SINGULAR FIBERS AND CHARACTERISTIC CLASSES

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Abstract. Let $f : M \to N$ be a proper generic map between smooth manifolds with $\dim N - \dim M = -1$. We explicitly calculate the cohomology class dual to the closure of the set of points in N over which lies a specific singular fiber in terms of characteristic classes of M and N.

1. Introduction

Let $f : M \to N$ be a smooth map between smooth manifolds. In this paper, for a point $y \in N$, the *fiber* over y means the map germ

$$f : (M, f^{-1}(y)) \to (N, y) \tag{1.1}$$

along the set $f^{-1}(y)$. When $y \in N$ is a regular value of f, we call it a *regular fiber*; otherwise, a *singular fiber*. Note that $f^{-1}(y)$ has positive dimension in general if the *codimension* $\ell = \dim N - \dim M$ of f is negative.

Instead of considering the map germ (1.1) along the whole inverse image, we can also consider the multi-germ

$$f : (M, S_y) \to (N, y) \tag{1.2}$$

along $S_y = f^{-1}(y) \cap S(f)$, where $S(f)$ is the set of singular points of f. Note that if f is proper and generic, then S_y is a finite set of points. For a given (contact) equivalence class $\underline{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_r)$ of multi-germs, let $\underline{\alpha}(f)$ be the closure of the set of points x_1 in M such that for some points $x_2, x_3, \ldots, x_r$ in $f^{-1}(f(x_1)) \subset M$ with $x_i \neq x_j$ for $1 \leq i < j \leq r$, the map germ $f : (M, \{x_1, x_2, \ldots, x_r\}) \to (N, f(x_1))$ is in the equivalence class $\underline{\alpha}$. Furthermore, set $\overline{\alpha}(f) = f(\underline{\alpha}(f))$. Then, according to Kazarian [3], for a certain family of equivalence classes $\underline{\alpha}^1, \underline{\alpha}^2, \ldots, \underline{\alpha}^s$, the union $\cup_j \underline{\alpha}^j(f)$ (or $\cup_j \overline{\alpha}^j(f)$) represents a $\mathbf{Z}_2$-homology class of closed support and its Poincaré dual is expressed as a polynomial of the characteristic classes of the forms $w_i(f^*TN - TM)$ and $f^*f_!w_I(f^*TN - TM)$ (resp. $f_!w_I(f^*TN - TM)$), where w_i is the i-th Stiefel-Whitney class, $w_I = w_1^{i_1} w_2^{i_2} \cdots w_k^{i_k}$ for a multi-index $I = (i_1, i_2, \ldots, i_k)$, and $f_!$ is the Gysin homomorphism in the cohomology. Furthermore, the polynomial expressions are universal in the sense that they do not depend on a particular proper generic map f. Note that the proof of this fact depends on the existence of a universal space, whose cohomology ring plays the key role.

In this paper, we consider the corresponding cohomology classes determined by the topological type of map germs of the form (1.1), instead of multi-germs as in (1.2). Recall that in [10] the first author has developed a theory of singular fibers of generic differentiable maps. In this paper, we first show that a similar universal expression in terms of characteristic classes exists for singular fibers as

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well, following Kazarian's argument. Furthermore, for some explicit singular fibers, we calculate such universal expressions.

The paper is organized as follows. In §2 we consider several equivalence relations for singular fibers. For the construction of a universal space for singular fibers, the stable $\mathcal{K}$-classification plays an important role. Since this equivalence relation *for singular fibers* has not been introduced or studied so far, we give several explicit definitions and clarify certain properties similar to the multi-germ case. In §3 we show the existence of a universal expression in terms of the characteristic classes, using Kazarian's universal space [3, 4] (see also [9]). The result itself may be implicit in Kazarian's works.

In §4, we first recall the Euler characteristic formula for Morse functions on surfaces in terms of singular fibers. Then, we generalize it to proper generic maps $f : M \to N$ of codimension -1. More precisely, we show that the closure of the set of points in N over which lies a fiber of "$\widetilde{\mathrm{I}}^2$-type" represents a $\mathbf{Z}_2$-homology class of closed support in N, and its Poincaré dual coincides with $f_! w_2(f^*TN - TM) + f_!(w_1(f^*TN - TM)^2)$ (Theorem 4.2).

Second, we recall the signature formula for stable maps of closed oriented 4-manifolds into $\mathbf{R}^3$ obtained in [11]. Then, we generalize it to proper generic *oriented* maps $f : M \to N$ of codimension -1. Recall that a smooth map between smooth manifolds of negative codimension is an oriented map if the regular parts of the fibers are consistently oriented (see also [1]). In this situation, we show that three times the Poincaré dual to the $\mathbf{Z}$-homology class represented by the closure of the "III^8-locus" coincides with $-f_! p_1(f^*TN - TM)$, where p_1 is the first Pontrjagin class (Theorem 4.6).

