INTRODUCTION TO SINGULAR FIBERS OF DIFFERENTIABLE MAPS

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Abstract. In this talk, I would like to present the theory of singular fibers of differentiable maps together with its applications in various explicit situations, giving concrete examples. For most of the terminologies and results, the reader is referred to [13, 14, 15].

Contents

1. Introduction

(1) What does "singular fiber" refer to?

Let $f : M \to N$ be a differentiable map ($= C^\infty$ map) between C^∞ manifolds. For a singular value $y \in N$ of f, the *singular fiber* over y means the map germ

$$f : (M, f^{-1}(y)) \to (N, y) \tag{1.1}$$

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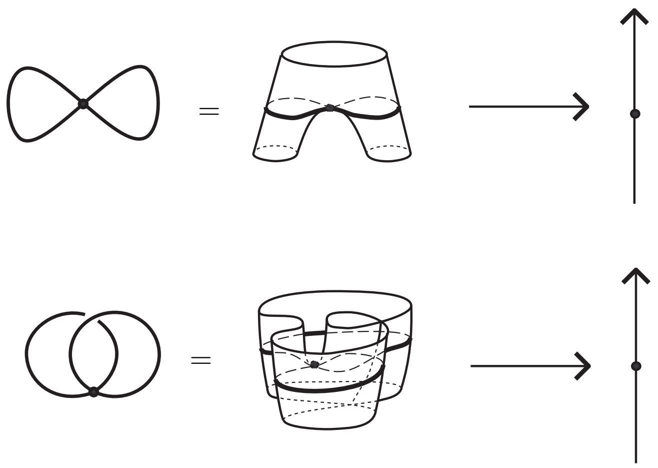

FIGURE 1. Two singular fibers corresponding to the same multi-germ

along the set $f^{-1}(y)$. Note that $f^{-1}(y)$ has positive dimension in general if $\dim M > \dim N$.

(2) <u>What is the advantage of considering such objects?</u>

Let S_y be the set of singular points lying in $f^{-1}(y)$. In singularity theory, we often consider the *multi-germ*

$$f : (M, S_y) \to (N, y) \tag{1.2}$$

along S_y, which is usually a finite set of points. However, this does not take into consideration the information on how the singular points in S_y are connected to each other in $f^{-1}(y)$. Note that the map germ (1.1) contains the information on the topology of the subset $f^{-1}(y)$ of M, and also the information carried by the multi-germ (1.2). So, when $\dim M > \dim N$, singular fibers generally carry more information than the corresponding multi-germs.

For example, each of the two singular fibers as depicted in Fig. 1 has exactly one singular point corresponding to the map germ

$$(x, y) \mapsto x^2 - y^2$$

at the origin. In other words, they cannot be distinguished as multi-germs. However, they are DIFFERENT as singular fibers, since the neighborhood of the first singular fiber is orientable, while that of the second one is nonorientable. (It would be an enjoyable exercise to show that the "regular" neighborhood of the second one is a once punctured Möbius band.)

(3) <u>What do singular fibers serve for?</u>

A singular fiber is associated with a point in the TARGET. So, it reflects a certain kind of geometric properties of a map into a fixed target manifold. In fact, we will see that

(a) we can define a cochain complex which reflects the adjacencies of singular fibers, and
(b) the cohomology classes of the cochain complex give rise to COBORDISM INVARIANTS of singular maps.

For a C^∞ map $f : M \to N$, let us denote by $\Sigma(f)$ the subset of M consisting of singular points of f of a given type Σ. Furthermore, let us denote by $\mathfrak{F}(f)$ the set of points in the target over which lies a singular fiber of a given type $\mathfrak{F}$. Then, in a certain sense, we have the following:

- homology class in the source manifold represented by $\overline{\Sigma(f)}$
  $\to$ *homotopy invariant* of f, [Horizontal object]
- homology class in the target manifold represented by $\overline{\mathfrak{F}(f)}$
  $\to$ *cobordism invariant* of f. [Vertical object]

The first one leads to the theory of Thom polynomials. As to the second one, no systematic theory has been established until now. However, in some situations, we will see that the cobordism invariants obtained as above give *complete* cobordism invariants.

(4) Are there any applications?

Cobordism theory has, in general, a lot of geometric applications. For example, we will see that the cobordism theory of singular maps can be used to prove the TOPOLOGICAL INVARIANCE of the number of certain singularities of a *generic perturbation of a C^∞ map germ.*[1]

2. Classification

For singular fibers, we consider the following equivalence relations.

Definition 2.1. Let $f_i : M_i \to N_i$ be C^∞ maps, $i = 0, 1$. For $y_i \in N_i$, we say that the fibers over y_0 and y_1 are C^∞ *equivalent* (or C^0 *equivalent*) if for some open neighborhoods U_i of y_i in N_i, there exist diffeomorphisms (resp. homeomorphisms) $\widetilde{\varphi} : (f_0)^{-1}(U_0) \to (f_1)^{-1}(U_1)$ and $\varphi : U_0 \to U_1$ with $\varphi(y_0) = y_1$ which make the following diagram commutative:

$$\begin{array}{ccc} ((f_0)^{-1}(U_0), (f_0)^{-1}(y_0)) & \xrightarrow{\widetilde{\varphi}} & ((f_1)^{-1}(U_1), (f_1)^{-1}(y_1)) \\ {\scriptstyle f_0}\big\downarrow & & \big\downarrow{\scriptstyle f_1} \\ (U_0, y_0) & \xrightarrow{\varphi} & (U_1, y_1). \end{array}$$

When $y \in N$ is a regular value of a C^∞ map $f : M \to N$, we call $f^{-1}(y)$ a *regular fiber*; otherwise, a *singular fiber*.

For lower dimensions, we can effectively classify the singular fibers as follows.[2]

Theorem 2.2. *Let $f : M \to N$ be a proper C^∞ stable map of an orientable 4-manifold M into a 3-manifold N. Then, every singular fiber of f is C^∞ equivalent to the disjoint union of one of the fibers as in Fig. 2 and a finite number of copies of a fiber of the trivial circle bundle.*

In Fig. 2, κ denotes the codimension of the set of points in the target over which lies a singular fiber of that type. Furthermore, the letters a and b correspond to a cusp point, the letter c corresponds to a definite swallow-tail, and the letters d and e correspond to an indefinite swallow-tail.

[1]The author was inspired by Ohsumi's talk at Hakodate, October 2004, to use the cobordism theory for proving the topological invariance of such numbers.

[2]In the following, a continuous map is said to be *proper* if the inverse image of a compact set is always compact.

