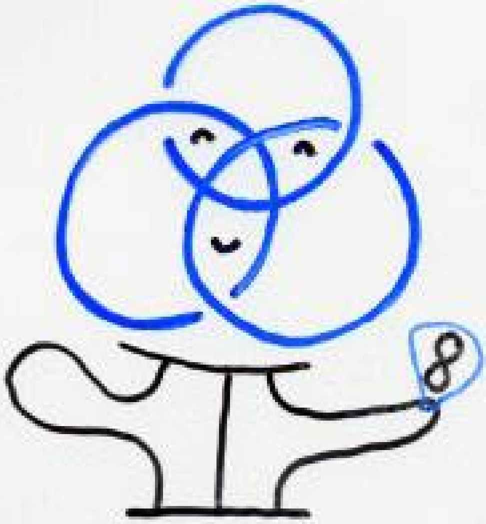
Introduction to Singular Fibers of Differentiable Maps



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§1. Introduction

- What does **singular fiber** refer to?

$f : M \to N$ $\quad C^\infty$ map

$\quad\quad\quad\;\; \rotatebox{90}{$\in$}$

$\quad\quad\quad\;\; y$: sing. value

sing. fiber over y means

the map germ

$f : (M, f^{-1}(y)) \to (N, y)$

dim > 0 if $\dim M > \dim N$

- What is the advantage of considering sing. fibers?

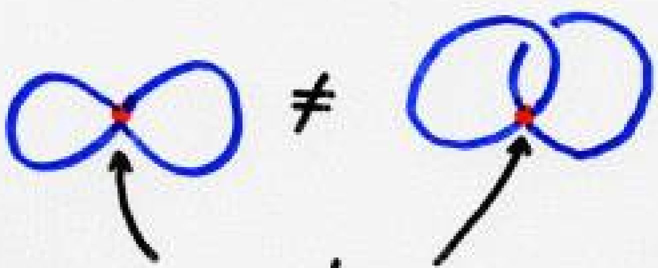
$f^{-1}(y) \supset \underbrace{S_y}_{\text{finite set}}$: set of sing. pts

multi-germ

$f : (M, S_y) \to (N, y)$

contains NO INFORMATION

on the topology of $f^{-1}(y)$



same multi-germ

(see Fig. 1)

abstract p.2

- What do sing. fibers serve for?

$$f : M \longrightarrow \underset{\text{TARGET}}{N} \ni y \rightsquigarrow f^{-1}(y)$$

Can serve for constructing certain <u>invariants</u> in the <u>target</u>.

$\Sigma(f) \subset M$: set of <u>sing.</u> of type Σ

$\mathcal{F}(f) \subset N$: set of pts in N over which lies a <u>sing. fiber</u> of type $\mathcal{F}$

$[\overline{\Sigma(f)}]^* \in H^*(M)$ <u>HOMOTOPY INV.</u>

$[\overline{\mathcal{F}(f)}]^* \in H^*(N)$ <u>COBORDISM INV.</u>

- Any applications ?

  Yes !

- Topological invariance of the number of certain sing. of a stable perturbation of a map germ

- Characteristic classes of surface bundles

§2. Classification

Def. $f_i : M_i \to N_i \ni y_i$ C^∞ map, $i=0,1$

$f_0^{-1}(y_0)$ and $f_1^{-1}(y_1)$ are C^∞ equiv.

(or C^0 equiv.)

$\Leftrightarrow$ $y_i \in {}^\exists U_i$ nbd in N_i

$$\begin{array}{ccc}
(f_0^{-1}(U_0), f_0^{-1}(y_0)) & \xrightarrow{\exists \tilde{\varphi}} & (f_1^{-1}(U_1), f_1^{-1}(y_1)) \\
f_0 \downarrow & \circlearrowleft & \downarrow f_1 \\
(U_0, y_0) & \xrightarrow{\exists \varphi} & (U_1, y_1)
\end{array}$$

$\tilde{\varphi}, \varphi$: C^∞ diffeo.

(resp. homeo.)