In §5, we pose a question concerning the cohomology class Poincaré dual to the closure of the set of singular points contained in specific singular fibers.

Throughout the paper, manifolds and maps are differentiable of class C^∞ unless otherwise indicated. For a topological space X, id_X denotes the identity map of X.

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2. Equivalences for singular fibers

For singular fibers of smooth maps between smooth manifolds, let us consider the following equivalence relations.

Definition 2.1. (1) Let $f_i : M_i \to N_i$ be smooth maps between smooth manifolds, $i = 0, 1$, with $\dim M_0 = \dim M_1$ and $\dim N_0 = \dim N_1$. For $y_i \in N_i$, $i = 0, 1$, we say that the fibers over y_0 and y_1 are *$\mathcal{K}$-equivalent*[1] if there exist diffeomorphism germs $s : (M_0, V_0) \to (M_1, V_1)$ and $H : (M_0 \times N_0, V_0 \times \{y_0\}) \to (M_1 \times N_1, V_1 \times \{y_1\})$ such that $H(x, y_0) = (s(x), y_1)$ and the following diagram is commutative:

$$\begin{array}{ccccc}
(M_0, V_0) & \xrightarrow{(\mathrm{id}_{M_0}, f_0)} & (M_0 \times N_0, V_0 \times \{y_0\}) & \xrightarrow{\pi_0} & (M_0, V_0) \\
{\scriptstyle s}\downarrow & & {\scriptstyle H}\downarrow & & {\scriptstyle s}\downarrow \\
(M_1, V_1) & \xrightarrow{(\mathrm{id}_{M_1}, f_1)} & (M_1 \times N_1, V_1 \times \{y_1\}) & \xrightarrow{\pi_1} & (M_1, V_1),
\end{array}$$

where $V_i = (f_i)^{-1}(y_i)$ and $\pi_i : M_i \times N_i \to M_i$ is the projection to the first factor, $i = 0, 1$.

[1] This definition is originally due to Toru Ohmoto.

(2) Let $f: M \to N$ be a smooth map between smooth manifolds. For $y \in N$, let us consider a smooth map germ

$$F: (M \times \mathbf{R}^k, V \times \{0\}) \to (N \times \mathbf{R}^k, \{y\} \times \{0\}),$$

$k > 0$, such that $F(x, 0) = f(x)$, where $V = f^{-1}(y)$. The fiber of F over $(y, 0)$ is called an *unfolding* of the fiber of f over y.

(3) Let $f_i : M_i \to N_i$ be smooth maps between smooth manifolds, $i = 0, 1$. We assume $\dim N_0 - \dim M_0 = \dim N_1 - \dim M_1$. For $y_i \in N_i$, we say that the fibers over y_0 and y_1 are *stably $\mathcal{K}$-equivalent* if they have $\mathcal{K}$-equivalent unfoldings.

(4) Let $f_i : M_i \to N_i$ be smooth maps between smooth manifolds, $i = 0, 1$, with $\dim M_0 = \dim M_1$ and $\dim N_0 = \dim N_1$. For $y_i \in N_i$, we say that the fibers over y_0 and y_1 are *C^∞ equivalent* if for some open neighborhoods U_i of y_i in N_i, there exist diffeomorphisms $\widetilde{\varphi} : (f_0)^{-1}(U_0) \to (f_1)^{-1}(U_1)$ and $\varphi : U_0 \to U_1$ with $\varphi(y_0) = y_1$ which make the following diagram commutative:

$$\begin{array}{ccc} ((f_0)^{-1}(U_0), V_0) & \xrightarrow{\widetilde{\varphi}} & ((f_1)^{-1}(U_1), V_1) \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f_1} \\ (U_0, y_0) & \xrightarrow{\varphi} & (U_1, y_1), \end{array}$$

where $V_i = (f_i)^{-1}(y_i)$, $i = 0, 1$. When the fibers over y_0 and y_1 are C^∞ equivalent, we also say that the map germs $f_0 : (M_0, V_0) \to (N_0, y_0)$ and $f_1 : (M_1, V_1) \to (N_1, y_1)$ are C^∞ *right-left equivalent*. Note that then they are $\mathcal{K}$-equivalent.