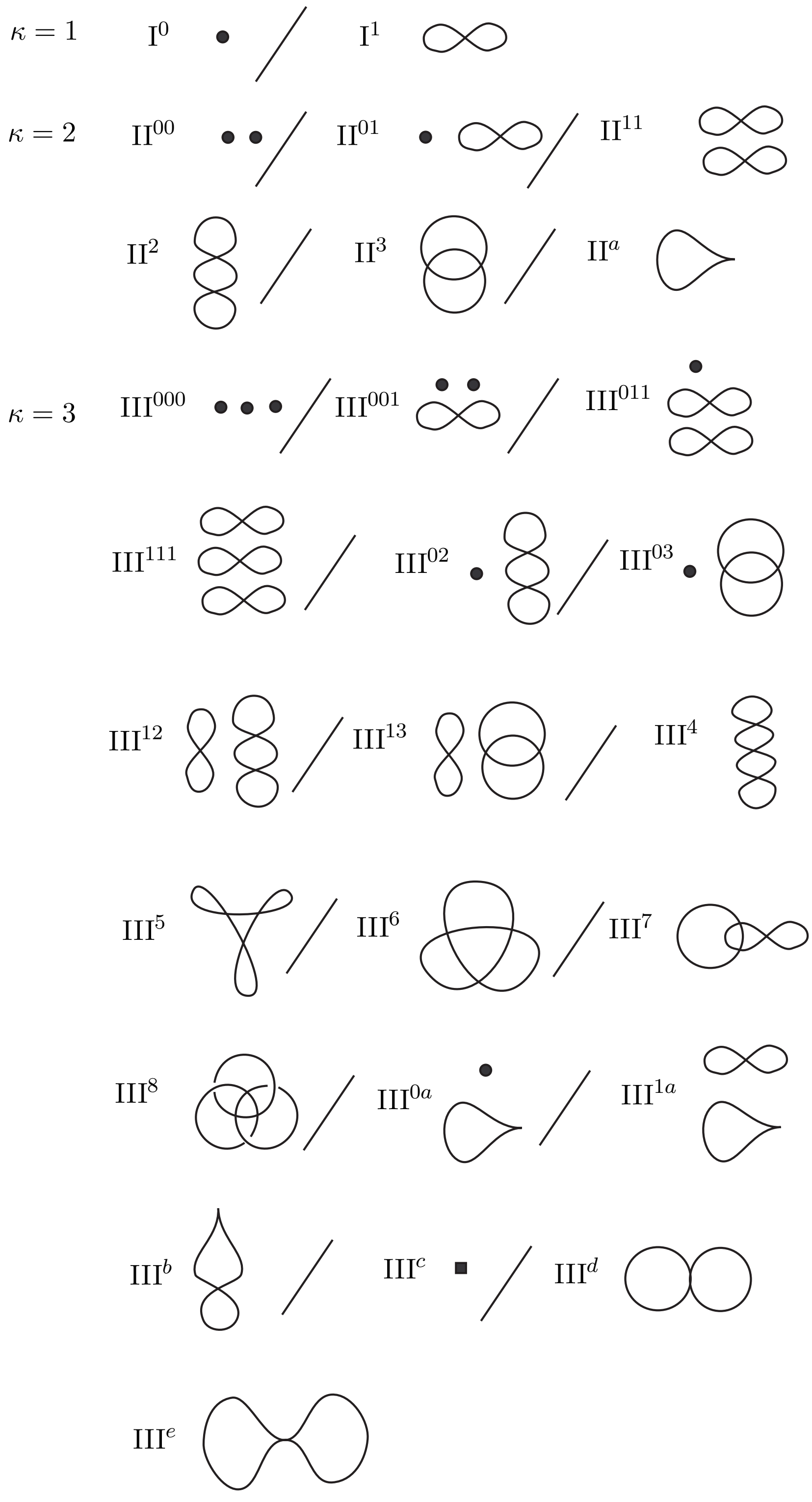

FIGURE 2. List of singular fibers of proper C^∞ stable maps of orientable 4-manifolds into 3-manifolds

Remark 2.3. For proper C^∞ stable maps of orientable 2-manifolds into 1-manifolds, the list consists of those with $\kappa = 1$ in Fig. 2. For proper C^∞ stable maps of orientable 3-manifolds into 2-manifolds, the list consists of those with $\kappa \leq 2$ in Fig. 2.

Idea for the proof of Theorem 2.2.
(1) Classify the multi-germs (Singularity Theory).
(2) List up all the possibilities for the topological types of the singular fibers by connecting the local singular fibers by nonsingular arcs (Combinatorial Argument).
(3) Show that two singular fibers corresponding to the same topological type are C^∞ equivalent by using (the relative version of) the Ehresmann fibration theorem.

As a direct consequence of the classification, we easily get the following.

Corollary 2.4. *For $n = 2, 3, 4$, two fibers of proper C^∞ stable maps of orientable n-manifolds into $(n-1)$-manifolds are C^∞ equivalent if and only if they are C^0 equivalent.*

Remark 2.5. Classification of singular fibers of proper C^∞ stable maps of general (not necessarily orientable) n-manifolds into $(n-1)$-manifolds has been obtained for $n = 2, 3, 4, 5$. For details, see [9, 13, 17], for example.

3. Universal complex of singular fibers

In order to construct a universal complex of singular fibers, we have to prepare the following two materials:

(1) a class of singular fibers τ, and
(2) an equivalence relation ϱ among the singular fibers in τ.

For (1), for technical reasons, we consider singular fibers of proper *Thom maps.*[3] Furthermore, τ should be *closed under the adjacency relation*, i.e. if a singular fiber is in τ, then any nearby singular fiber should also lie in τ.

For (2), we consider an equivalence relation which is weaker than the C^0 equivalence: i.e. each equivalence class with respect to ϱ is a union of C^0 equivalence classes. This implies that for an equivalence class $\mathfrak{F}$ with respect to ϱ and a given proper Thom map $f : M \to N$, the set $\mathfrak{F}(f)$ of points in the target N over which lies a singular fiber of type $\mathfrak{F}$ is a C^0 submanifold of N of constant *codimension*, which we denote by $\kappa(\mathfrak{F})$.

Furthermore, ϱ should satisfy the following.

(2-1) For any two proper Thom maps $f_i : M_i \to N_i$ and any points $y_i \in N_i$, $i = 0, 1$, such that the fibers over y_i lie in τ and are equivalent with respect to ϱ, there exist neighborhoods U_i of y_i in N_i, $i = 0, 1$, and a homeomorphism $\varphi : U_0 \to U_1$ such that $\varphi(y_0) = y_1$ and $\varphi(U_0 \cap \mathfrak{F}(f_0)) = U_1 \cap \mathfrak{F}(f_1)$ for every equivalence class $\mathfrak{F}$ of fibers with respect to ϱ.

The universal complex of singular fibers $\mathcal{C}^*(\tau, \varrho)$ is constructed as follows. For $\kappa \in \mathbf{Z}$, let $C^\kappa(\tau, \varrho)$ be the $\mathbf{Z}_2$-vector space consisting of all formal linear combinations,

$$\sum_{\kappa(\mathfrak{F})=\kappa} m_{\mathfrak{F}}\mathfrak{F} \quad (m_{\mathfrak{F}} \in \mathbf{Z}_2),$$

which may possibly contain infinitely many terms, of the equivalence classes $\mathfrak{F}$ of singular fibers in τ with $\kappa(\mathfrak{F}) = \kappa$.

For equivalence classes $\mathfrak{F}$ and $\mathfrak{G}$ with $\kappa(\mathfrak{G}) = \kappa(\mathfrak{F}) + 1$, we take a proper Thom map f with $\mathfrak{G}(f) \neq \emptyset$. Then we take a top dimensional stratum $\Sigma \subset \mathfrak{G}(f)$, and let B_Σ be

[3]A Thom map has nice properties with respect to Whitney stratifications of the source and the target manifolds. For a precise definition, see [2].

a small disk which intersects Σ transversely exactly at its center and whose dimension coincides with the codimension of Σ. Then $B_\Sigma \cap \overline{\mathfrak{F}(f)}$ consists of a finite number of arcs which have $B_\Sigma \cap \Sigma$ as a common end point. Let $[\mathfrak{F} : \mathfrak{G}] \in \mathbf{Z}_2$ denote the number of such arcs modulo two, which clearly does not depend on the choice of B_Σ, Σ or f by the property (2-1) above. We call $[\mathfrak{F} : \mathfrak{G}] \in \mathbf{Z}_2$ an *incidence coefficient*.

Then the $\mathbf{Z}_2$-linear map $\delta_\kappa : C^\kappa(\tau, \varrho) \to C^{\kappa+1}(\tau, \varrho)$ is defined by

$$\delta_\kappa(\mathfrak{F}) = \sum_{\kappa(\mathfrak{G})=\kappa+1} [\mathfrak{F} : \mathfrak{G}]\mathfrak{G},$$

for $\mathfrak{F}$ with $\kappa(\mathfrak{F}) = \kappa$. Note that the map δ_κ is well-defined by virtue of the local finiteness of Whitney stratifications. Furthermore, we can prove that

$$\delta_{\kappa+1} \circ \delta_\kappa = 0.$$

We call the resulting complex $\mathcal{C}^*(\tau, \varrho) = (C^\kappa(\tau, \varrho), \delta_\kappa)_\kappa$ the *universal complex of singular fibers for τ with respect to the equivalence relation ϱ*, and we denote its cohomology group of dimension κ by $H^\kappa(\tau, \varrho)$.

Remark 3.1. The above constructed cochain complex is an analogy of the cochain complexes for singularities constructed by Vassiliev et al in [6, 11, 18].

In the following, a proper Thom map $f : M \to N$ is called a $\underline{\tau\text{-map}}$ if its fibers all lie in τ.

Definition 3.2. Let

$$c = \sum_{\kappa(\mathfrak{F})=\kappa} n_{\mathfrak{F}}\mathfrak{F}$$

be a κ-dimensional cochain of the complex $\mathcal{C}^*(\tau, \varrho)$, where $n_{\mathfrak{F}} \in \mathbf{Z}_2$. For a τ-map $f : M \to N$, we define $c(f)$ to be the closure of the set of points $y \in N$ such that the fiber over y belongs to some $\mathfrak{F}$ with $n_{\mathfrak{F}} \neq 0$. If c is a cocycle, then $c(f)$ is a $\mathbf{Z}_2$-cycle of closed support of codimension κ of the target manifold N. If in addition, M is closed and $\kappa > 0$, then $c(f)$ is a $\mathbf{Z}_2$-cycle in the usual sense.