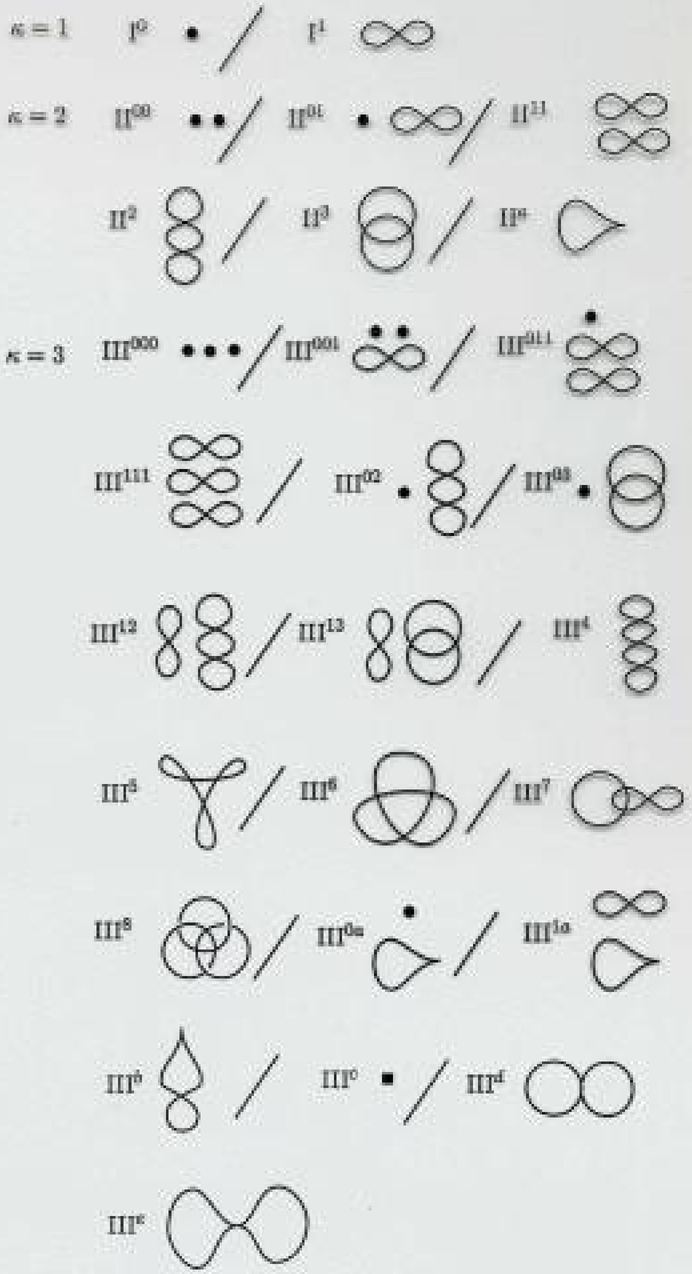


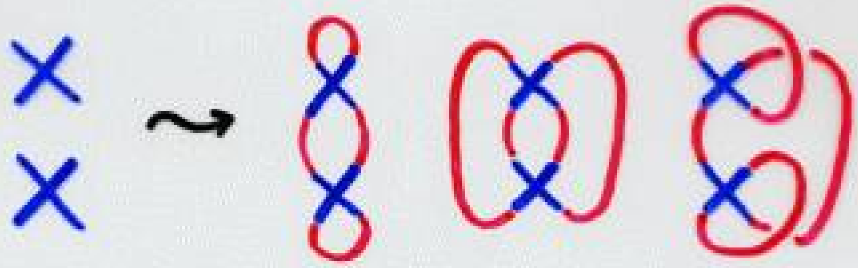
FIGURE 2. List of singular fibers of proper C^∞ stable maps of orientable 4-manifolds into 3-manifolds

How to carry out the classification?

(1) Classify the multi-germs

[Sing. Theory]

(2) List up all possible topological types of sing. fibers



etc.

[Combinatorial Argument]

(3) Same topol. type $\Rightarrow C^\infty$ equiv.

[Ehresmann Fibration Theorem]

Cor. For sing. fibers of proper C^∞-stable maps

$M^n \to N^{n-1}$ $(n = 2, 3, 4)$

orientable

C^∞ equiv. $\Longleftrightarrow$ C^0 equiv.

§3. Universal complex of singular fibers

Let us consider

τ : class of sing. fibers
ρ : equiv. relation for sing. fibers in τ

$\underline{\tau}$: certain set of sing. fibers of proper "Thom maps" closed under adjacency

$A \in \tau \Rightarrow B \in \tau$

ρ : weaker than C^0-equiv.

& consistent with adjacency

$\mathcal{F}$: equiv. class w.r.t. ρ

$f: M \to N$ proper Thom map

$$\mathcal{F}(f) = \{ y \in N \mid f^{-1}(y) \in \mathcal{F} \}$$

submfd of N of

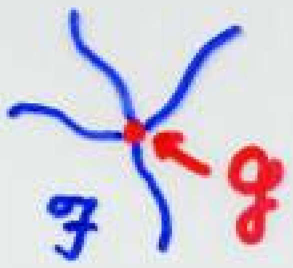
const. codimension $\kappa(\mathcal{F})$

$C^\kappa(\tau, \rho)$: $\mathbb{Z}_2$-vector space

spanned by equiv. classes

$\mathcal{F}$ with $\kappa(\mathcal{F}) = \kappa$

$\mathcal{F}, \mathcal{G}$ s.t. $\kappa(\mathcal{G}) = \kappa(\mathcal{F}) + 1$



TARGET

$[\mathcal{F} : \mathcal{G}] \in \mathbb{Z}_2$

incidence coeff.

$$\delta_\kappa : C^\kappa(\tau, \rho) \to C^{\kappa+1}(\tau, \rho)$$

$$\delta_\kappa(\mathcal{F}) = \sum_{\kappa(\mathcal{G}) = \kappa+1} [\mathcal{F} : \mathcal{G}]\, \mathcal{G}$$

Lemma $\delta_{\kappa+1} \circ \delta_\kappa = 0$

$$C^*(\tau, \rho) = \left(C^\kappa(\tau, \rho), \delta_\kappa \right)_\kappa$$

Universal Complex of Singular Fibers

(Analogy of Vassiliev's univ. cpx of singularities)

What is the geometric meaning of the cohomology $H^*(\tau, \rho)$?

Def. $f : M \to N$ proper Thom map

f is a τ-map

$\Leftrightarrow \quad f^{-1}(y) \in \tau \quad (\forall y \in N)$

Def. $c = \sum n_{\mathcal{F}} \mathcal{F} \in C^k(\tau, \rho)$

$$c(f) := \overline{\{ y \in N \mid f^{-1}(y) \in \mathcal{F} \text{ with } n_{\mathcal{F}} \neq 0 \}}$$

Lemma (1) c : cocycle $(\delta_k(c) = 0)$

$\Rightarrow c(f)$: cycle $(\dim = \dim N - k)$

(2) $c \sim c'$ cohomologous

$\Rightarrow c(f) \sim c(f')$ homologous

Def 3.4 $\alpha = [C] \in H^k(\tau, \rho)$

$f : M \to N^p$ τ-map

$\alpha(f) := [C(f)] \in H^c_{p-k}(N; \mathbb{Z}_2)$

$$\varphi_f : H^k(\tau, \rho) \to H^k(N; \mathbb{Z}_2)$$

$\alpha \in H^k(\tau,\rho) \longmapsto$ Poincaré dual of $\alpha(f)$ $\in H^k(N;\mathbb{Z}_2)$

homomorphism induced by f

Intuitively …

$f \rightsquigarrow$ Target N is stratified according to sing. fibers

$\rightsquigarrow$ find a cycle made up of certain strata

$\rightsquigarrow$ "intersection homo." defines an element of $H^*(N; \mathbb{Z}_2)$

§4. Cobordism Invariance

Def. 4.1 For $f : M \to N$, the map

$$\Sigma f = f \times \mathrm{id}_{\mathbb{R}} : M \times \mathbb{R} \to N \times \mathbb{R}$$

is called the suspension of f.