(5) Let $f : M \to N$ be a smooth map between smooth manifolds. The fiber over $y \in N$ of f is *stable* if for every unfolding

$$F : (M \times \mathbf{R}^k, V \times \{0\}) \to (N \times \mathbf{R}^k, \{y\} \times \{0\})$$

of the fiber over y of f with $V = f^{-1}(y)$, there exist diffeomorphism germs

$$h : (M \times \mathbf{R}^k, V \times \{0\}) \to (M \times \mathbf{R}^k, V \times \{0\}) \quad \text{and}$$
$$H : (N \times \mathbf{R}^k, \{y\} \times \{0\}) \to (N \times \mathbf{R}^k, \{y\} \times \{0\})$$

such that $h(x, 0) = (x, 0)$, $H(X, 0) = (X, 0)$ and the following diagram is commutative:

$$\begin{array}{ccccc} (M \times \mathbf{R}^k, V \times \{0\}) & \xrightarrow{F} & (N \times \mathbf{R}^k, \{y\} \times \{0\}) & \xrightarrow{\pi_2} & (\mathbf{R}^k, 0) \\ {\scriptstyle h}\downarrow & & {\scriptstyle H}\downarrow & & {\scriptstyle \mathrm{id}_{\mathbf{R}^k}}\downarrow \\ (M \times \mathbf{R}^k, V \times \{0\}) & \xrightarrow{f \times \mathrm{id}_{\mathbf{R}^k}} & (N \times \mathbf{R}^k, \{y\} \times \{0\}) & \xrightarrow{\pi_2} & (\mathbf{R}^k, 0), \end{array}$$

where $\pi_2 : N \times \mathbf{R}^k \to \mathbf{R}^k$ is the projection to the second factor.

Lemma 2.2. *Let $f_i : M_i \to N_i$ be proper smooth maps between smooth manifolds, $i = 0, 1$, with $\dim M_0 = \dim M_1$ and $\dim N_0 = \dim N_1$. We suppose that the fibers over y_0 and y_1 are stable. If they are stably $\mathcal{K}$-equivalent, then they are C^∞ equivalent.*

Proof. Since the fibers over y_i are stable and $V_i = (f_i)^{-1}(y_i)$ are compact, V_i contain only finitely many singular points. Let S_i denote the set of singular points contained in V_i. Then, the multi-germs

$$f_i : (M_i, S_i) \to (N_i, y_i) \tag{2.1}$$

are stable and are $\mathcal{K}$-equivalent. Then by [7], they are C^∞ right-left equivalent.

On the other hand, since f_i have $\mathcal{K}$-equivalent unfoldings, there exists a diffeomorphism $V_0 \to V_1$ in the sense of [10, Definition 1.1]. We may assume that this diffeomorphism is compatible with the source diffeomorphism which gives the C^∞ right-left equivalence between the multi-germs (2.1). Then, by using the relative version of Ehresmann's fibration theorem (for example, see [10, Chap. 1]), we can construct a diffeomorphism between neighborhoods of V_i which gives a C^∞ equivalence between the fibers over y_i (for this, imitate the argument in the proof of [10, Theorem 3.5]). □

3. Thom polynomials for singular fibers

Let $EO(n) \to BO(n)$ be the universal real n-plane bundle over the classifying space $BO(n)$. We denote by $MO(n)$ its Thom space. As Kazarian made an essential observation [3, 4], for each (possibly negative) integer ℓ, the iterated loop space $\mathbf{N}_\ell = \Omega^K MO(K+\ell)$, with K sufficiently large, serves as the classifying space of multi-singularities of smooth maps of codimension ℓ. Let us recall Kazarian's observation in the context of singular fibers.[2]

We may first replace $BO(K+\ell)$ by a suitable Grassmannian manifold, which is a smooth finite dimensional approximation. Furthermore, the elements of $\mathbf{N}_\ell$ may be assumed to be smooth in a neighborhood of the inverse image of the zero section $BO(K+\ell) \subset MO(K+\ell)$, as maps from S^K to $MO(K+\ell)$. We may further assume, up to infinite codimension, that for each point of S^K whose image lies in $BO(K+\ell)$, the map germ defined by the composition $S^K \to MO(K+\ell) \to \mathbf{R}^{K+\ell}$ is $\mathcal{K}$-finite at the point, where the second map is a projection to a fiber of $EO(K+\ell)$. Note that to each element of $\mathbf{N}_\ell$ is associated a well-defined stable $\mathcal{K}$-equivalence class of a fiber.

In the following, we fix a "G-classification" or a stratification of $\mathbf{N}_\ell$ in the sense of Kazarian [4] or Vassiliev [15], where G is an appropriate group acting on $\mathbf{N}_\ell$ or an equivalence relation on it. For example, when we consider the stable $\mathcal{K}$-equivalence, two elements of $\mathbf{N}_\ell$ are equivalent if the associated fibers are stably $\mathcal{K}$-equivalent, and the connected components of the equivalence classes form the strata. In general, we can also consider connected components of certain unions of stable $\mathcal{K}$-equivalence classes as strata.