Lemma 3.3. *Suppose that c and c' are κ-dimensional cocycles of the complex $\mathcal{C}^*(\tau, \varrho)$ which are cohomologous. Then $c(f)$ and $c'(f)$ are homologous in N for every τ-map $f : M \to N$.*

Proof. There exists a $(\kappa - 1)$-dimensional cochain d of the complex such that $c - c' = \delta_{\kappa-1}d$. Then we see easily that $c(f) - c'(f) = \partial d(f)$, where $d(f)$ is defined similarly. Hence the result follows. $\square$

Definition 3.4. Let α be a κ-dimensional cohomology class of the complex $\mathcal{C}^*(\tau, \varrho)$. For a τ-map $f : M \to N$, we define $\alpha(f) \in H^c_{p-\kappa}(N; \mathbf{Z}_2)$ to be the homology class represented by the cycle $c(f)$ of closed support, where c is a cocycle representing α and $p = \dim N$. By Lemma 3.3, this is well-defined. When M is closed and $\kappa > 0$, we can also regard $\alpha(f)$ as an element of $H_{p-\kappa}(N; \mathbf{Z}_2)$.

Then we can define the map

$$\varphi_f : H^\kappa(\tau, \varrho) \to H^\kappa(N; \mathbf{Z}_2)$$

by $\varphi_f(\alpha) = \alpha(f)^*$, where $\alpha(f)^* \in H^\kappa(N; \mathbf{Z}_2)$ is the Poincaré dual to the homology class $\alpha(f) \in H^c_{p-\kappa}(N; \mathbf{Z}_2)$. This is clearly a homomorphism and we call it the *homomorphism induced by the τ-map f*. When M is closed and $\kappa > 0$, we can also regard φ_f as a homomorphism into the cohomology group $H^\kappa_c(N; \mathbf{Z}_2)$ of compact support.

Intuitively, the homomorphism induced by a τ-map gives the information on the position of the set of points in the TARGET over which lies a prescribed type of singular fiber.

4. Cobordism invariance

Definition 4.1. Let $f : M \to N$ be a proper Thom map. Then we call the map

$$f \times \mathrm{id}_{\mathbf{R}} : M \times \mathbf{R} \to N \times \mathbf{R}$$

(or $f \times \mathrm{id}_I : M \times I \to N \times I$ for any interval I in $\mathbf{R}$) the *suspension* of f. Furthermore, to the fiber of f over a point $y \in N$, we can associate the fiber of $f \times \mathrm{id}_{\mathbf{R}}$ over $y \times \{0\}$. We say that the latter fiber is obtained from the original fiber by *suspension*.

In the following, we assume that a class of singular fibers τ consists of certain singular fibers of proper Thom maps of an n-dimensional manifold into a p-dimensional manifold for a fixed dimension pair (n, p). In this case, we often write $\tau = \tau(n, p)$.

Let us consider two classes of singular fibers $\tau(n, p)$ and $\tau(n+1, p+1)$ and their associated equivalence relations $\varrho_{n,p}$ and $\varrho_{n+1,p+1}$ respectively. In addition to the conditions as in the previous section, let us impose the following conditions:

1. the suspension of any element of $\tau(n, p)$ is an element of $\tau(n+1, p+1)$, and
2. if two singular fibers are equivalent with respect to $\varrho_{n,p}$, then so are their suspensions with respect to $\varrho_{n+1,p+1}$.

The suspension induces a natural map

$$s^\sharp_\kappa : C^\kappa(\tau(n+1, p+1), \varrho_{n+1,p+1}) \to C^\kappa(\tau(n, p), \varrho_{n,p})$$

for each κ. More precisely, for an equivalence class $\mathfrak{F} \in C^\kappa(\tau(n+1, p+1), \varrho_{n+1,p+1})$ of fibers with respect to $\varrho_{n+1,p+1}$, we define $s^\sharp_\kappa(\mathfrak{F}) \in C^\kappa(\tau(n, p), \varrho_{n,p})$ to be the (possibly infinite) sum of all those equivalence classes of fibers of codimension κ with respect to $\varrho_{n,p}$ whose suspensions are contained in $\mathfrak{F}$. Note that $s^\sharp_\kappa$ is a well-defined $\mathbf{Z}_2$-linear map.

Remark 4.2. We warn the reader that the codimension may decrease by suspension.

Lemma 4.3. *The system of* $\mathbf{Z}_2$*-linear maps* $\{s^\sharp_\kappa\}_\kappa$ *defines a cochain map*

$$\mathcal{C}^*(\tau(n+1, p+1), \varrho_{n+1,p+1}) \to \mathcal{C}^*(\tau(n, p), \varrho_{n,p}).$$

In other words, we have $\delta_\kappa \circ s^\sharp_\kappa = s^\sharp_{\kappa+1} \circ \delta_\kappa$ *for all* κ.

Definition 4.4. Set $\tau = \tau(n, p) \cup \tau(n+1, p+1)$. For a C^∞ p-dimensional manifold N, two $\tau(n, p)$-maps $f_0 : M_0 \to N$ and $f_1 : M_1 \to N$ of closed manifolds M_0 and M_1 are said to be *τ-cobordant* if there exist a compact manifold W with boundary the disjoint union of M_0 and M_1, and a $\tau(n+1, p+1)$-map $F : W \to N \times [0, 1]$ such that F restricted to a collar neighborhood of M_i in W is identified with the suspension of $f_i : M_i \to N \times \{i\}$, $i = 0, 1$. We call F a *τ-cobordism* between f_0 and f_1.

When M_i are oriented and W can be taken to be oriented so that $\partial W = (-M_0) \amalg M_1$, then we say that f_0 and f_1 are *oriented τ-cobordant*.

Remark 4.5. The notion of τ-maps and that of τ-cobordisms were essentially introduced by Rimányi and Szűcs [12], although they considered only the case with $n \le p$. In their case, Rimányi and Szűcs constructed a universal τ-map and this gives rise to a lot of τ-cobordism invariants. Our aim in this section is to construct invariants of τ-cobordisms even in the case with $n > p$.

Proposition 4.6. *Let $f_i : M_i \to N$, $i = 0, 1$, be $\tau(n,p)$-maps, where we assume that M_i are closed. If they are τ-cobordant, then for every κ we have*

$$\varphi_{f_0} \circ s_\kappa^* = \varphi_{f_1} \circ s_\kappa^* : H^\kappa(\tau(n+1,p+1), \varrho_{n+1,p+1}) \to H^\kappa(N; \mathbf{Z}_2),$$

where $s_\kappa^* : H^\kappa(\tau(n+1,p+1), \varrho_{n+1,p+1}) \to H^\kappa(\tau(n,p), \varrho_{n,p})$ is the homomorphism induced by suspension. (In other words, we have $\varphi_{f_0}|_{\operatorname{Im} s_\kappa^*} = \varphi_{f_1}|_{\operatorname{Im} s_\kappa^*}$.)

Proof. Let $F : W \to N \times [0,1]$ be a τ-cobordism between f_0 and f_1. Let c be an arbitrary κ-dimensional cocycle of the complex $\mathcal{C}^*(\tau(n+1,p+1), \varrho_{n+1,p+1})$ and set $\bar{c} = s_\kappa^\sharp(c) \in C^\kappa(\tau(n,p), \varrho_{n,p})$. Then we see easily that $\partial c(F) = \bar{c}(f_1) \times \{1\} - \bar{c}(f_0) \times \{0\}$, since c is a cocycle. Then the result follows immediately. □

Thus, *every element of $H^\kappa(\tau(n+1,p+1), \varrho_{n+1,p+1})$ gives rise to a τ-cobordism invariant* for $\tau(n,p)$-maps. (Note that NOT all elements of $H^\kappa(\tau(n,p), \varrho_{n,p})$ give a τ-cobordism invariant.)

5. Several variants

5.1. **Co-orientable singular fibers.** Let τ and ϱ be as above. An equivalence class $\mathfrak{F}$ is *co-orientable* if for a given τ-map $f : M \to N$ and a point $y \in \mathfrak{F}(f)$ in the target, any local homeomorphism at y preserving the adjacent equivalence classes preserves the orientation of the normal direction to the submanifold $\mathfrak{F}(f)$ at y.

Then, taking into account the co-orientations, we can define the incidence coefficients as integers, and the *universal complex of co-orientable singular fibers*, denoted by

$$\mathcal{CO}^*(\tau, \varrho),$$

is defined over the integers.

Almost all the theory developed so far also hold for co-orientable singular fibers, by replacing the coefficient $\mathbf{Z}_2$ by $\mathbf{Z}$.

5.2. **Chiral singular fibers.**

Definition 5.1. Let $\mathfrak{F}$ be a C^0 equivalence class of a fiber of a proper Thom map of an *oriented* manifold. We say that $\mathfrak{F}$ is *achiral* if there exist homeomorphisms $\widetilde{\varphi}$ and φ which make the diagram

$$\begin{array}{ccc} (f^{-1}(U_0), f^{-1}(y)) & \xrightarrow{\widetilde{\varphi}} & (f^{-1}(U_1), f^{-1}(y)) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ (U_0, y) & \xrightarrow{\varphi} & (U_1, y) \end{array}$$

commutative such that the homeomorphism $\widetilde{\varphi}$ reverses the orientation and that the homeomorphism

$$\varphi|_{\mathfrak{F}(f)\cap U_0} : \mathfrak{F}(f) \cap U_0 \to \mathfrak{F}(f) \cap U_1 \tag{5.1}$$

preserves the local orientation of $\mathfrak{F}(f)$ at y, where f is a proper Thom map such that the fiber over y belongs to $\mathfrak{F}$, and U_i are open neighborhoods of y.

