Special Generic Maps on Open 4-Manifolds

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§1. Introduction

Special generic map

Definition 1.1 M^n, N^p: smooth manifolds $(n \geq p)$
A singularity of a smooth map $M^n \to N^p$ that has the normal form

$$(x_1, x_2, \ldots, x_n) \mapsto (x_1, x_2, \ldots, x_{p-1}, x_p^2 + x_{p+1}^2 + \cdots + x_n^2) \qquad (1)$$

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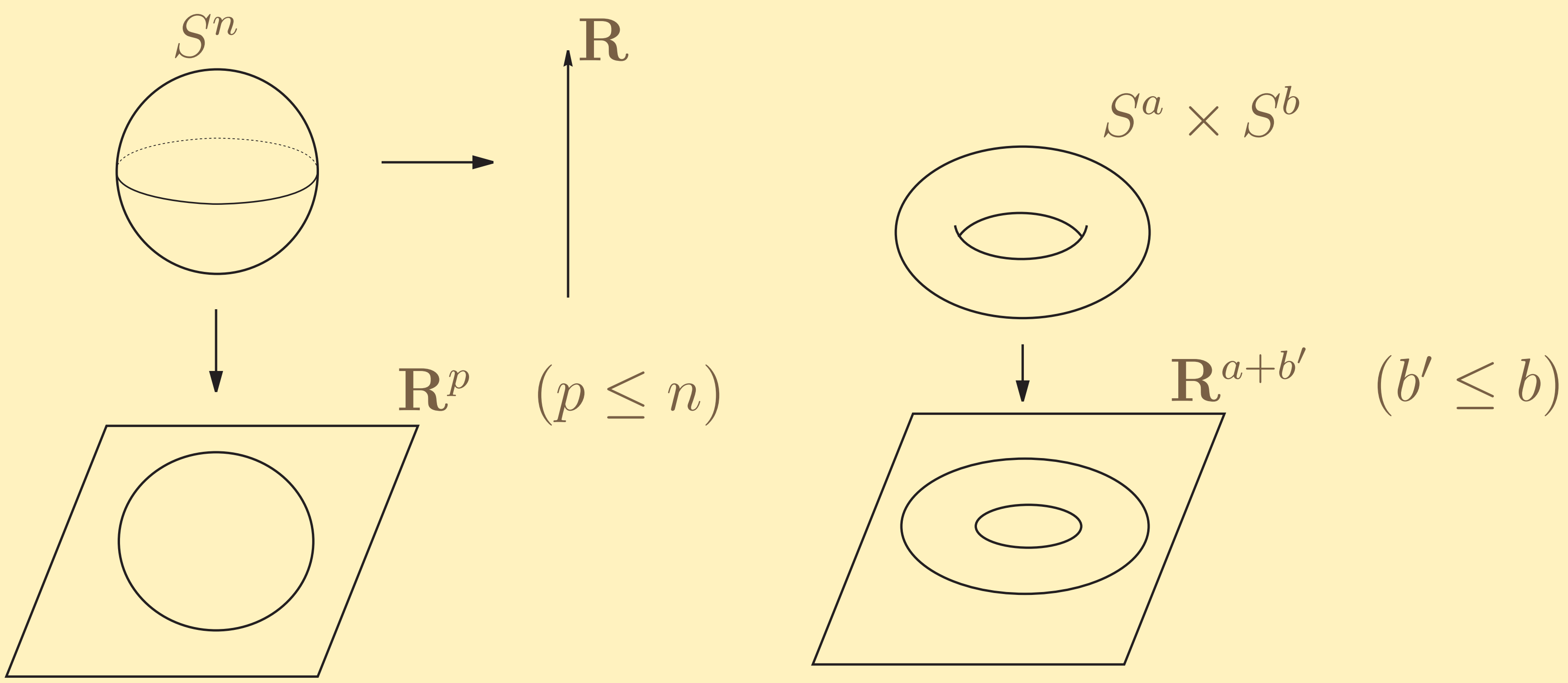