Now, there exists a space $\mathbf{M}_\ell$ and a map $\mathbf{f}_\ell : \mathbf{M}_\ell \to \mathbf{N}_\ell$ such that every proper smooth map $f : M \to N$ of codimension ℓ is induced from $\mathbf{f}_\ell$: i.e. there are maps Ψ_f and Φ_f which make the following diagram commutative:

$$\begin{array}{ccc} M & \xrightarrow{\Psi_f} & \mathbf{M}_\ell \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathbf{f}_\ell} \\ N & \xrightarrow{\Phi_f} & \mathbf{N}_\ell. \end{array}$$

Furthermore, the fiber of f over a point $y \in N$ is stably $\mathcal{K}$-equivalent to that corresponding to $\Phi_f(y)$. The map Φ_f is called the *classifying map* of f. Moreover, for a given manifold N, two maps $f_i : M_i \to N$, with M_i being closed, $i = 0, 1$, are cobordant if and only if the corresponding maps $\Phi_{f_i} : N \to \mathbf{N}_\ell$ are homotopic. (See also Stong [12].)

We say that a proper smooth map $f : M \to N$ is *generic* if its classifying map $\Phi_f : N \to \mathbf{N}_\ell$ is transverse to the strata of $\mathbf{N}_\ell$. Note that the set of generic maps is residual in the space $C^\infty_{\mathrm{pr}}(M, N)$ of proper smooth maps[3] by virtue of the parametrized multi-transversality theorem.

All these arguments are valid also for oriented maps if we work with $MSO(K+\ell)$ instead of $MO(K+\ell)$.

[2] The authors are indebted to Toru Ohmoto [8] and Maxim Kazarian for many of the arguments in this section.

[3] A subset of a topological space is said to be *residual* if it is the intersection of a countable family of open and dense subsets. Since $C^\infty_{\mathrm{pr}}(M, N)$ is an open set of the Baire space $C^\infty(M, N)$, its residual subset is always dense.

In the following, for a multi-index $I = (i_1, i_2, \dots, i_k)$, we put $w_I = w_1^{i_1} w_2^{i_2} \cdots w_k^{i_k}$ and $p_I = p_1^{i_1} p_2^{i_2} \cdots p_k^{i_k}$, where w_i and p_j are the Stiefel-Whitney and the Pontrjagin classes, respectively. Let $f : M \to N$ be a smooth map between smooth manifolds of codimension ℓ. Let $w_i(f) = w_i(f^*TN - TM)$ be the i-th Stiefel-Whitney class of the formal difference bundle, and $s_I^w(f) = f_! w_I(f) \in H^*(N; \mathbf{Z}_2)$ the *Landweber-Novikov class* associated with the Stiefel-Whitney classes, where $f_! : H^*(M; \mathbf{Z}_2) \to H^{*+\ell}(N; \mathbf{Z}_2)$ is the Gysin homomorphism induced by f. Furthermore, when f is an oriented map, we similarly set $s_I^p(f) = f_! p_I(f) \in H^*(N; \mathbf{Z})$, which is the *Landweber-Novikov class* associated with the Pontrjagin classes, where $p_I(f)$ is the monomial of the Pontrjagin classes $p_j(f) = p_j(f^*TN - TM)$.

Now suppose that a proper smooth map $f : M \to N$ is generic so that N has a stratification induced from that of $\mathbf{N}_\ell$. Let $\varphi : N' \to N$ be a smooth map of a smooth manifold N' which is transverse to all the strata of N. Then, it induces the commutative diagram

$$\begin{array}{ccc} M' & \xrightarrow{\psi} & M \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ N' & \xrightarrow{\varphi} & N, \end{array}$$

where $M' = \{(x, y) \in N' \times M \, : \, \varphi(x) = f(y)\}$, and f' and ψ are the projections to the first and the second factors of $N' \times M$, respectively, restricted to M'. Note that M' is a smooth manifold and f' is a proper smooth map which is generic. In this case, we say that f' is the *pull back* of f by φ. Note that the classifying map $\Phi_{f'}$ coincides with the composition $\Phi_f \circ \varphi$.

Recall that the Landweber-Novikov classes are functorial with respect to the pull back operation: i.e. $s_I^w(f') = \varphi^* s_I^w(f)$ holds. Furthermore, if f is an oriented map, then so is f' and $s_I^p(f') = \varphi^* s_I^p(f)$ holds.

Definition 3.1. A *characteristic class* v of generic smooth maps means the following. For each generic smooth map $f : M \to N$, a cohomology class $v(f) \in H^*(N; \mathbf{Z}_2)$ is defined, and it satisfies the following *naturality condition*: if f' is the pull back of f by φ, then $v(f') = \varphi^*(v(f))$ holds.

A characteristic class for generic oriented smooth maps can similarly be defined for coefficients in $\mathbf{Z}$ or in $\mathbf{Q}$.

Remark 3.2. Suppose that $f : M \to N$ is a proper generic (oriented) map of codimension $\ell \leq 0$ and N is connected. Then, the $(-\ell)$-dimensional (oriented) cobordism class of a regular fiber of f is well-defined. Furthermore, if a map is a pull back of another map, then the (oriented) cobordism classes of their regular fibers coincide.