Note that if the codimension of $\mathfrak{F}$ coincides with the dimension of the target of f, then the condition about the homeomorphism (5.1) is redundant. Note also that the above definition does not depend on the choice of f or y.

Moreover, we say that $\mathfrak{F}$ is *chiral* if it is not achiral.

We also call any fiber belonging to a chiral (resp. achiral) C^0 equivalence class a *chiral fiber* (resp. *achiral fiber*).

For example, a regular fiber is achiral if and only if the fiber manifold admits an orientation reversing homeomorphism. The disjoint union of an achiral fiber and an achiral regular fiber is clearly achiral. The disjoint union of a chiral fiber and an achiral regular fiber is always chiral.

For chiral singular fibers and appropriate equivalence relations, we can define the associated *universal complex of chiral singular fibers* with coefficients in $\mathbf{Z}$. For this universal complex, an induced homomorphism into the cohomology of the target with coefficients in $\mathbf{Z}$ is defined for *oriented maps*.[4]

5.3. **Universal homology complex.** In [6], Kazarian introduced the notion of a universal homology complex of singularities, which combines the universal cohomology complex of co-orientable singularities and that of usual (not necessarily co-orientable) singularities, and which is constructed by reversing the arrows. In this subsection, we explain the same procedure in our situation of singular fibers.

Let τ and ϱ be as in the previous section. Let

$$\mathcal{C}_*(\tau, \varrho)$$

be the chain complex defined as follows. For each κ, the κ-dimensional chain group, denoted by $C_\kappa(\tau, \varrho)$, is the direct sum, over all equivalence classes of codimension κ, of the groups $\mathbf{Z}$ for co-orientable classes and the groups $\mathbf{Z}_2$ for non co-orientable classes.

Let $\mathfrak{F}$ and $\mathfrak{G}$ be two equivalence classes of singular fibers such that $\kappa(\mathfrak{F}) = \kappa(\mathfrak{G}) + 1$. Then the incidence coefficient $[\mathfrak{G} : \mathfrak{F}]$ is defined as before so that

$$\begin{aligned}
&[\mathfrak{G} : \mathfrak{F}] \in \mathbf{Z}_2, \quad \text{if } \mathfrak{G} \text{ is not co-orientable,}\\
&[\mathfrak{G} : \mathfrak{F}] \in \mathbf{Z}, \quad \text{if } \mathfrak{F} \text{ and } \mathfrak{G} \text{ are co-orientable, and}\\
&[\mathfrak{G} : \mathfrak{F}] = 0 \in \mathbf{Z}, \quad \text{otherwise.}
\end{aligned}$$

Then the boundary homomorphism $\partial_\kappa : C_\kappa(\tau, \varrho) \to C_{\kappa-1}(\tau, \varrho)$ is defined by the formula

$$\partial_\kappa(\mathfrak{F}) = \sum_{\kappa(\mathfrak{G}) = \kappa(\mathfrak{F}) - 1} [\mathfrak{G} : \mathfrak{F}]\mathfrak{G}$$

for the generators $\mathfrak{F}$ of $C_\kappa(\tau, \varrho)$. Note that this is a well-defined homomorphism.[5]

It is easy to check that $\partial_{\kappa-1} \circ \partial_\kappa = 0$ as before.

As in [6], we can check that the universal cochain complex of singular fibers $\mathcal{C}^*(\tau, \varrho)$ and the universal cochain complex of co-orientable singular fibers $\mathcal{CO}^*(\tau, \varrho)$ are isomorphic to

$$\mathrm{Hom}(\mathcal{C}_*(\tau, \varrho), \mathbf{Z}_2) \quad \text{and} \quad \mathrm{Hom}(\mathcal{C}_*(\tau, \varrho), \mathbf{Z})$$

respectively. In this sense, the universal homology complex $\mathcal{C}_*(\tau, \varrho)$ unifies the universal complex of usual singular fibers with coefficients in $\mathbf{Z}_2$ and that of co-orientable ones with coefficients in $\mathbf{Z}$.

Note that each chain group $C_\kappa(\tau, \varrho)$ is not free in general. Then by using a free approximation[6] of $\mathcal{C}_*(\tau, \varrho)$, we can define the *hypercohomology*

$$\mathbb{H}^*(\mathcal{C}_*(\tau, \varrho); G)$$

[4] A smooth map $f : M \to N$ is *oriented* if the fibers of f restricted to the complement of the singular point set is consistently oriented.

[5] Note that the similar definition for the *coboundary homomorphism* is NOT well-defined in general. This is why we should work with the HOMOLOGY complex.

[6] For the definition of a free approximation, see [16, Chap. 5, Sec. 2] or [6].

for any abelian group G. Then we can construct a homomorphism

$$\widetilde{\varphi}_f : \mathbb{H}^*(\mathcal{C}_*(\tau, \varrho); G) \to H^*(N; G)$$

induced by a τ-map $f : M \to N$ and a homomorphism induced by suspension so that the cobordism invariance similar to Proposition 4.6 holds also in this case.

It is expected that we can obtain more information by using the hypercohomologies than by using the usual cohomologies. In fact, in the singularity case, Kazarian found an additional class, which he called a "hidden singularity class". We do not know at present if there is a "hidden singular fiber" in our case.

6. Example 1 – Cobordism group of Morse functions on surfaces

A real-valued C^∞ function on a C^∞ manifold is called a *Morse function* if its critical points are all non-degenerate. We do not assume that the values at the critical points are all distinct: distinct critical points may have the same value. If the critical values are all distinct, then such a Morse function is said to be *stable*.

For a positive integer n, we denote by $M^{SO}(n)$ (or $M(n)$) the set of all Morse functions on closed oriented (resp. possibly nonorientable) n-dimensional manifolds. We adopt the convention that the function on the empty set $\emptyset$ is an element of $M^{SO}(n)$ and of $M(n)$ for all n.

In the following, a C^∞ map is called a *fold map* if it has only fold singularities.[7]

Definition 6.1. Two Morse functions $f_0 : M_0 \to \mathbf{R}$ and $f_1 : M_1 \to \mathbf{R}$ in $M^{SO}(n)$ are said to be *oriented cobordant* if there exist a compact oriented $(n+1)$-dimensional manifold X and a fold map $F : X \to \mathbf{R} \times [0, 1]$ such that the oriented boundary ∂X of X is the disjoint union $M_0 \amalg (-M_1)$, and the map F restricted to a collar neighborhood of M_i in X is identified with the suspension of $f_i : M_i \to \mathbf{R} \times \{i\}$, $i = 0, 1$. In this case, we call F an *oriented cobordism* between f_0 and f_1.

If a Morse function in $M^{SO}(n)$ is oriented cobordant to the function on the empty set, then we say that it is *oriented null-cobordant*.

It is easy to show that the above relation defines an equivalence relation on the set $M^{SO}(n)$ for each n. Furthermore, we see easily that the set of all equivalence classes forms an additive group under disjoint union: the neutral element is the class corresponding to oriented null-cobordant Morse functions, and the inverse of a class represented by a Morse function $f : M \to \mathbf{R}$ is given by the class of $-f : -M \to \mathbf{R}$. We denote by $\mathcal{M}^{SO}(n)$ the group of all oriented cobordism classes of elements of $M^{SO}(n)$ and call it the *oriented cobordism group of Morse functions on manifolds of dimension* n, or the n*-dimensional oriented cobordism group of Morse functions*.

We can also define the unoriented versions of all the objects defined above by forgetting the orientations and by using $M(n)$ instead of $M^{SO}(n)$. For the terminologies, we omit the term "oriented" (or use "unoriented" instead) for the corresponding unoriented versions. The unoriented cobordism group of Morse functions on manifolds of dimension n is denoted by $\mathcal{M}(n)$ by omitting the superscript SO.

Remark 6.2. We see easily that two Morse functions on a manifold connected by a one-parameter family of Morse functions are always cobordant. In particular, every Morse function is (oriented) cobordant to a stable Morse function.

[7] A *fold singularity* corresponds to an iterated suspension of a nondegenerate critical point of a C^∞ function.