$(\Sigma f)^{-1}(y \times \{0\})$: suspension of $f^{-1}(y)$

$\tau(n,p)$: class of sing. fibers of proper Thom maps $M^n \to N^p$

Let us consider $C^*(\tau(n,p), \rho_{n,p})$ & $C^*(\tau(n+1,p+1), \rho_{n+1,p+1})$ which are consistent with suspension

(1) suspensim of any element of $\tau(n,p)$ belongs to $\tau(n+1,p+1)$

(2) equiv. w.r.t. $\rho_{n,p}$

$\Rightarrow$ suspensions are equiv. w.r.t. $\rho_{n+1,p+1}$

$$\Downarrow$$

$$\exists S_k^{\#} : C^k(\tau(n+1,p+1), \rho_{n+1,p+1}) \to C^k(\tau(n,p), \rho_{n,p})$$

<u>cohain map</u>

i.e. $\delta_k \circ S_k^{\#} = S_{k+1}^{\#} \circ \delta_k$

Def 4.4 $\tau = \tau(n,p) \cup \tau(n+1, p+1)$

$f_i : M_i^n \to N^p$ $(i=0,1)$: $\tau(n,p)$-maps

closed $\boxed{\text{TARGET is FIXED!}}$

f_0 & f_1 are τ-cobordant

$\Leftrightarrow$ $^\exists W^{n+1}$: cpt. s.t. $\partial W^{n+1} = M_0 \amalg M_1$

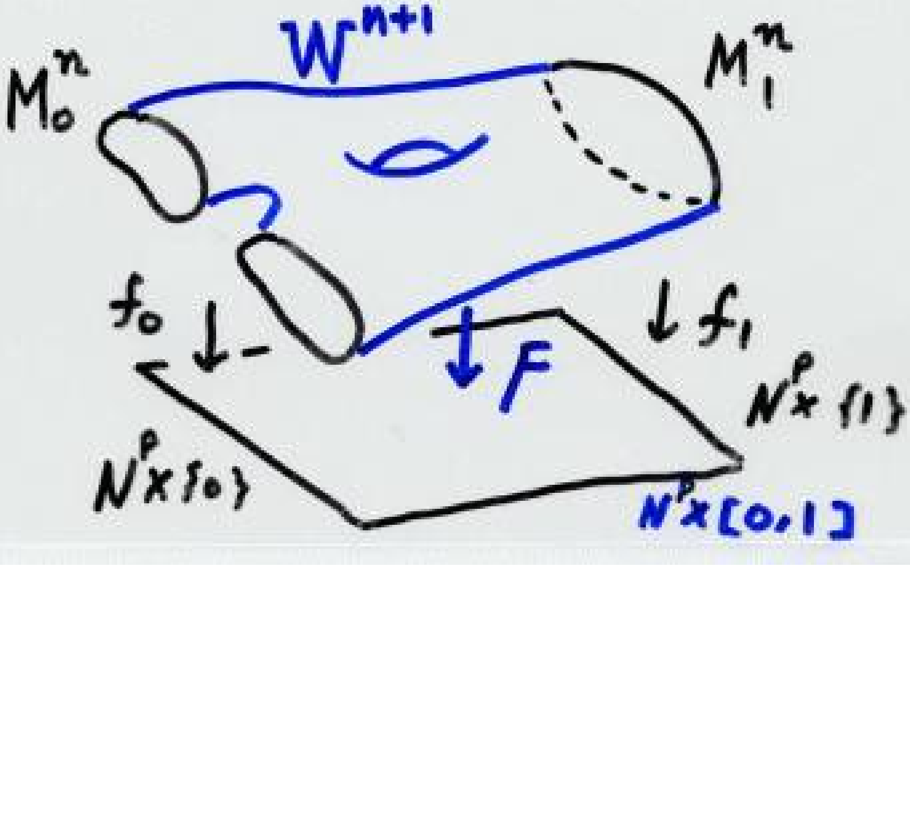
$^\exists F : W^{n+1} \longrightarrow N^p \times [0,1]$

$\tau(n+1, p+1)$-map

s.t. $F|_{\text{collar nbd of } M_i}$

$=$ suspension of $f_i : M_i \to N \times \{i\}$

$(i=0,1)$



Rem Notion of τ-cobordism was introduced by Rimányi-Szűcs.

$n \leq p \Rightarrow {}^{\exists}$universal τ-map

How about the case $n > p$?

Prop f_0 & f_1 are τ-cob.

$$\Rightarrow \varphi_{f_0} \circ S_k^* = \varphi_{f_1} \circ S_k^* :$$

$$H^k(\tau(n+1, p+1), \rho_{n+1,p+1}) \to H^k(N; \mathbb{Z}_2)$$

$\Downarrow$

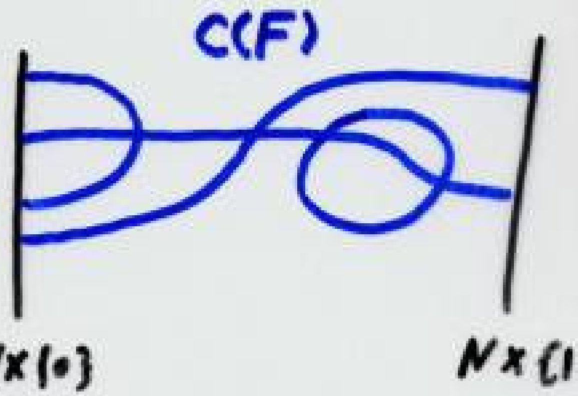
τ-cobordism invariants

(proof)

$$^\forall[c] \in H^k(\tau(n+1,p+1), \rho_{n+1,p+1})$$

$$\bar{c} := S_k^*(c) \in C^k(\tau(n,p), \rho_{n,p})$$

$$\Rightarrow \partial c(F) = \bar{c}(f_1)\times\{1\} - \bar{c}(f_0)\times\{0\}$$



c : cocycle

$$\Rightarrow [\bar{c}(f_0)] = [\bar{c}(f_1)] \text{ in } H^c_{p-k}(N;\mathbb{Z}_2)$$

$$\therefore \varphi_{f_0}(S_k^*[c]) = \varphi_{f_1}(S_k^*[c])$$

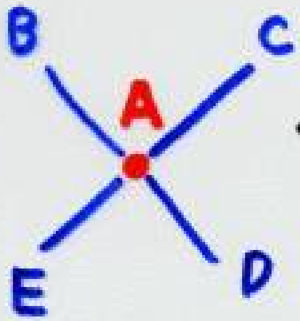
//

§5. Several Variants

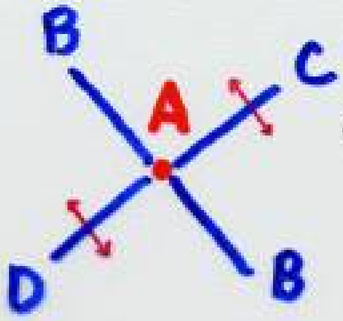
5.1 Co-orientable Sing. Fibers

Def $\mathcal{F}$: eq. class w.r.t. f, $\mathcal{F}$ is co-ori.