Proposition 3.3. *Every characteristic class of proper generic smooth maps with a prescribed cobordism class of regular fibers is a polynomial of Landweber-Novikov classes associated with the Stiefel-Whitney classes. Furthermore, every characteristic class with rational coefficients of proper generic oriented smooth maps with a prescribed oriented cobordism class of regular fibers is a polynomial of Landweber-Novikov classes associated with the rational Pontrjagin classes.*

Proof. We consider the case where the codimension ℓ is nonpositive. The case where $\ell > 0$ is similar.

First note that if $\ell \leq 0$, then the classifying space $\Omega^K MO(K + \ell) = \mathbf{N}_\ell$ may have two or more connected components. They are homotopy equivalent to each other, and each of them corresponds to a $(-\ell)$-dimensional cobordism class γ of regular fibers.[4] Let $\mathbf{N}^\gamma$ denote the corresponding component.

[4] The authors are indebted to Maxim Kazarian for this important observation.

By the naturality property, every characteristic class is the pull back of a cohomology class in $H^*(\mathbf{N}^\gamma; \mathbf{Z}_2)$. On the other hand, it is known that the cohomology of each component of $\Omega^K MO(K+\ell) = \mathbf{N}_\ell$, with K sufficiently large, is multiplicatively generated by the Landweber-Novikov classes (for example, see [3, 4, 13]). A similar argument is valid also for oriented maps and the connected components of $\Omega^K MSO(K+\ell)$ with rational coefficients. □

Definition 3.4. Let τ be a set of fibers of codimension ℓ. If for every fiber in τ, its nearby fibers also belong to τ, then τ is called *ascending* (see [10, §8.6]). A proper smooth map whose fibers all belong to τ is called a *τ-map*. Finally, τ is said to be *big* if it is ascending, the set of τ-maps is always residual in the space of proper smooth maps, and τ is closed under stable $\mathcal{K}$-equivalence (see [10, Remark 13.6]).

Let τ be a big set of fibers for proper generic smooth maps of codimension $\ell \leq 0$ as above. Let us consider the universal complex $\mathcal{C}^*(\tau, \varrho)$ of singular fibers associated with τ in the sense of [10], where ϱ represents an appropriate equivalence relation for fibers which is weaker than the stable $\mathcal{K}$-equivalence.

In view of Remark 3.2, the cochain complex $\mathcal{C}^*(\tau, \varrho)$ splits into the product

$$\prod_\gamma \mathcal{C}^*_\gamma(\tau, \varrho),$$

where $\mathcal{C}^*_\gamma(\tau, \varrho)$ is the subcomplex corresponding to those singular fibers whose nearby regular fibers belong to the $(-\ell)$-dimensional cobordism class γ. (Precisely speaking, for this splitting, the equivalence relation ϱ should be able to distinguish singular fibers corresponding to different cobordism classes of regular fibers.)

For a proper τ-map $f : M \to N$ of codimension ℓ with associated cobordism class γ of regular fibers, we have the homomorphism $\varphi_f : H^*(\mathcal{C}^*_\gamma(\tau, \varrho)) \to H^*(N; \mathbf{Z}_2)$, which assigns to each cohomology class α the Poincaré dual to the homology class represented by the closure of the set of points in the target over which lies a fiber appearing in a cycle representing α.

Then, by means of φ_f, each α defines a characteristic class for τ-maps, and hence for proper generic smooth maps, since τ is big. Therefore, we have the following.

Theorem 3.5. *Suppose that τ is a big set of fibers of codimension ℓ. For any cohomology class α of the universal complex $\mathcal{C}^*_\gamma(\tau, \varrho)$ of singular fibers whose associated regular fibers belong to the cobordism class γ, there exists a universal polynomial $P_\alpha(s^w_I)$ such that for every proper generic τ-map $f : M \to N$ of codimension ℓ with cobordism class γ of regular fibers, the cohomology class $\varphi_f(\alpha)$ coincides with $P_\alpha(f_! w_I(f^*TN - TM))$ in $H^*(N; \mathbf{Z}_2)$ (when M is closed, in $H^*_c(N; \mathbf{Z}_2)$).*

We call the polynomial P_α the *Thom polynomial* for α.

In order to discuss characteristic classes with coefficients in $\mathbf{Z}$ or in $\mathbf{Q}$, let us introduce the following definition.

Definition 3.6. A fiber of codimension ℓ is said to be *co-orientable* if the normal bundle of the corresponding strata in the classifying space $\mathbf{N}_\ell$ has an orientation consistent with the action of the group G or with the equivalence relation G. When an orientation is fixed, we say that the fiber is *co-oriented*.