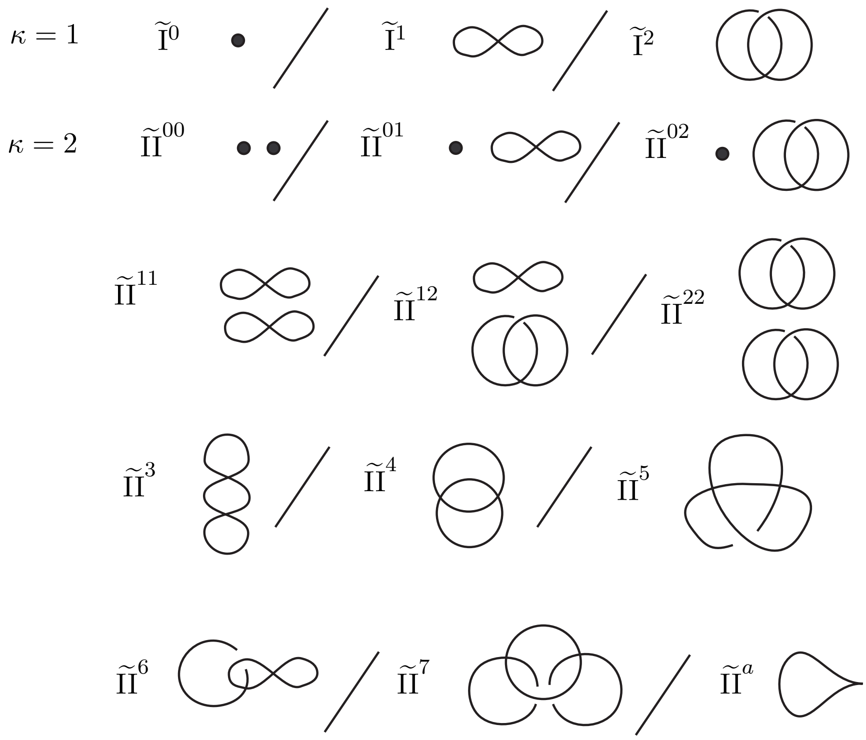

FIGURE 3. List of singular fibers of proper C^∞ stable maps of 3-manifolds into surfaces

In order to describe the cobordism group of Morse functions on surfaces, let us consider proper C^∞ stable maps of 3-manifolds into surfaces. Then we have the list of C^∞ (or C^0) equivalence classes of singular fibers of such maps as in Fig. 3: every singular fiber of such a map is C^∞ (or C^0) equivalent to the disjoint union of one of the fibers as in Fig. 3 and a finite number of copies of a fiber of the trivial circle bundle.

The equivalence class of fibers of codimension zero corresponds to the class of regular fibers. We denote this codimension zero equivalence class by $\widetilde{\mathbf{0}}$.

We note that the fiber $\widetilde{\mathrm{II}}^a$ corresponds to a cusp singular point.

If the source 3-manifold is orientable, then the singular fibers of types $\widetilde{\mathrm{I}}^2$, $\widetilde{\mathrm{II}}^{02}$, $\widetilde{\mathrm{II}}^{12}$, $\widetilde{\mathrm{II}}^{22}$, $\widetilde{\mathrm{II}}^{5}$, $\widetilde{\mathrm{II}}^{6}$ and $\widetilde{\mathrm{II}}^{7}$ do not appear.

Note also that the list of C^∞ (or C^0) equivalence classes of singular fibers of proper stable Morse functions on surfaces is nothing but those appearing in Fig. 3 with $\kappa = 1$.

Let $\varrho^0_{n,n-1}(2)$ be the C^0 *equivalence relation modulo two circle components* for fibers of proper C^∞ stable maps of n-dimensional manifolds into $(n-1)$-dimensional manifolds.[8] For a C^0 equivalence class $\widetilde{\mathfrak{F}}$ of singular fibers, we denote by $\widetilde{\mathfrak{F}}_{\mathrm{o}}$ (or $\widetilde{\mathfrak{F}}_{\mathrm{e}}$) the equivalence class with respect to $\varrho^0_{n,n-1}(2)$ containing a singular fiber of type $\widetilde{\mathfrak{F}}$ whose total number of components is odd (resp. even).

We easily get the following for $n = 3$.

Lemma 6.3. *Those equivalence classes with respect to $\varrho^0_{3,2}(2)$ which are co-orientable are $\widetilde{\mathbf{0}}_*$, $\widetilde{\mathrm{I}}^0_*$, $\widetilde{\mathrm{I}}^1_*$, $\widetilde{\mathrm{II}}^{01}_*$ and $\widetilde{\mathrm{II}}^a_*$, where $* = \mathrm{o}$ and* e.

[8] Two fibers are equivalent with respect to $\varrho^0_{n,n-1}(2)$ if one is C^0 equivalent to the other after adding an even number of regular circle components.

Remark 6.4. If we consider $\varrho^0_{3,2}(1)$ (C^0 equivalence modulo regular components) instead of $\varrho^0_{3,2}(2)$, then no co-orientable equivalence class appears. That is why we have chosen the C^0 equivalence modulo two circle components.

Let us denote by $\tau^0(n,p)$ (or $\tau^0(n,p)^{\mathrm{ori}}$) the set of all C^0-equivalence classes of fibers for proper C^∞ stable fold maps of (orientable) n-dimensional manifolds into p-dimensional manifolds. Let

$$\mathcal{CO}^*(\tau^0(n,n-1),\varrho^0_{n,n-1}(2)) \quad \text{and} \quad \mathcal{CO}^*(\tau^0(n,n-1)^{\mathrm{ori}},\varrho^0_{n,n-1}(2)) \tag{6.1}$$

be the universal complexes of co-orientable singular fibers for the respective classes of maps with respect to the C^0 equivalence modulo two circle components. Note that these complexes are defined over the integers $\mathbf{Z}$.

Then by Lemma 6.3, we see that the following equivalence classes constitute a basis of the κ-dimensional cochain group for the two cochain complexes in (6.1) with $n=3$, where $*=\mathrm{o}$ and e:

$$\widetilde{\mathbf{0}}_* \quad (\kappa=0), \quad \widetilde{\mathrm{I}}^0_*,\ \widetilde{\mathrm{I}}^1_* \quad (\kappa=1), \quad \widetilde{\mathrm{II}}^{01}_* \quad (\kappa=2).$$

Note that $\widetilde{\mathrm{II}}^a_*$ do not appear, since fold maps have no cusps. Note also that for $n=2$, we have the same bases for $\kappa\le 1$.

Let us fix a co-orientation for each of the above equivalence classes. Then we see that the coboundary homomorphism is given by the following formulae:

$$\begin{aligned}
\delta_0(\widetilde{\mathbf{0}}_{\mathrm{o}}) &= \widetilde{\mathrm{I}}^0_{\mathrm{o}}+\widetilde{\mathrm{I}}^0_{\mathrm{e}}+\widetilde{\mathrm{I}}^1_{\mathrm{o}}+\widetilde{\mathrm{I}}^1_{\mathrm{e}},\\
\delta_0(\widetilde{\mathbf{0}}_{\mathrm{e}}) &= -\widetilde{\mathrm{I}}^0_{\mathrm{o}}-\widetilde{\mathrm{I}}^0_{\mathrm{e}}-\widetilde{\mathrm{I}}^1_{\mathrm{o}}-\widetilde{\mathrm{I}}^1_{\mathrm{e}},\\
\delta_1(\widetilde{\mathrm{I}}^0_{\mathrm{o}}) &= \widetilde{\mathrm{II}}^{01}_{\mathrm{o}}-\widetilde{\mathrm{II}}^{01}_{\mathrm{e}},\\
\delta_1(\widetilde{\mathrm{I}}^0_{\mathrm{e}}) &= \widetilde{\mathrm{II}}^{01}_{\mathrm{o}}-\widetilde{\mathrm{II}}^{01}_{\mathrm{e}},\\
\delta_1(\widetilde{\mathrm{I}}^1_{\mathrm{o}}) &= -\widetilde{\mathrm{II}}^{01}_{\mathrm{o}}+\widetilde{\mathrm{II}}^{01}_{\mathrm{e}},\\
\delta_1(\widetilde{\mathrm{I}}^1_{\mathrm{e}}) &= -\widetilde{\mathrm{II}}^{01}_{\mathrm{o}}+\widetilde{\mathrm{II}}^{01}_{\mathrm{e}}.
\end{aligned} \tag{6.2}$$

In the following, we denote by $[c]$ the (co)homology class represented by a (co)cycle c. Then, by a straightforward calculation, we get the following.