$\Leftrightarrow$ Normal bdle to $\mathcal{F}(f)$ is "canonically" oriented



← A is co-ori.



← A is NOT co-ori.

Incidence coeff. $[\mathcal{F} : \mathcal{G}] \in \mathbb{Z}$

$CO^*(\tau, \rho)$ univ. cpx of

co-ori. sing. fibers

defined over $\mathbb{Z}$

$\varphi_f : H^k(CO^*(\tau, \rho)) \to H^k(N; \mathbb{Z})$

τ-cobordism invariance

also holds

5.2 Chiral Sing. Fibers

Def. $\mathcal{F}$: C^0 equiv. class of a fiber of $f : M \to N \ni y$

(M orientable)

- $\mathcal{F}$ is **achiral**

$\Leftrightarrow$

$$
\begin{array}{ccc}
(f^{-1}(V), f^{-1}(y)) & \xrightarrow{\exists \tilde{\varphi}} & (f^{-1}(V), f^{-1}(y)) \\
f \downarrow & \circlearrowleft & \downarrow f \\
(V, y) & \xrightarrow{\exists \varphi} & (V, y)
\end{array}
$$

$\tilde{\varphi}, \varphi$: homeo.

s.t. $\left\{ \begin{array}{l} \tilde{\varphi} \text{ reverses the ori.} \\ \varphi|_{\mathcal{F}(f) \cap V} \text{ preserves the ori.} \end{array} \right.$

- $\mathcal{F}$ is **chiral**

$\Leftrightarrow$ NOT achiral

⇓

Univ. cpx of chiral sing. fibers defined over $\mathbb{Z}$

$f : M \to N$ oriented map

DEF $\Leftrightarrow$ fibers of $f|_{M-S(f)} : M - S(f) \to N$ are consistently oriented

sing. pt set of f

⇓

$$\varphi_f : H^k(\text{Univ. cpx}) \to H^k(N; \mathbb{Z})$$

Sensitive to orientation of the source manifold.

Oriented τ-cob. invariance holds.

5.3 Universal Homology Complex

$$C_*(\tau,\rho) = (C_k(\tau,\rho), \partial_k)_k$$

$$C_k(\tau,\rho) = \Big(\bigoplus_{\substack{\mathcal{F}: \\ \text{co-ori.}}} \mathbb{Z}\langle \mathcal{F}\rangle\Big) \oplus \Big(\bigoplus_{\substack{\mathcal{F} \\ \text{NOT co-ori.}}} \mathbb{Z}_2\langle \mathcal{F}\rangle\Big)$$

$$\partial_k : C_k(\tau,\rho) \to C_{k-1}(\tau,\rho)$$

$\delta_k : C_k(\tau,\rho) \to C_{k+1}(\tau,\rho)$ is NOT well-defined in general !

$$C^*(\tau,\rho) = \mathrm{Hom}(C_*(\tau,\rho), \mathbb{Z}_2)$$

$$CO^*(\tau,\rho) = \mathrm{Hom}(C_*(\tau,\rho), \mathbb{Z})$$

$C_*(\tau,\rho)$: NOT free

$\Downarrow$

use free approximations

$\Downarrow$

hyper-cohomology $\mathbb{H}^*$

$^{\exists}\tilde{\varphi}_f : \mathbb{H}^k(C_*(\tau,\rho);\overset{\forall}{G}) \to H^k(N;G)$

$^{\exists}$ cobordism invariance

Problem. $\exists ?$ cob. invariant obtained by $C_*(\tau,\rho)$ and not obtained by $C^*(\tau,\rho)$ or $CO^*(\tau,\rho)$? (Hidden Sing. Fiber)

(Analogy of Kazarian's construction)

§6. Example 1

Cobordism group of Morse functions on surfaces

$f_i : M_i^n \to \mathbb{R}$ $(i=0,1)$ Morse functions

closed

f_0 & f_1 are cobordant

DEF $\Leftrightarrow$ $\exists W^{n+1}$: cpt. mfd with $\partial W^{n+1} = M_0 \amalg M_1$

$\exists F : W^{n+1} \to \mathbb{R} \times [0,1]$ fold map

s.t. $F|$ collar nbd of M_i = susp. of f_i $(i=0,1)$

$\mathcal{M}(n)$: set of all cob. classes of Morse functions on n-dim. mfds

abelian grp w.r.t. disjoint union

$\mathcal{M}^{SO}(n)$: oriented version

(cf. Ikegami [3])

$\mathcal{T}^0(n, n-1)$: fibers of proper

C^∞ stable fold maps $M^n \to N^{n-1}$

(regular fiber = disj. union of circles)

$\rho^0_{n,n-1}(2)$: C^0 equiv. modulo

two circle components

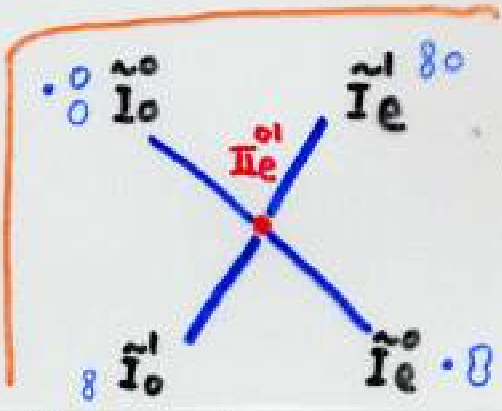
$8 \sim 800 \qquad 8 \not\sim 80$

List of co-ori. fibers for $n=3$

$k=0$ $\quad \bigcirc\ \tilde{0}_0 \qquad \bigcirc\bigcirc\ \tilde{0}_e$

$k=1$ $\quad \bullet\ \tilde{\mathrm{I}}^0_* \qquad 8\ \tilde{\mathrm{I}}^1_*$

$k=2$ $\quad \bullet 8\ \tilde{\mathrm{II}}^{01}_*$



Rem. No co-ori. fiber for $\rho_{3,2}(1)$

Lemma $H^k(CO^*(\tau^0(3,2), \rho^0_{3,2}(2)))$

$$\cong \begin{cases} \mathbb{Z} \ (\text{gen. by } [\hat{O}_0 + \hat{O}_e]) & k=0 \\ \mathbb{Z} \oplus \mathbb{Z} \ (\text{gen. by } \alpha_1 = -[\tilde{I}^0_0 + \tilde{I}^1_e] = [\tilde{I}^0_e + \tilde{I}^1_0], \\ \quad \alpha_2 = [-\tilde{I}^0_0 + \tilde{I}^0_e],\ \alpha_3 = [\tilde{I}^1_0 - \tilde{I}^1_e] \\ \quad \text{with } 2\alpha_1 = \alpha_2 + \alpha_3\ ) & k=1 \end{cases}$$