Note that the above notion is closely related to the notion of a chiral singular fiber introduced in [11].

For co-orientable fibers, we have the following.

Theorem 3.7. *Suppose that τ is a big set of fibers of codimension ℓ. For any cohomology class α of the universal complex $\mathcal{CO}^*_\gamma(\tau, \varrho)$ of co-orientable singular fibers whose associated regular fibers belong to the oriented cobordism class γ, with* *coefficients in* $\mathbf{Q}$*, there exists a universal polynomial* $P_\alpha(s_I^p)$ *such that for every proper generic oriented* τ*-map* $f : M \to N$ *of codimension* ℓ *with oriented cobordism class* γ *of regular fibers, the cohomology class* $\varphi_f(\alpha)$ *coincides with* $P_\alpha(f_! p_I(f^*TN - TM))$ *in* $H^*(N;\mathbf{Q})$ *(when* M *is closed, in* $H_c^*(N;\mathbf{Q})$*).*

We again call the polynomial P_α the *Thom polynomial* for α.

Remark 3.8. In a sense, Theorems 3.5 and 3.7 may be implicit in the works of Kazarian [3, 4], although he does not mention them explicitly. Note also that the polynomial corresponding to $\cup_j \overline{\alpha}^j(f)$ mentioned in §1 coincides with the "sum" of our polynomials over all singular fibers which have $\underline{\alpha}^j$ as their associated multi-germ for some j. In other words, our polynomials are refinements of those considered by Kazarian [3] for multi-germs.

Remark 3.9. We do not know if Theorem 3.5 or 3.7 holds if τ is not big.

Remark 3.10. The above observations show that if τ is big, then there exists a homomorphism $H^*(\tau,\rho) \to H^*(\mathbf{N}_\ell;\mathbf{Z}_2)$ (in the co-oriented case, a homomorphism $H^*(\mathcal{CO}(\tau,\rho)) \to H^*(\mathbf{N}_\ell;\mathbf{Z}))$ which maps each cohomology class to its Thom polynomial. As observed in [2, 4], we have a natural filtration of $\mathbf{N}_\ell$ according to the codimensions of the strata, and this leads to a spectral sequence converging to $H^*(\mathbf{N}_\ell;\mathbf{Z}_2)$ whose first term $E_1^{*,0}$ corresponds to the universal complex. Then, the above homomorphism coincides with the composition

$$H^*(\tau,\rho) \cong E_2^{*,0} \to E_\infty^{*,0} \to H^*(\mathbf{N}_\ell;\mathbf{Z}_2).$$

(The same observation holds in the co-orientable case as well.)

When τ is not big, we have a homomorphism $H^*(\tau,\rho) \to H^*(T_\tau;\mathbf{Z}_2)$, where T_τ is a subspace of $\mathbf{N}_\ell$ corresponding to τ-maps. We do not know if this homomorphism can be lifted to $H^*(\mathbf{N}_\ell;\mathbf{Z}_2)$ in general.

The following proposition is useful in explicitly determining Thom polynomials.

Proposition 3.11. *Let* α *be a* κ*-dimensional cohomology class of the universal complex* $\mathcal{C}_\gamma^*(\tau,\varrho)$ *of singular fibers, where we assume that* τ *is big. Suppose that a polynomial* $P(s_I^w)$ *satisfies that for every* τ*-map* $f' : M' \to N'$ *whose regular fibers belong to the cobordism class* γ *with* $\dim N' = \kappa$ *and with* M' *and* N' *being closed, the cohomology class* $\varphi_{f'}(\alpha)$ *coincides with* $P(f'_! w_I((f')^*TN' - TM'))$ *in* $H^\kappa(N';\mathbf{Z}_2)$*. Then, the Thom polynomial for* α *is given by the polynomial* P*.*

Proof. Let $f : M \to N$ be a proper τ-map whose regular fibers belong to the cobordism class γ with $\dim M = n$ and $\dim N = p$. We may assume that f is generic. In order to show that the Thom polynomial for α coincides with P, we have only to show that

$$\langle \varphi_f(\alpha), \xi \rangle = \langle P(f_! w_I(f^*TN - TM)), \xi \rangle \in \mathbf{Z}_2$$

for all $\xi \in H_\kappa(N;\mathbf{Z}_2)$ by virtue of Poincaré duality.