Lemma 6.5. *For the cohomology groups of the two cochain complexes in* (6.1) *with* $n=3$, *we have*

$$H^0 \cong \mathbf{Z} \quad (\textit{generated by } [\widetilde{\mathbf{0}}_{\mathrm{o}}+\widetilde{\mathbf{0}}_{\mathrm{e}}]), \quad \textit{and}$$

$$H^1 \cong \mathbf{Z}\oplus\mathbf{Z} \quad (\textit{generated by } \alpha_1=-[\widetilde{\mathrm{I}}^0_{\mathrm{o}}+\widetilde{\mathrm{I}}^1_{\mathrm{e}}]=[\widetilde{\mathrm{I}}^0_{\mathrm{e}}+\widetilde{\mathrm{I}}^1_{\mathrm{o}}],\ \alpha_2=[-\widetilde{\mathrm{I}}^0_{\mathrm{o}}+\widetilde{\mathrm{I}}^0_{\mathrm{e}}],$$
$$\textit{and } \alpha_3=[\widetilde{\mathrm{I}}^1_{\mathrm{o}}-\widetilde{\mathrm{I}}^1_{\mathrm{e}}] \textit{ with } 2\alpha_1=\alpha_2+\alpha_3).$$

Furthermore, for $n=2$, *the same isomorphism holds for* H^0, *and for* H^1, *we have*

$$H^1 \cong \mathbf{Z}\oplus\mathbf{Z}\oplus\mathbf{Z} \quad (\textit{generated by } \beta_1=-[\widetilde{\mathrm{I}}^0_{\mathrm{o}}+\widetilde{\mathrm{I}}^1_{\mathrm{e}}]=[\widetilde{\mathrm{I}}^0_{\mathrm{e}}+\widetilde{\mathrm{I}}^1_{\mathrm{o}}],\ \beta_2=[\widetilde{\mathrm{I}}^0_{\mathrm{o}}],$$
$$\textit{and } \beta_3=[\widetilde{\mathrm{I}}^1_{\mathrm{o}}]).$$

Let

$$s^*_\kappa : H^\kappa(\mathcal{CO}^*(\tau^0(3,2),\varrho^0_{3,2}(2))) \to H^\kappa(\mathcal{CO}^*(\tau^0(2,1),\varrho^0_{2,1}(2)))$$

etc. be the homomorphism induced by suspension. Then for $\kappa=1$, we have

$$s^*_1\alpha_1=\beta_1,\ s^*_1\alpha_2=\beta_1-\beta_2-\beta_3 \text{ and } s^*_1\alpha_3=\beta_1+\beta_2+\beta_3.$$

In particular, we see that s_1^* is injective and its image is isomorphic to $\mathbf{Z} \oplus \mathbf{Z}$.

Then, by an easy argument we see that $s_1^*\alpha_1(f) = \beta_1(f) \in H_0(\mathbf{R}; \mathbf{Z})$ always vanishes for any stable Morse function f on a closed surface (see Remark 7.2). Furthermore, we have the following.

Lemma 6.6. *For a stable Morse function f as above, we have*

$$s_1^*\alpha_2(f) = -s_1^*\alpha_3(f) = \max(f) - \min(f)$$

under the natural identification $H_0(\mathbf{R}; \mathbf{Z}) = \mathbf{Z}$, where $\max(f)$ (or $\min(f)$) is the number of local maxima (resp. minima) of the Morse function f.

Then by [4], we get the following.

Theorem 6.7. *The map*

$$\Phi^{SO} : \mathcal{M}^{SO}(2) \to \mathbf{Z}$$

which sends the cobordism class of a stable Morse function f to $s_1^*\alpha_2(f) = \max(f) - \min(f) \in \mathbf{Z}$ is an isomorphism.

In the unoriented case, the corresponding map does not give an isomorphism. In order to get an isomorphism, let us consider the universal complex of singular fibers

$$\mathcal{C}^*(\tau^0(n, n-1), \varrho^0_{n,n-1}(2)) \tag{6.3}$$

with coefficients in $\mathbf{Z}_2$. The coboundary homomorphisms satisfy the following.

$$\begin{aligned}
\delta_0(\widetilde{\mathbf{0}}_{\mathrm{o}}) &= \widetilde{\mathrm{I}}^0_{\mathrm{o}} + \widetilde{\mathrm{I}}^0_{\mathrm{e}} + \widetilde{\mathrm{I}}^1_{\mathrm{o}} + \widetilde{\mathrm{I}}^1_{\mathrm{e}}, \\
\delta_0(\widetilde{\mathbf{0}}_{\mathrm{e}}) &= \widetilde{\mathrm{I}}^0_{\mathrm{o}} + \widetilde{\mathrm{I}}^0_{\mathrm{e}} + \widetilde{\mathrm{I}}^1_{\mathrm{o}} + \widetilde{\mathrm{I}}^1_{\mathrm{e}}, \\
\delta_1(\widetilde{\mathrm{I}}^0_{\mathrm{o}}) &= \widetilde{\mathrm{II}}^{01}_{\mathrm{o}} + \widetilde{\mathrm{II}}^{01}_{\mathrm{e}}, \\
\delta_1(\widetilde{\mathrm{I}}^0_{\mathrm{e}}) &= \widetilde{\mathrm{II}}^{01}_{\mathrm{o}} + \widetilde{\mathrm{II}}^{01}_{\mathrm{e}}, \\
\delta_1(\widetilde{\mathrm{I}}^1_{\mathrm{o}}) &= \widetilde{\mathrm{II}}^{01}_{\mathrm{o}} + \widetilde{\mathrm{II}}^{01}_{\mathrm{e}}, \\
\delta_1(\widetilde{\mathrm{I}}^1_{\mathrm{e}}) &= \widetilde{\mathrm{II}}^{01}_{\mathrm{o}} + \widetilde{\mathrm{II}}^{01}_{\mathrm{e}}, \\
\delta_1(\widetilde{\mathrm{I}}^2_{\mathrm{o}}) &= \widetilde{\mathrm{II}}^{02}_{\mathrm{o}} + \widetilde{\mathrm{II}}^{02}_{\mathrm{e}} + \widetilde{\mathrm{II}}^{12}_{\mathrm{o}} + \widetilde{\mathrm{II}}^{12}_{\mathrm{e}} + \widetilde{\mathrm{II}}^{6}_{\mathrm{o}} + \widetilde{\mathrm{II}}^{6}_{\mathrm{e}}, \\
\delta_1(\widetilde{\mathrm{I}}^2_{\mathrm{e}}) &= \widetilde{\mathrm{II}}^{02}_{\mathrm{o}} + \widetilde{\mathrm{II}}^{02}_{\mathrm{e}} + \widetilde{\mathrm{II}}^{12}_{\mathrm{o}} + \widetilde{\mathrm{II}}^{12}_{\mathrm{e}} + \widetilde{\mathrm{II}}^{6}_{\mathrm{o}} + \widetilde{\mathrm{II}}^{6}_{\mathrm{e}}.
\end{aligned} \tag{6.4}$$

By a straightforward calculation, we get the following.

Lemma 6.8. *For the cohomology groups of the cochain complex (6.3) with $n = 3$, we have*

$$\begin{aligned}
&H^0 \cong \mathbf{Z}_2 \quad (\textit{generated by } [\widetilde{\mathbf{0}}_{\mathrm{o}} + \widetilde{\mathbf{0}}_{\mathrm{e}}]), \quad \textit{and} \\
&H^1 \cong \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \quad (\textit{generated by } \widehat{\alpha}_1 = [\widetilde{\mathrm{I}}^0_{\mathrm{o}} + \widetilde{\mathrm{I}}^1_{\mathrm{e}}] = [\widetilde{\mathrm{I}}^0_{\mathrm{e}} + \widetilde{\mathrm{I}}^1_{\mathrm{o}}], \\
&\qquad \widehat{\alpha}_2 = [\widetilde{\mathrm{I}}^0_{\mathrm{o}} + \widetilde{\mathrm{I}}^0_{\mathrm{e}}] = [\widetilde{\mathrm{I}}^1_{\mathrm{o}} + \widetilde{\mathrm{I}}^1_{\mathrm{e}}] \textit{ and } \widehat{\alpha}_3 = [\widetilde{\mathrm{I}}^2_{\mathrm{o}} + \widetilde{\mathrm{I}}^2_{\mathrm{e}}]).
\end{aligned}$$

Furthermore, for $n = 2$, the same isomorphism holds for H^0, and for H^1, we have

$$\begin{aligned}
&H^1 \cong \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \quad (\textit{generated by } \widehat{\beta}_1 = [\widetilde{\mathrm{I}}^0_{\mathrm{o}} + \widetilde{\mathrm{I}}^1_{\mathrm{e}}] = [\widetilde{\mathrm{I}}^0_{\mathrm{e}} + \widetilde{\mathrm{I}}^1_{\mathrm{o}}], \ \widehat{\beta}_2 = [\widetilde{\mathrm{I}}^0_{\mathrm{o}}], \\
&\qquad \widehat{\beta}_3 = [\widetilde{\mathrm{I}}^1_{\mathrm{o}}], \ \widehat{\beta}_4 = [\widetilde{\mathrm{I}}^2_{\mathrm{o}}] \textit{ and } \widehat{\beta}_5 = [\widetilde{\mathrm{I}}^2_{\mathrm{e}}]).
\end{aligned}$$

We can also describe the homomorphisms induced by suspension with respect to the above generators.