Lemma $f : M^2 \to \mathbb{R}$ stable Morse fct.
(M^2 closed)

$s_1^*\alpha_1(f) = 0$

$s_1^*\alpha_2(f) = -s_1^*\alpha_3(f) = \max(f) - \min(f)$

in $H_0(\mathbb{R}; \mathbb{Z}) \cong \mathbb{Z}$

Corollary

$\max(f) - \min(f)$ is a cobordism invariant of f

As a generator of $H^1(\tau^0(3,2), \rho^0_{3,2}(2))$

NON co-ori. version

we get

$\hat{\alpha}_3 = [\tilde{I}^2_0 + \tilde{I}^2_e]$  $\tilde{I}^2_*$

Corollary

$|\tilde{I}^2(f)| \in \mathbb{Z}_2$ is a cobordism invariant of f

Theorem

$$\begin{array}{ccc} \mathcal{M}(2) & \xrightarrow{\cong} & \mathbb{Z} \oplus \mathbb{Z}_2 \\ \psi & & \psi \\ [f] & \mapsto & (\max(f) - \min(f), |\tilde{I}^2(f)|) \end{array}$$

$$\left(\begin{array}{ccc} \mathcal{M}^{SO}(2) & \xrightarrow{\cong} & \mathbb{Z} \\ \psi & & \psi \\ [f] & \longmapsto & (\max(f) - \min(f)) \end{array} \right)$$

univ. cpx of sing. fibers $\rightsquigarrow$ COMPLETE cob. inv. !

§7. Application to Map Germs

Def. $g, g' : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ C^∞ map germs

topologically $\mathcal{A}$-equiv.

$\overset{\text{DEF}}{\Longleftrightarrow}$

$$\begin{array}{ccc} (\mathbb{R}^3,0) & \xrightarrow{\exists \Phi} & (\mathbb{R}^3,0) \\ g\downarrow & \circlearrowleft & \downarrow g' \\ (\mathbb{R}^2,0) & \xrightarrow{\exists \varphi} & (\mathbb{R}^2,0) \end{array}$$

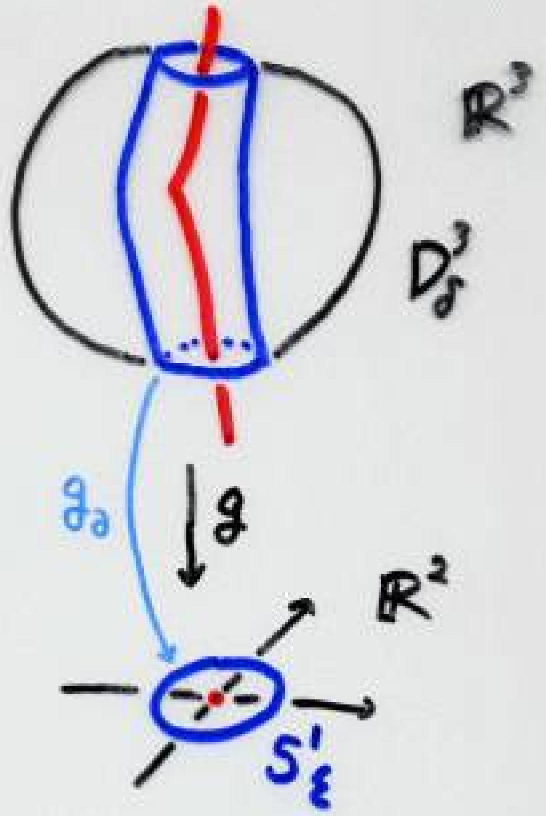
Φ, φ: homeo.

topologically $\mathcal{A}_+$-equiv. if φ preserves the ori. of $\mathbb{R}^2$

Def. $g : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ generic

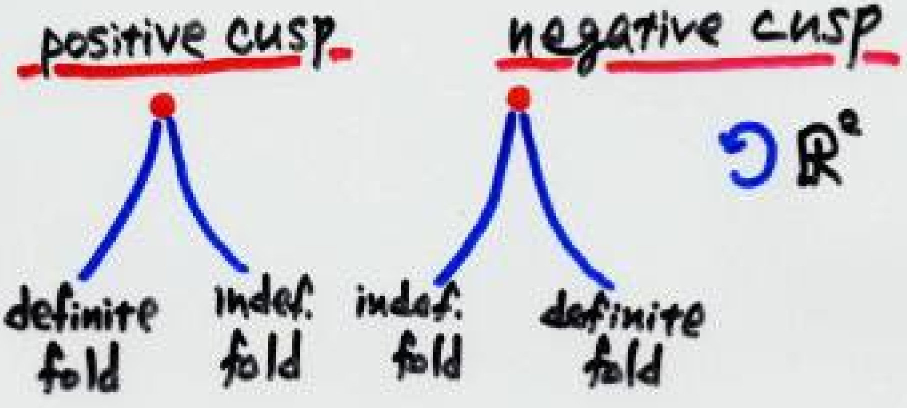
$\overset{\text{DEF}}{\Longleftrightarrow}$ for $0 < \varepsilon \ll \delta \ll 1$

$$\begin{cases} D^3_\delta \cap g^{-1}(S^1_\varepsilon) : C^\infty \text{ mfd with boundary} \\ g_\partial = g|_{D^3_\delta \cap g^{-1}(S^1_\varepsilon)} : D^3_\delta \cap g^{-1}(S^1_\varepsilon) \to S^1_\varepsilon : C^\infty \text{ stable} \\ g|_{\partial D^3_\delta \cap g^{-1}(D^2_\varepsilon)} : \partial D^3_\delta \cap g^{-1}(D^2_\varepsilon) \to D^2_\varepsilon : \text{submersion} \end{cases}$$



Rem Non generic map germs have "infinite codimension".