By Thom's result [13], there exists a closed κ-dimensional manifold N' and a smooth map $h : N' \to N$ such that $\xi = h_*[N']$, where $[N'] \in H_\kappa(N';\mathbf{Z}_2)$ is the fundamental class of N'. We may assume that h is transverse to the strata of N. Let $f' : M' \to N'$ be the pull back of f with respect to h. Since f' is a generic τ-map with $\dim N' = \kappa$ and with M' being closed, we see that $\langle \varphi_{f'}(\alpha), [N'] \rangle = \langle P(f'_! w_I((f')^*TN' - TM')), [N'] \rangle$ holds by our assumption. Therefore, we have

$$\begin{aligned} & \langle \varphi_f(\alpha), \xi \rangle = \langle \varphi_f(\alpha), h_*[N'] \rangle = \langle h^*(\varphi_f(\alpha)), [N'] \rangle = \langle \varphi_{f'}(\alpha), [N'] \rangle \\ = \ & \langle P(f'_! w_I((f')^*TN' - TM')), [N'] \rangle = \langle h^* P(f_! w_I(f^*TN - TM)), [N'] \rangle \\ = \ & \langle P(f_! w_I(f^*TN - TM)), h_*[N'] \rangle = \langle P(f_! w_I(f^*TN - TM)), \xi \rangle, \end{aligned}$$

which completes the proof. □

Note that the above proposition holds also for co-oriented singular fibers, if we work with rational coefficients.

4. Explicit calculations

For stable maps between low dimensional manifolds of codimension -1, singular fibers have been classified up to C^∞ equivalence (see [10, 16] and [5, 6]). In the following, we will use the notation of [10]. Note that the 1-dimensional (oriented) cobordism group of manifolds is trivial, so that we do not have to worry about the cobordism classes of regular fibers.

Definition 4.1. Let $f_i : (M_i, (f_i)^{-1}(y_i)) \to (N_i, y_i)$ be proper smooth map germs along fibers with $n = \dim M_i$ and $p = \dim N_i$, $i = 0, 1$, with $n \geq p$. We may assume that N_i is the open disk $\mathrm{Int}\, D^p$ and y_i is its center 0, $i = 0, 1$. We say that the two fibers are C^∞ *equivalent modulo regular fibers* if there exist $(n-p)$-dimensional smooth closed manifolds T_i, $i = 0, 1$, such that the unions $f_i \cup \pi_i : (M_i \cup (T_i \times \mathrm{Int}\, D^p), (f_i)^{-1}(y_i) \cup (T_i \times \{0\})) \to (N_i, y_i)$ of f_i and the map germ $\pi_i : (T_i \times \mathrm{Int}\, D^p, T_i \times \{0\}) \to (\mathrm{Int}\, D^p, 0)$ defined by the projection to the second factor, $i = 0, 1$, are C^∞ equivalent to each other.

In the following, I^0, $\widetilde{\mathrm{I}}^2$, etc., which were originally introduced in [10], will denote the corresponding equivalence class of singular fibers with respect to the C^∞ equivalence modulo regular fibers. Furthermore, for a given map f, $\widetilde{\mathrm{I}}^2(f)$ etc. will denote the set of points in the target over which lies a singular fiber of the relevant type.

Let $f : M \to N$ be a C^∞ stable map of a closed surface M into a 1-dimensional manifold N. In [10, Corollary 2.4], we have seen that if $N = \mathbf{R}$, then the Euler characteristic $\chi(M)$ of M has the same parity as the number of singular fibers of f of type $\widetilde{\mathrm{I}}^2$. By a similar argument, we can show that the same congruence holds for C^∞ stable maps into any 1-dimensional manifold N. Recall that the parity of $\chi(M)$ is given by $\langle f_! w_2(f) + f_!(w_1(f)^2), [N] \rangle$, where $f_! w_2(f) + f_!(w_1(f)^2)$ is an element of the 1st $\mathbf{Z}_2$-cohomology group of N of compact support, and $[N]$ is the $\mathbf{Z}_2$-fundamental class of N of closed support. Combining these observations with Proposition 3.11, we get the following.

Theorem 4.2. *Let $f : M \to N$ be a proper generic smooth map of codimension -1. Then, the closure of $\widetilde{\mathrm{I}}^2(f)$ forms a $\mathbf{Z}_2$-cycle of closed support in N, and the Poincaré dual to its $\mathbf{Z}_2$-homology class coincides with $f_! w_2(f) + f_!(w_1(f)^2)$ in $H^1(N; \mathbf{Z}_2)$ (when M is closed, in $H^1_c(N; \mathbf{Z}_2)$).*

Remark 4.3. We have another simpler proof as follows. It is known that the Poincaré dual to the homology class represented by the singular point set of f coincides with $w_2(f) + w_1(f)^2$ (see [14]). Applying the Gysin homomorphism, we see that $f_!(w_2(f) + w_1(f)^2)$ coincides with the Poincaré dual to the homology class represented by the closure of $\widetilde{\mathrm{I}}^0(f) \cup \widetilde{\mathrm{I}}^1(f) \cup \widetilde{\mathrm{I}}^2(f)$. Let us consider the closure of the set of points in $N \smallsetminus f(S(f))$ over which lies an odd number of regular fiber components. Then, as observed in [10], its boundary coincides with the closure of $\widetilde{\mathrm{I}}^0(f) \cup \widetilde{\mathrm{I}}^1(f)$. Thus, the result follows.[5]

Let us now consider generic oriented maps of codimension -1. Let us first consider singular fibers of proper C^∞ stable maps of oriented 4-manifolds into 3-manifolds. In [11], we have seen that the singular fiber of type III^8 is co-oriented. In fact, we can define a sign ($= \pm 1$) for each such singular fiber by using the orientation of the source 4-manifold. For a C^∞ stable map f of a closed oriented

[5]The authors are indebted to the referee for this simple proof.