Figure 1: Example of special generic maps

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Theorem 1.4 (S, 1993)
$M^n \cong S^n$ *(diffeomorphic)*
$\Longleftrightarrow 1 \leq \forall p \leq n, \exists f : M^n \to \mathbf{R}^p$ *special generic map*

Theorem 1.5 (Sakuma-S, 1990's) $\exists (M_1^4, M_2^4)$ *such that*
$M_1^4 \approx M_2^4$ *(homeomorphic)*
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M^4: *closed* 1*-connected* 4*-manifold*
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4-Dimensional case

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Note. Usually an open 4-manifold admits *uncountably many* smooth structures.

§2. Main Results

Theorem 2.1

M^4: open 1-connected 4-manifold of "finite type"
$\exists f : M^4 \to N^3$ proper special generic map
for some 3-manifold N^3 with $S(f) \neq \emptyset$
$\Longleftrightarrow$ M^4 is diffeomorphic to the connected sum
of a finite number of copies of the following manifolds:
$\mathbf{R}^4(= S^4 \setminus \{\text{point}\})$, $\mathrm{Int}\left(\natural^k(S^2 \times D^2)\right) = S^4 \setminus (\vee^k S^1)$,
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Corollary 2.2 $M^4 \approx \mathbf{R}^4$ *(homeomorphic)*

$\exists f : M^4 \to \mathbf{R}^p$ *proper special generic map for* $1 \leq \exists p \leq 3$

$\Longleftrightarrow$ $M^4 \cong \mathbf{R}^4$ *(diffeomorphic)*

Theorem 2.3 L^3 : *closed orientable* 3*-manifold*
$M^4 \approx L^3 \times \mathbf{R}$ *(homeomorphic)*
$\exists f : M^4 \to \mathbf{R}^3$ *proper special generic map*
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Remark 2.4 "$\Longleftarrow$" is easy.
Consider $f = g \times \mathrm{id}_{\mathbf{R}} : L^3 \times \mathbf{R} \to \mathbf{R}^2 \times \mathbf{R}$.

§3. Ends of Open Manifolds

Definition 3.1 (Siebenmann, 1965) X: Hausdorff space
ε: collection of subsets of X such that

(i) Each $G \in \varepsilon$ is a connected open non-empty set with compact frontier $\overline{G} - G$,

(ii) $G, G' \in \varepsilon \Longrightarrow \exists G'' \in \varepsilon$ with $G'' \subset G \cap G'$,

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A maximal such collection ε is called an **end** of X.
A **neighborhood** of an end ε is any set $N \subset X$ that contains some member of ε.