Def. $\tilde{g} : \mathbb{R}^3 \to \mathbb{R}^2$ stable



Theorem $g : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ generic
$\Rightarrow$ Algebraic number of cusps of a stable perturbation $\tilde{g}$ of g is an invariant of the topological $\mathcal{A}_+$-equiv. class of g.

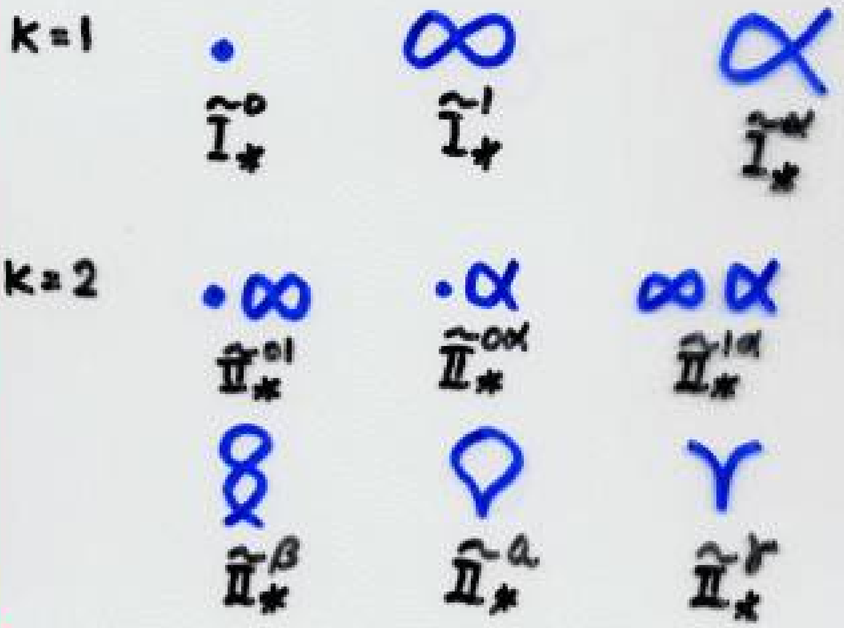
Idea for the Proof

(1) Express the # of cusps of $\tilde{g}$ in terms of the sing. fibers of g_∂

(2) Show the cob. invariance of the quantity obtained in (1)

List of $\underbrace{\text{sing.}}_{\text{co-ori.}}$ fibers for

$$\tilde{g}|_{D^3_\partial \cap g^{-1}(D^2_\varepsilon)} : D^3_\partial \cap g^{-1}(D^2_\varepsilon) \to D^2_\varepsilon$$



Lemma

(1) alg. # of cusps of $\tilde{g}$

$= -\|\tilde{I}^0_o(g_\partial)\| + \|\tilde{I}^0_e(g_\partial)\|$

$= \|\tilde{I}^\alpha_o(g_\partial)\| - \|\tilde{I}^\alpha_e(g_\partial)\| + \|\tilde{I}^1_o(g_\partial)\| - \|\tilde{I}^1_e(g_\partial)\|$

(2) the above integer is a (fold) cobordism invariant of g_∂

Problem

(1) Find an "algebraic" formula for the alg. # of cusps of $\tilde{g}$ in terms of g.

(2) Can we obtain a NEW topological inv. of a map germ by counting certain sing. fibers of a stable perturbation?

§8. Example 2

Euler characteristic formula

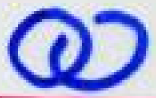
$f : M^2 \to \mathbb{R}$ stable Morse fct.

closed surface

$\chi(M^2) \equiv \#$ of critical pts of f (mod 2)

Euler chara.

$$= |\tilde{\mathrm{I}}^0_o(f)| + |\tilde{\mathrm{I}}^0_e(f)| + |\tilde{\mathrm{I}}^1_o(f)| + |\tilde{\mathrm{I}}^1_e(f)| + |\tilde{\mathrm{I}}^2_o(f)| + |\tilde{\mathrm{I}}^2_e(f)|$$

• $\tilde{\mathrm{I}}^0_*$ ∞ $\tilde{\mathrm{I}}^1_*$  $\tilde{\mathrm{I}}^2_*$

On the other hand,

$$\delta_o(\hat{\tilde{O}}_o) = \tilde{\mathrm{I}}^0_o + \tilde{\mathrm{I}}^0_e + \tilde{\mathrm{I}}^1_o + \tilde{\mathrm{I}}^1_e$$

$$\Rightarrow |\tilde{\mathrm{I}}^0_o(f)| + |\tilde{\mathrm{I}}^0_e(f)| + |\tilde{\mathrm{I}}^1_o(f)| + |\tilde{\mathrm{I}}^1_e(f)| \equiv 0 \pmod 2$$

Theorem $f: M^2 \to \mathbb{R}$ stable Morse fct

$\Rightarrow \chi(M^2) \equiv |\tilde{I}^2(f)| \pmod 2$

In fact, $\hat{\alpha}_2' = [\tilde{I}_0^2 + \tilde{I}_1^2]$ is a

generator of $H^1(\underbrace{\tau(3,2)}_{\text{fibers of stable maps}}, \mathcal{A}_{3,2}(2))$

$\Rightarrow$ $\mathfrak{N}_2$: 2-dim. unoriented cobordism grp

$$\begin{array}{ccccl} \Phi : & \mathfrak{N}_2 & \longrightarrow & \mathbb{Z}_2 & \\ & \Psi & & \Psi & \\ & [M] & \longmapsto & |\tilde{I}^2(f)| & \text{for } f: M \to \mathbb{R} \end{array}$$

well-defined homomorphism

On the other hand

$$\begin{array}{cccc} \Phi' : & \mathfrak{N}_2 & \xrightarrow{\cong} & \mathbb{Z}_2 \\ & \Psi & & \Psi \\ & [M] & \longmapsto & \chi(M) \bmod 2 \end{array}$$

We can check $\Phi(\mathbb{R}P^2) = \Phi'(\mathbb{R}P^2) = 1$.