4-manifold into a 3-manifold, we call the sum of the signs over all singular fibers of f of type III^8 the *algebraic number of* III^8*-type fibers of* f. Note also that each point of $\mathrm{III}^8(f) \subset N$ has its own sign, and then $\mathrm{III}^8(f)$ can naturally be regarded as a 0-dimensional $\mathbf{Z}$-cycle of N. In [11], we have proved the following.

Theorem 4.4. *Let M be a closed oriented 4-manifold and N a 3-manifold. Then, for any C^∞ stable map $f : M \to N$, the algebraic number of III^8-type fibers of f coincides with the signature of M. In particular, if N is also oriented, then three times the Poincaré dual to the 0-dimensional $\mathbf{Z}$-homology class represented by $\mathrm{III}^8(f)$ coincides with $f_! p_1(M)$ in $H^3_c(N; \mathbf{Z})$.*

Remark 4.5. By using the universal complex of co-orientable singular fibers (see [11, §8]), we see that the fiber which satisfies the property as in Theorem 4.4 should necessarily be the fiber of type III^8.

Let $f : M \to N$ be a proper generic oriented map of codimension -1. Then, we see that the set $\mathrm{III}^8(f)$ is a codimension three regular submanifold of N. Furthermore, we can naturally orient its normal bundle in N by using the sign of a fiber in the 4-dimensional case, since f is an oriented map. Therefore, the closure of $\mathrm{III}^8(f)$ represents a homology class of closed support in $H^c_{n-4}(N; \mathcal{O}_N)$, where $\mathcal{O}_N$ denotes the orientation local system of N and $\dim N = n - 1$. Note that then its Poincaré dual lies in the usual cohomology group $H^3(N; \mathbf{Z})$.

Theorem 4.6. *Let $f : M \to N$ be a proper generic oriented smooth map of codimension -1. Then, the closure of $\mathrm{III}^8(f)$ forms a co-oriented cycle of closed support in N, and three times the Poincaré dual to its $\mathbf{Z}$-homology class coincides with $-f_! p_1(f)$ in $H^3(N; \mathbf{Z})$ (when M is closed, in $H^3_c(N; \mathbf{Z})$).*

Proof. By the rational coefficient version of Proposition 3.11, we see that three times the Poincaré dual to the homology class coincides with $-f_! p_1(f)$ modulo torsion.

On the other hand, since $\Omega^{K+1} MSO(K)$, with K sufficiently large, is 2-connected and its 3rd homotopy group is isomorphic to the 4-dimensional oriented cobordism group $\Omega_4 \cong \mathbf{Z}$, we see that $H^3(\mathbf{N}_{-1}; \mathbf{Z}) = H^3(\Omega^{K+1} MSO(K); \mathbf{Z})$ is isomorphic to $\mathbf{Z}$ by the Hurewicz theorem. Furthermore, three times a generator coincides with $f_! p_1$. Hence, the desired result follows. □

5. Related problem

Let us end this paper by posing a problem. For a generic map $f : M \to N$ as in Theorem 4.2, let us denote by $\widetilde{1}^2(f)$ ($\subset M$) the set of singular points *in the source* which are contained in a fiber of type $\widetilde{\mathrm{I}}^2$. In other words, $\widetilde{1}^2(f) = f^{-1}(\widetilde{\mathrm{I}}^2(f)) \cap S(f)$. Similarly, for a generic map $f : M \to N$ as in Theorem 4.6, we set $3^8(f) = f^{-1}(\mathrm{III}^8(f)) \cap S(f)$.

Problem 5.1. Let $f : M \to N$ be a proper generic smooth map of codimension -1. Then, does the closure of $\widetilde{1}^2(f)$ form a $\mathbf{Z}_2$-cycle of closed support in M? If yes, then describe the Poincaré dual to its $\mathbf{Z}_2$-homology class in terms of characteristic classes of M and N. Similarly, when f is oriented, does the closure of $3^8(f)$ form a co-oriented cycle of closed support in M? If yes, then describe the Poincaré dual to its homology class.

It is probable that we can construct a universal *source* complex of singular fibers as in [10] corresponding to the above types of strata in the source manifolds, taking into account the singular fibers in which the relevant points are contained.

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