Example

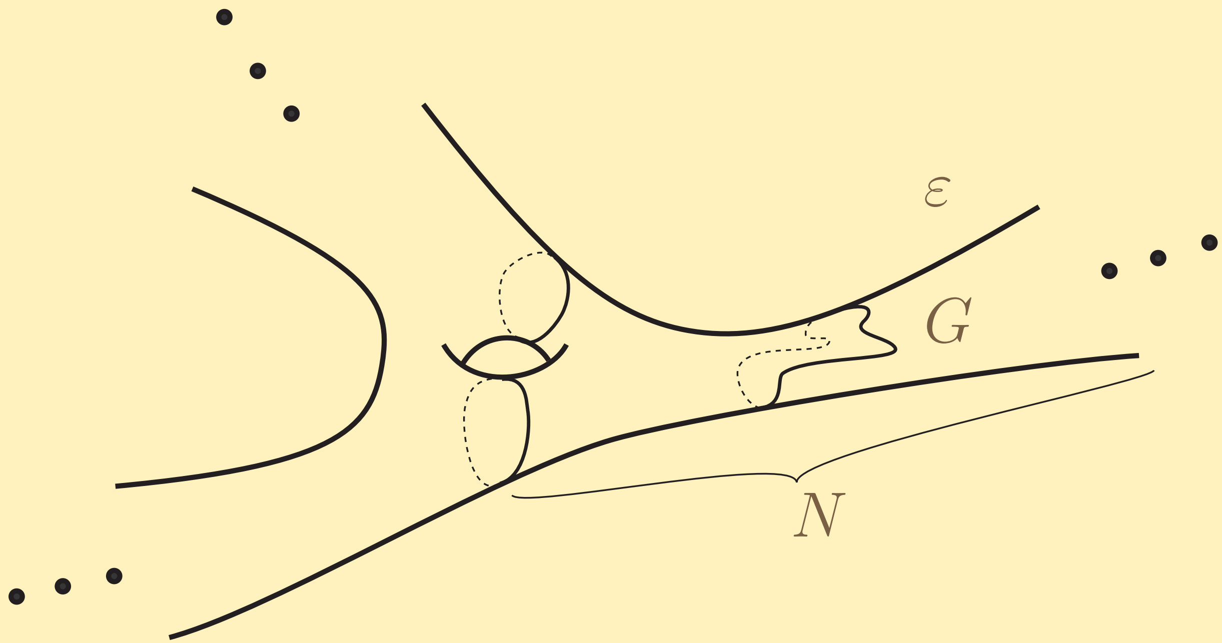

Figure 2: Ends of a manifold

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$\exists\, X_1 \supset X_2 \supset \cdots$ a sequence of path connected neighborhoods of ε such that $\bigcap \overline{X}_i = \emptyset$ and the sequence

$$\mathcal{G}: \qquad \pi_1(X_1) \xleftarrow{f_1} \pi_1(X_2) \xleftarrow{f_2} \cdots$$

induced by the inclusions induces isomorphisms

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According to Siebenmann, $\pi_1(\varepsilon)$ is well defined up to isomorphism.

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Lemma 3.5 (Husch–Price, 1970)
W^3: *open orientable* 3*-manifold of finite type*
$\Longrightarrow \exists \widetilde{W}^3$ *compact orientable* 3*-manifold and*
$\exists h : W^3 \to \widetilde{W}^3$ *embedding*
such that $h(\mathrm{Int}\, W^3) = \mathrm{Int}\, \widetilde{W}^3$.

§4. Stein Factorization

Definition 4.1 $f : M \to N$ smooth map
For $x, x' \in M$, define $x \sim_f x'$ if
(i) $f(x) = f(x')(= y)$, and
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$\exists! \bar{f} : W_f \to N$ that makes the diagram commutative:

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The above diagram is called the **Stein factorization** of f.