$\Rightarrow \Phi = \Phi'$

Prop. $f : \underset{\uparrow}{M^2} \to \mathbb{R}$ stable Morse fct.
closed surface

For $\widehat{\alpha_2'} = [\tilde{\mathrm{I}}_0^2 + \tilde{\mathrm{I}}_e^2] \in H^1(\tau(3,2), \rho_{3,2}(2))$,

$$\varphi_f \circ s_1^*(\widehat{\alpha_2'}) = f_! w_2(M) \in H^1_c(\mathbb{R};\mathbb{Z}_2) = \mathbb{Z}_2$$

cohomology with cpt support

$\Big(w_2(M) \in H^2(M;\mathbb{Z}_2)$: 2nd Stiefel-Whitney class of M

$f_! : H^2(M;\mathbb{Z}_2) = H^2_c(M;\mathbb{Z}_2) \to H^1_c(\mathbb{R};\mathbb{Z}_2)$
Gysin homo. induced by f $\Big)$

Cor. $f : M^n \to N^{n-1}$ proper,
Thom-Boardman generic

For $\alpha = [\tilde{\mathrm{I}}_0^2 + \tilde{\mathrm{I}}_e^2] \in H^1(\tau(n+1,n), \rho_{n+1,n}(2))$,

$$\varphi_f \circ s_1^*(\alpha) = f_! w_2(M) + \underline{(f_! w_1(M)) \cup w_1(N)} \in H^1(N;\mathbb{Z}_2)$$

Please correct the abstract, Corollary 8.3, p.19.

Idea for the Proof

$^\forall x \in H_1(N;\mathbb{Z}_2)$

$\exists g : S^1 \cup \cdots \cup S^1 \to N \quad C^\infty$ map

s.t. $g_* \underbrace{[S^1 \cup \cdots \cup S^1]_2}_{\text{fundamental class}} = x$

may assume $g \pitchfork f$

$$\rightsquigarrow \quad \begin{tikzcd} {}^\exists \tilde{M}^2 \arrow[r, "\tilde{g}"] \arrow[d, "{}^\exists \tilde{f}"'] \arrow[dr, phantom, "\circlearrowleft"] & M^n \arrow[d, "f"] \\ S^1 \cup \cdots \cup S^1 \arrow[r, "g"] & N^{n-1} \end{tikzcd}$$

stable → $^\exists \tilde{f}$

By applying the prop. to $\hat{f}$, we get

$\langle \varphi_f \circ S_1^*(\alpha), x \rangle = \langle \varphi_{\tilde{f}} \circ S_1^*(\hat{\alpha}_2'), [S^1 \cup \cdots \cup S^1]_2 \rangle$

$= \langle \tilde{f}_! w_2(\tilde{M}^2), [S^1 \cup \cdots \cup S^1]_2 \rangle$

$= \langle g^*(f_! w_2(M) + (f_! w_1(M)) \cup w_1(N)),$
$[S^1 \cup \cdots \cup S^1]_2 \rangle$

$= \langle f_! w_2(M) + (f_! w_1(M) \cup w_1(N)), x \rangle$

//

§9. Example 3

Signature Formula

C^* : universal complex of CHIRAL sing. fibers for proper C^∞ stable maps $\underset{\text{ori.}}{M^5} \to N^4$

$$\begin{array}{ccccccc} C^1 & \to & C^2 & \to & C^3 & \xrightarrow{\delta_3} & C^4 \\ \| & & \| & & & & \\ 0 & & 0 & & \mathrm{rk}\,3 & & \mathrm{rk}\,14 \end{array}$$

[**Prop.** $H^3 \cong \mathbb{Z}$ generated by $[\mathrm{III}^8]$

III^8

Why is III^8-fiber chiral ?

$f : M^4 \to N^3 \ni y$

oriented

fix an ori. of N^3 around y

→ regular parts of fibers get ori.

→ cyclic order for three sing. pts

→

$\subset N^3$

compare the induced ori.
with the originally chosen ori.

same ori. → +1
different ori. → −1

sign

Lemma (1) Sign does not depend on the choice of local ori. of N^3 at y.

(2) If we change the ori. of M^4, then the sign changes.

Theorem (T. Yamamoto - S)

$f : M^4 \longrightarrow N^3$ C^∞ stable

closed oriented

$\Rightarrow \| \mathrm{III}^8(f) \| =$ signature of M^4 (*)

Idea for the Proof

- Both $\| \mathrm{III}^8(f) \|$ & sign. of M^4 are ori. cob. inv.
- ori. cob. grp $\Omega_4 \xrightarrow[\cong]{\text{sign.}} \mathbb{Z}$

$\Rightarrow$ suffices to check (*) for a generator. //

Corollary (T. Yamamoto - S)

$f : M^n \to N^{n-1}$ proper,

Thom-Boardman generic

f is an oriented map

(i.e. fibers of $f|M^n - S(f)$ are consistently oriented)

$$\Rightarrow \quad 3[\mathrm{III}^8(f)]^* = f_! \, P_1(M)$$

in $H^3(N;\mathbb{Z})$ mod. torsion

In the following, for $I = (i_1, \cdots, i_n)$,

$$w_I = w_1^{i_1} \cup \cdots \cup w_n^{i_n},$$

$$p_I = p_1^{i_1} \cup \cdots \cup p_n^{i_n}.$$

Conjecture

← Please correct the abstract Conjecture 9.4, p.20

(1) $\forall \alpha \in H^*(\tau(n+1, p+1), \mathcal{S}_{n+1, p+1})$

$\exists P_\alpha(\tilde{w}_I, w_j)$: univ. polynomial

s.t. $\varphi_f \circ S^*(\alpha) = P_\alpha(f_! w_I(M), w_j(N))$

in $H^*(N; \mathbb{Z}_2)$

for $\forall f : M^n \to N^p$ proper $\tau(n,p)$-map

(2) $\forall \alpha \in H^*$(univ. cpx of chiral sing. fibers)