Let $f : M \to \mathbf{R}$ be a stable Morse function on a closed surface M. Then we see that $s_1^*\widehat{\alpha}_1(f) \in H_0(\mathbf{R}; \mathbf{Z}_2) \cong \mathbf{Z}_2$ always vanishes as before. Furthermore, $s_1^*\widehat{\alpha}_2(f)$ coincides with $\min(f) + \max(f)$ modulo two. Finally, $s_1^*\widehat{\alpha}_3(f)$ gives the number (modulo 2) of singular fibers of type $\widetilde{\mathrm{I}}^2$ of f.

Then by [3, 5], we get the following.

Theorem 6.9. *The map*

$$\Phi : \mathcal{M}(2) \to \mathbf{Z} \oplus \mathbf{Z}_2$$

which sends the cobordism class of a stable Morse function f to

$$(s_1^*\alpha_2(f), s_1^*\widehat{\alpha}_3(f)) = (\max(f) - \min(f), |\widetilde{\mathrm{I}}^2(f)|) \in \mathbf{Z} \oplus \mathbf{Z}_2$$

is an isomorphism, where $|*|$ denotes the number of elements modulo two.

Note that as will be seen in §8, $|\widetilde{\mathrm{I}}^2(f)| \in \mathbf{Z}_2$ coincides with the parity of the Euler characteristic $\chi(M)$ of the source surface M.

As the above observations show, the cohomology classes of universal complexes of singular fibers can give complete cobordism invariants for singular maps.

We note that by using the universal homology complex of singular fibers as introduced in Subsection 5.3, we get the same isomorphisms as in Theorems 6.7 and 6.9.

7. Application to map germs

In this section, as an application of the theory of universal complexes of singular fibers we give a new topological invariant for generic C^∞ map germs $(\mathbf{R}^3, 0) \to (\mathbf{R}^2, 0)$.

First, let us consider the following situation. Let $F : W \to D^2$ be a C^∞ map of a compact 3-dimensional manifold W with nonempty boundary $\partial W = M$ with the following properties:

(1) $F^{-1}(\partial D^2) = M$,
(2) $f = F|_M : M \to \partial D^2 = S^1$ is a C^∞ stable map,
(3) $F|_{M\times[0,1)} = f \times \mathrm{id}_{[0,1)}$, where we identify the small open collar neighborhood of M (or ∂D^2) in W (resp. in D^2) with $M \times [0, 1)$ (resp. $\partial D^2 \times [0, 1)$),
(4) $F|_{\mathrm{Int}\,W} : \mathrm{Int}\,W \to \mathrm{Int}\,D^2$ is a proper C^∞ stable map.

Note that F may have cusp singular points. Thus, in general, F has singular fibers as depicted in Fig. 3.

In §6, we have seen that the fibers $\widetilde{\mathbf{0}}_*$, $\widetilde{\mathrm{I}}^0_*$, $\widetilde{\mathrm{I}}^1_*$ and $\widetilde{\mathrm{II}}^{01}_*$ are co-orientable. If cusp singular points are allowed, then $\widetilde{\mathrm{II}}^a_*$ is also co-orientable. We give co-orientations to $\widetilde{\mathrm{II}}^a_*$ as depicted in Fig. 4.

In the following, we orient D^2 and $S^1 = \partial D^2$ consistently so that S^1 gets the counterclockwise orientation. Then we have the following.

Lemma 7.1. *For the algebraic numbers of singular fibers of F and f, we have*

$$\begin{aligned}
||\widetilde{\mathrm{I}}^0_{\mathrm{o}}(f)|| &= -||\widetilde{\mathrm{II}}^{01}_{\mathrm{o}}(F)|| + ||\widetilde{\mathrm{II}}^{01}_{\mathrm{e}}(F)|| - ||\widetilde{\mathrm{II}}^a_{\mathrm{e}}(F)||,\\
||\widetilde{\mathrm{I}}^0_{\mathrm{e}}(f)|| &= -||\widetilde{\mathrm{II}}^{01}_{\mathrm{o}}(F)|| + ||\widetilde{\mathrm{II}}^{01}_{\mathrm{e}}(F)|| + ||\widetilde{\mathrm{II}}^a_{\mathrm{o}}(F)||,\\
||\widetilde{\mathrm{I}}^1_{\mathrm{o}}(f)|| &= ||\widetilde{\mathrm{II}}^{01}_{\mathrm{o}}(F)|| - ||\widetilde{\mathrm{II}}^{01}_{\mathrm{e}}(F)|| - ||\widetilde{\mathrm{II}}^a_{\mathrm{o}}(F)||,\\
||\widetilde{\mathrm{I}}^1_{\mathrm{e}}(f)|| &= ||\widetilde{\mathrm{II}}^{01}_{\mathrm{o}}(F)|| - ||\widetilde{\mathrm{II}}^{01}_{\mathrm{e}}(F)|| + ||\widetilde{\mathrm{II}}^a_{\mathrm{e}}(F)||.
\end{aligned}$$

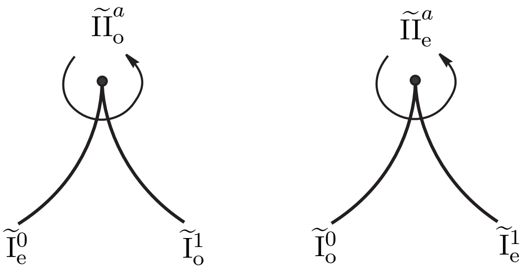

FIGURE 4. Co-orientations for $\widetilde{\mathrm{II}}^a_{\mathrm{o}}$ and $\widetilde{\mathrm{II}}^a_{\mathrm{e}}$

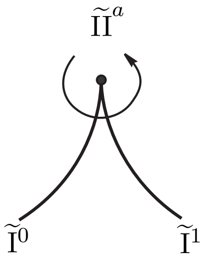

FIGURE 5. Co-orientation for $\widetilde{\mathrm{II}}^a$

Proof. Let us consider the closures of $\widetilde{\mathrm{I}}^0_{\mathrm{o}}(F)$, $\widetilde{\mathrm{I}}^0_{\mathrm{e}}(F)$, $\widetilde{\mathrm{I}}^1_{\mathrm{o}}(F)$ and $\widetilde{\mathrm{I}}^1_{\mathrm{e}}(F)$ as 1-dimensional chains in D^2 with coefficients in $\mathbf{Z}$. Then by observing the adjacencies for the singular fibers as we did to obtain the formulae for the coboundary homomorphism in (6.2), we get the following equalities as 0-dimensional chains:

$$\begin{aligned}
\partial\overline{\widetilde{\mathrm{I}}^0_{\mathrm{o}}(F)} &= \widetilde{\mathrm{II}}^{01}_{\mathrm{o}}(F) - \widetilde{\mathrm{II}}^{01}_{\mathrm{e}}(F) + \widetilde{\mathrm{II}}^{a}_{\mathrm{e}}(F) + \widetilde{\mathrm{I}}^0_{\mathrm{o}}(f),\\
\partial\overline{\widetilde{\mathrm{I}}^0_{\mathrm{e}}(F)} &= \widetilde{\mathrm{II}}^{01}_{\mathrm{o}}(F) - \widetilde{\mathrm{II}}^{01}_{\mathrm{e}}(F) - \widetilde{\mathrm{II}}^{a}_{\mathrm{o}}(F) + \widetilde{\mathrm{I}}^0_{\mathrm{e}}(f),\\
\partial\overline{\widetilde{\mathrm{I}}^1_{\mathrm{o}}(F)} &= -\widetilde{\mathrm{II}}^{01}_{\mathrm{o}}(F) + \widetilde{\mathrm{II}}^{01}_{\mathrm{e}}(F) + \widetilde{\mathrm{II}}^{a}_{\mathrm{o}}(F) + \widetilde{\mathrm{I}}^1_{\mathrm{o}}(f),\\
\partial\overline{\widetilde{\mathrm{I}}^1_{\mathrm{e}}(F)} &= -\widetilde{\mathrm{II}}^{01}_{\mathrm{o}}(F) + \widetilde{\mathrm{II}}^{01}_{\mathrm{e}}(F) - \widetilde{\mathrm{II}}^{a}_{\mathrm{e}}(F) + \widetilde{\mathrm{I}}^1_{\mathrm{e}}(f).
\end{aligned}$$

Since the algebraic number of points in the boundary of a 1-dimensional chain is always equal to zero, we get the desired equalities. $\square$

Remark 7.2. By the above lemma, we see easily that

$$||\widetilde{\mathrm{I}}^0_{\mathrm{o}}(f)|| + ||\widetilde{\mathrm{I}}^1_{\mathrm{e}}(f)|| = 0 \quad \text{and} \quad ||\widetilde{\mathrm{I}}^0_{\mathrm{e}}(f)|| + ||\widetilde{\mathrm{I}}^1_{\mathrm{o}}(f)|| = 0.$$

This gives an alternative proof of the fact that $\beta_1(f) = 0$ for a C^∞ stable map f of a closed surface into S^1, where β_1 is the cohomology class described in Lemma 6.5 (see also [13, Lemma 14.1]).

