Japan Society for the Promotion of Science.

Problem 9 ([10]). For a smooth closed connected orientable 3-manifold M and a positive integer g, are the following two conditions equivalent?

(1) There exists a Morse function $f : M^3 \to \mathbf{R}$ such that the genus of every component of every regular fiber is at most g.
(2) M^3 is diffeomorphic to the connected sum of finitely many closed orientable 3-manifolds of Heegaard genus at most g.

It is known that they are equivalent for $g = 1$.

Problem 10. Let M^4 be a closed oriented 4-dimensional manifold. For a C^∞ stable map $f : M^4 \to \mathbf{R}^3$, it is known that the number of singular fibers of type III^8, counted with signs, coincides with the signature $\sigma(M^4)$ of M^4 (see [15]). Does there always exist a stable map $M^4 \to \mathbf{R}^3$ such that the number of singular fibers of type III^8 (counted without signs) coincides with $|\sigma(M^4)|$?

Problem 11. Let M^4 be a simply connected smooth closed 4-dimensional manifold. If M^4 admits a simple fold map into $\mathbf{R}^2$, then does it admit a special generic map into $\mathbf{R}^3$?

Problem 12. Let G be a finitely presentable group. Does there exist a closed orientable 4-dimensional manifold M^4 and a simple stable map $M^4 \to \mathbf{R}^3$ such that $\pi_1(M) \cong G$? Or does there exist a closed orientable 4-dimensional manifold M^4 and a stable map $f : M^4 \to \mathbf{R}^2$ such that every component of every regular fiber is diffeomorphic to S^2 ? (See [14].)

Problem 13. Let M_1^4 and M_2^4 be smooth 4-dimensional manifolds that are homeomorphic. If there exist proper special generic maps $f_1 : M_1^4 \to \mathbf{R}^3$ and $f_2 : M_2^4 \to \mathbf{R}^3$, then are M_1^4 and M_2^4 diffeomorphic? (This would mean that the differentiable structure on a topological 4-manifold that allows the existence of a proper special generic map into $\mathbf{R}^3$ is unique.) See [12].

Problem 14. Does every closed non-orientable 4-dimensional manifold admit a fold map into $\mathbf{R}^3$?

Problem 15. It is known that closed manifolds whose tangent bundles satisfy certain conditions admit fold maps for which all the fold indices appear [1, 5]. Study the existence of fold maps with restricted set of fold indices. The extremal case corresponds to that of special generic maps.

Problem 16 (Gay–Kirby [6]). Let M^n be a closed connected n-dimensional manifold ($n \geq 3$). It is known that every smooth map $M^n \to S^2$ is homotopic to an excellent map (i.e. a smooth map with only folds and cusps as its singularities) without definite folds [11]. If M^n is 1-connected, is every smooth map $M^n \to S^2$ homotopic to an excellent map without folds of index $0, 1, n-2, n-1$?

Problem 17. Characterize those surface links that appear as the singular set of a stable map $S^4 \to \mathbf{R}^3$.

Problem 18. Let C be a plane projective curve in $\mathbf{C}P^2$. Study the condition for C to be topologically equivalent to a plane projective curve defined by a polynomial of *real* coefficients.

Problem 19. Let

$$f(z) = \sum_{j=1}^{n+1} z_j^{a_j} \quad \text{and} \quad g(z) = \sum_{j=1}^{n+1} z_j^{b_j}$$

be Brieskorn–Pham type polynomials. It is known that if their associated algebraic knots are cobordant then their Seifert forms are Witt equivalent over $\mathbf{R}$. Furthermore, their Seifert forms are Witt equivalent over $\mathbf{R}$ if and only if

$$\prod_{j=1}^{n+1} \cot \frac{\pi \ell}{2a_j} = \prod_{j=1}^{n+1} \cot \frac{\pi \ell}{2b_j}$$

holds for all odd integer ℓ (see [3]). Does it imply that $a_i = b_j$ up to renumbering the indices?

Problem 20. Let $f(z)$ be a Brieskorn–Pham type polynomial as above. Describe the condition on the exponents a_j such that $H_{n-1}(K_f; \mathbf{Z})$ is torsion free, where $K_f = f^{-1}(0) \cap S^{2n+1}$ is the $(2n-1)$-dimensional closed manifold called the *link* of f. The condition for the vanishing of $H_{n-1}(K_f; \mathbf{Z})$ has been described in [4].

Problem 21. ([2]) Is the multiplicity of a complex holomorphic function germ at an isolated singular point a cobordism invariant of the associated algebraic knot? This is known to be true for the case of algebraic 1-knots.

References

[1] Y. Ando, *Existence theorems of fold-maps*, Japan. J. Math. (N.S.) **30** (2004), 29–73.

[2] V. Blanlœil and O. Saeki, *Cobordism of fibered knots and related topics*, in "Singularities in Geometry and Topology 2004", Advanced Stud. in Pure Math. **46** (2007), 1–47.

[3] V. Blanlœil and O. Saeki, *Cobordism of algebraic knots defined by Brieskorn polynomials*, arXiv:0903.4253v1 [math.GT].

[4] E. Brieskorn, *Beispiele zur Differentialtopologie von Singularitäten*, Invent. Math. **2** (1966), 1–14.

[5] J.M. Èliašberg, *Surgery of singularities of smooth mappings*, Math. USSR Izv. **6** (1972), 1302–1326.

[6] D.T. Gay and R. Kirby, *Indefinite Morse 2-functions; broken fibrations and generalizations*, arXiv:1102.0750v2 [math.GT].

[7] M.È. Kazarian, *The Chern-Euler number of circle bundle via singularity theory*, Math. Scand. **82** (1998), 207–236.

[8] Y. Masumoto and O. Saeki, *A smooth function on a manifold with given Reeb graph*, Kyushu J. Math. **65** (2011), 75–84.

[9] O. Saeki, *Simple stable maps of 3-manifolds into surfaces*, Topology **35** (1996), 671–698.

[10] O. Saeki, *Morse functions with sphere fibers*, Hiroshima Math. J. **36** (2006), 141–170.

[11] O. Saeki, *Elimination of definite fold*, Kyushu J. Math. **60** (2006), 363–382.

[12] O. Saeki, *Special generic maps on open 4-manifolds*, J. of Singularities **1** (2010), 1–12.

[13] O. Saeki, *Lifting special generic maps*, preprint.

[14] O. Saeki and K. Suzuoka, *Generic smooth maps with sphere fibers*, J. Math. Soc. Japan **57** (2005), 881–902.

[15] O. Saeki and T. Yamamoto, *Singular fibers of stable maps and signatures of 4-manifolds*, Geometry & Topology **10** (2006), 359–399.

Institute of Mathematics for Industry, Kyushu University, Motooka 744, Nishi-ku, Fukuoka 819-0395, Japan

E-mail address: saeki@imi.kyushu-u.ac.jp