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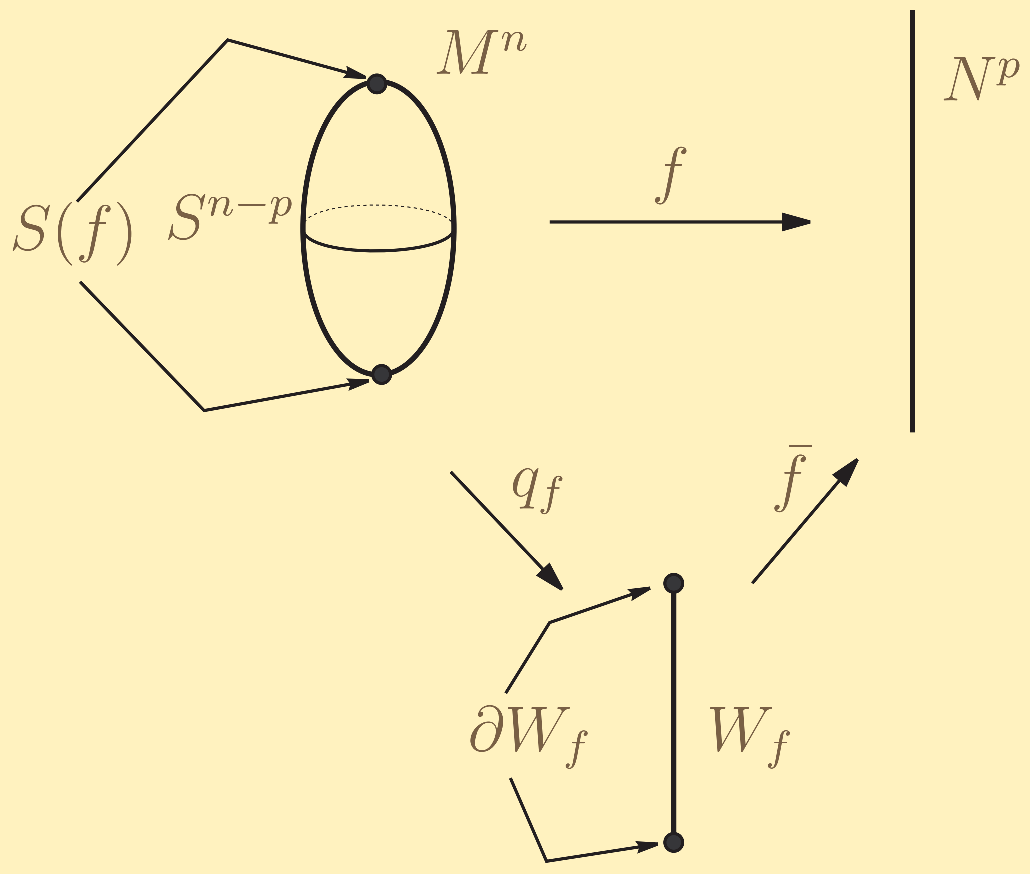

Figure 3: Stein factorization of a SGM

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Disk Bundle Theorem

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Theorem 4.2 (S, 1993)
$f : M^n \to N^p$ *proper special generic map with* $n - p = 1, 2, 3$
s.t. $S(f) \neq \emptyset$
$\Longrightarrow$
M^n *is diffeomorphic to the boundary of a* D^{n-p+1}*-bundle over* W_f.

§5. Proofs of Theorems

Let us prove the following.

Theorem 2.1:
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Then we easily get the desired conclusion.

Remark

Remark 5.1 Every 4-manifold as in Theorem 2.1 admits infinitely many (or uncountably many) distinct smooth structures.
Theorem 2.1 implies that among them there is exactly one structure that allows the existence of a proper special generic map into an orientable 3-manifold.

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Since $M^4 \approx L^3 \times \mathbf{R}$, we see $W_f \cong F_1 \times \mathbf{R}$.

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$\pi_1(L') \cong \pi_1(L^3)$ is free

Proof of Theorem 2.3:

$M^4 \approx L^3 \times \mathbf{R}$, $f : M^4 \to N^3$ proper special generic map

$\Longrightarrow$

W_f is of "finite type" and has exactly two ends $F_i \times [0, \infty)$, $i = 1, 2$

$F_i \times \{0\} \hookrightarrow W_f$ induce isomorphisms of fundamental groups

$W_f \cong (F_1 \times \mathbf{R})\sharp(\sharp^k D^3)$

Since $M^4 \approx L^3 \times \mathbf{R}$, we see $W_f \cong F_1 \times \mathbf{R}$.

$\Longrightarrow M^4 \cong L' \times \mathbf{R}$ for some 3-manifold L'

$\pi_1(L') \cong \pi_1(L^3)$ is free

$L' \cong L^3 \cong \sharp^\ell(S^1 \times S^2)$, and hence

$\exists g : L^3 \to \mathbf{R}^2$ special generic map (Burlet–de Rham, 1974)

Conjecture 5.2

M^4: *topological* 4*-manifold*
$\Longrightarrow$ *There exists at most one smooth structure on* M^4 *that allows the existence of a proper special generic map into* $\mathbf{R}^3$.

Thank you!