$\exists P_\alpha(\hat{p}_I, p_j)$: univ. polynomial

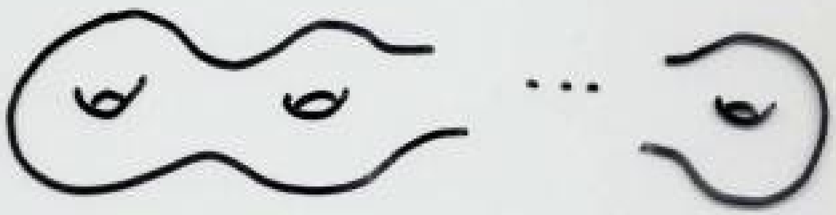
s.t. $\varphi_f \circ S^*(\alpha) = P_\alpha(f_! p_I(M), p_j(N))$

in $H^*(N; \mathbb{Z})$ (mod. torsion)

for $\forall f : M^n \to N^p$ proper oriented τ-map

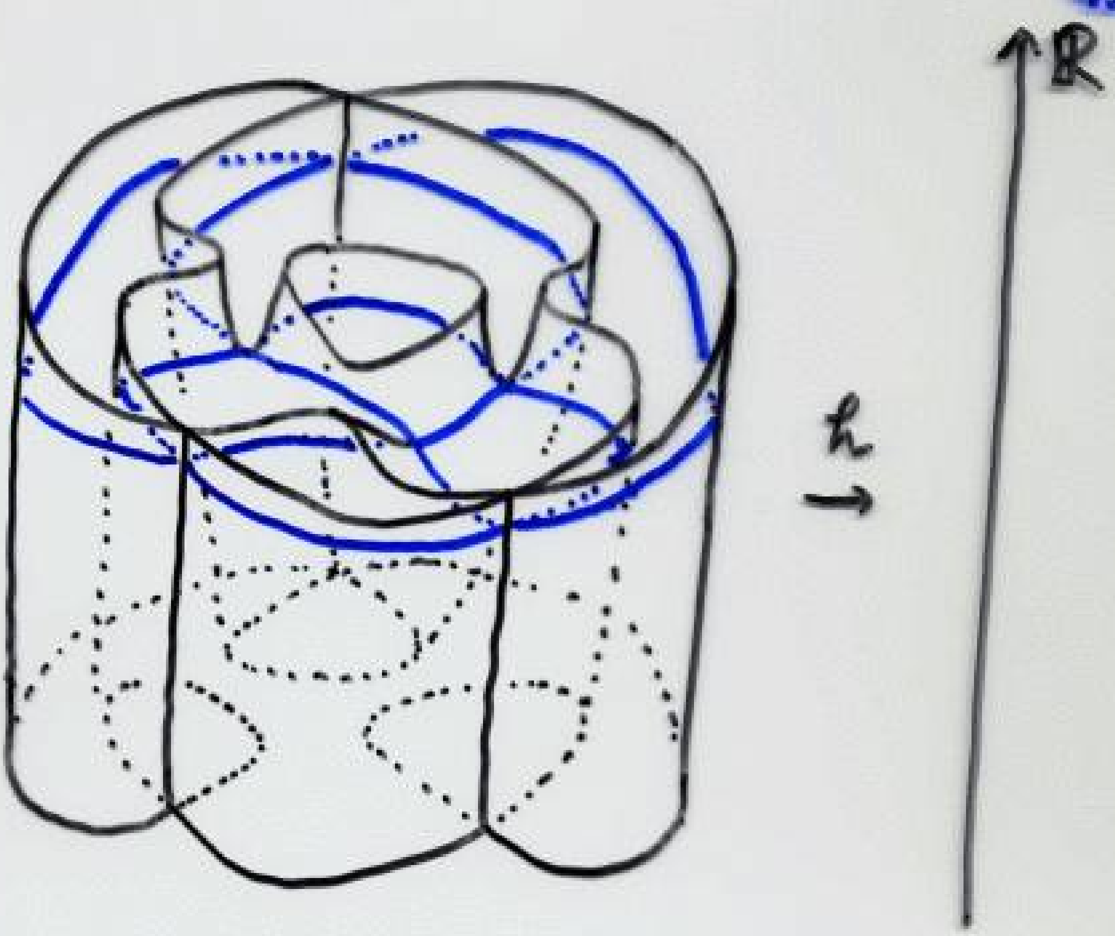
§10 Application to Surface Bundles

Σ_g : closed conn. ori. surface of genus $g \geq 0$

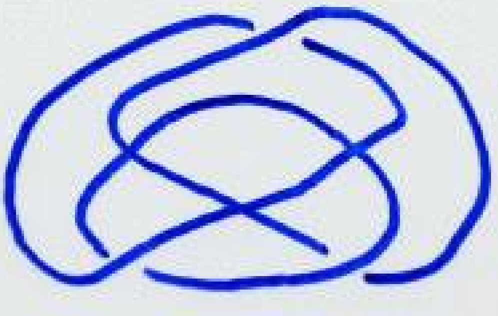


$$\Gamma := \{ h : \Sigma_g \to \mathbb{R} \ \text{Morse} \mid (*) \}$$

(*) h has exactly one sing. fiber of III^8-type and no other degenerate sing. fiber



(Fig. 7, p. 21)



III^8-type fiber

$\pi : E \to B$ C^∞ Σ_g-bundle

C^∞ mfds   oriented

$f : E \to \mathbb{R}$ "generic" fct

$y \in B$

$f_y = f|_{\pi^{-1}(y)} : \pi^{-1}(y) \cong \Sigma_g \to \mathbb{R}$

$\Gamma(f) := \{ y \in B \mid f_y \in \Gamma \}$

- $\Gamma(f)$ is a codim. 2 submfd of B if f is generic enough
- $\Gamma(f)$ is co-oriented if π is oriented.

Theorem (T. Yamamoto - S)

$\overline{\Gamma(f)}$ forms a codim. 2 cycle of closed support in twisted coefficients in B, and

$$[\overline{\Gamma(f)}]^* \in H^2(B; \mathbb{Z})$$

coincides with the first Miller - Morita - Mumford class $e_1(\pi)$ of $\pi: E \to B$.

Def. $\pi : E \to B$ ori. Σ_g-bdle

ξ : vertical tangent bdle of π (over E)

$e = \chi(\xi) \in H^2(E;\mathbb{Z})$ Euler class

$e_i(\pi) := \pi_!(e^{i+1}) \in H^{2i}(B;\mathbb{Z})$

i-th Miller-Morita-Mumford class

We can regard, for $g \geq 2$,

$e_i \in H^{2i}(BDiff_+\Sigma_g;\mathbb{Z}) = H^{2i}(\mathfrak{M}_g;\mathbb{Z})$

今日もまた

特異ファイバー

輪になって

踊っているよ

同境音頭