For $\widetilde{\mathrm{II}}^a$, we consider the co-orientation as depicted in Fig. 5. Since we have

$$||\widetilde{\mathrm{II}}^a(F)|| = ||\widetilde{\mathrm{II}}^a_{\mathrm{o}}(F)|| + ||\widetilde{\mathrm{II}}^a_{\mathrm{e}}(F)||,$$

we immediately get the following.

Proposition 7.3. *The algebraic number of singular fibers of F containing cusps is equal to*

$$-||\widetilde{\mathrm{I}}^0_{\mathrm{o}}(f)|| + ||\widetilde{\mathrm{I}}^0_{\mathrm{e}}(f)||. \tag{7.1}$$

Note that the integer given by (7.1) is a (fold) cobordism invariant as shown in §6.

Now, let $g : (\mathbf{R}^3, 0) \to (\mathbf{R}^2, 0)$ be a C^∞ map germ which is generic in the sense of Fukuda [1]. (In what follows, we will not distinguish the map germ g from its representative when there is no confusion.) Suppose furthermore that the origin is isolated in $g^{-1}(0)$, i.e. $0 \notin \overline{g^{-1}(0) \setminus \{0\}}$. Then for $\varepsilon > 0$ sufficiently small, $\widetilde{S}^2_\varepsilon = g^{-1}(S^1_\varepsilon)$ is diffeomorphic to S^2, and g is topologically equivalent to the cone of the C^∞ stable map

$$g_\partial = g|_{g^{-1}(S^1_\varepsilon)} : \widetilde{S}^2_\varepsilon \to S^1_\varepsilon,$$

where S^1_ε is the circle in $\mathbf{R}^2$ with radius ε centered at the origin, and the cone of a map $h : X \to Y$ refers to the map $Ch : X \times [0, 1)/X \times \{0\} \to Y \times [0, 1)/Y \times \{0\}$ defined by $Ch(x, t) = (h(x), t)$ (for details, see [1]).

Definition 7.4. Let g and $g' : (\mathbf{R}^3, 0) \to (\mathbf{R}^2, 0)$ be C^∞ map germs. We say that they are *topologically $\mathcal{A}$-equivalent* if there exist homeomorphism germs $\Phi : (\mathbf{R}^3, 0) \to (\mathbf{R}^3, 0)$ and $\varphi : (\mathbf{R}^2, 0) \to (\mathbf{R}^2, 0)$ such that $g' = \varphi \circ g \circ \Phi^{-1}$. Furthermore, if the homeomorphism germ φ can be chosen so that it preserves the orientation of $\mathbf{R}^2$, then we say that g and g' are *topologically $\mathcal{A}_+$-equivalent.*

Let $\widetilde{g}$ be a stable perturbation of a representative of g. In the following, the algebraic number of singular fibers of type $\widetilde{\mathrm{II}}^a$ of $\widetilde{g}$ appearing near the origin is simply called the *algebraic number of cusps* of $\widetilde{g}$. Then we have the following.

Theorem 7.5. *Let $g : (\mathbf{R}^3, 0) \to (\mathbf{R}^2, 0)$ be a generic C^∞ map germ such that 0 is isolated in $g^{-1}(0)$. Then the algebraic number of cusps of a C^∞ stable perturbation $\widetilde{g}$ of a representative of g is an invariant of the topological $\mathcal{A}_+$-equivalence class of g, and is equal to*

$$-||\widetilde{\mathrm{I}}^0_{\mathrm{o}}(g_\partial)|| + ||\widetilde{\mathrm{I}}^0_{\mathrm{e}}(g_\partial)||, \tag{7.2}$$

where $\varepsilon > 0$ is sufficiently small. In particular, the absolute value of the algebraic number of cusps of $\widetilde{g}$ is an invariant of the topological $\mathcal{A}$-equivalence class of g.

Essential idea for the proof consists of the following two steps:

1. to find a formula for the algebraic number of cusps of a stable perturbation $\widetilde{g}$ in terms of the <u>singular fibers of g_∂</u>, and
2. to show the <u>cobordism invariance</u> of the quantity obtained in (1).

Proof of Theorem 7.5. Let $g' : (\mathbf{R}^3, 0) \to (\mathbf{R}^2, 0)$ be a C^∞ map germ which is generic in the sense of [1] and which is topologically $\mathcal{A}_+$-equivalent to g. Then we see that the C^∞ stable maps g_∂ and g'_∂ are cobordant in a sense similar to Definition 6.1. Then by the results obtained in §6, we have

$$-||\widetilde{\mathrm{I}}^0_{\mathrm{o}}(g_\partial)|| + ||\widetilde{\mathrm{I}}^0_{\mathrm{e}}(g_\partial)|| = -||\widetilde{\mathrm{I}}^0_{\mathrm{o}}(g'_\partial)|| + ||\widetilde{\mathrm{I}}^0_{\mathrm{e}}(g'_\partial)||.$$

Therefore, the integer (7.2) is an invariant of the topological $\mathcal{A}_+$-equivalence class.

Furthermore, the integer (7.2) is equal to the algebraic number of cusps of a C^∞ stable perturbation $\widetilde{g}$ of a representative of g by Proposition 7.3.

It is easy to observe that if we reverse the orientation of $\mathbf{R}^2$, then the algebraic number of cusps changes the sign. Thus the last assertion of the theorem follows. □

In order to generalize the above result to the case where the origin may not necessarily be isolated in $g^{-1}(0)$, let us consider the following situation. Let $F : W \to D^2$ be a C^∞ map of a compact 3-dimensional manifold W with nonempty boundary ∂W with the following properties:

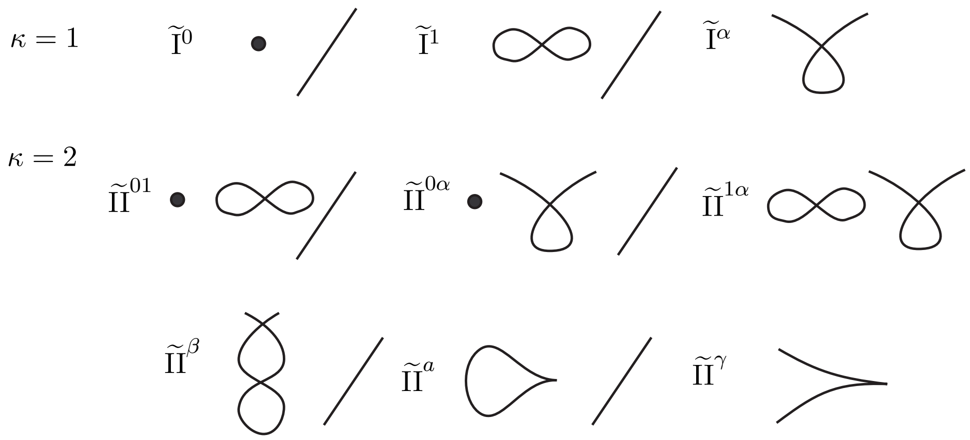

FIGURE 6. List of co-orientable singular fibers for F

(1) $\partial W = M \cup P$, where M is a compact surface with boundary, P is a finite disjoint union of 2-dimensional disks, and $M \cap P = \partial M = \partial P$,
(2) $F^{-1}(\partial D^2) = M$,
(3) $F|_P : P \to D^2$ is a submersion,
(4) $f = F|_M : M \to \partial D^2 = S^1$ is a C^∞ stable map,
(5) $F|_{M \times [0,1)} = f \times \mathrm{id}_{[0,1)}$, where we identify the small open "collar neighborhood" of M (or ∂D^2) in W (resp. in D^2) with $M \times [0,1)$ (resp. $\partial D^2 \times [0,1)$),
(6) $F|_{W \smallsetminus M} : W \smallsetminus M \to \mathrm{Int}\, D^2$ is a proper C^∞ stable map.

In what follows, for simplicity we assume that W and M are orientable, which is enough for our purpose.

Then we can get a list of the C^0 equivalence classes of singular fibers that appear for F as above, which is similar to Fig. 3. For these fibers, let us consider the following equivalence relation: two fibers are equivalent if one is C^0 equivalent to the other after adding even numbers of regular components to both of the fibers. Note that in contrast to the case where $\partial M = \emptyset$, regular fibers consist of circles and intervals. However, when we count the number of regular components, we do not distinguish them.

Then we easily get the following.

Lemma 7.6. *Those equivalence classes of singular fibers which are co-orientable are $\widetilde{\mathfrak{F}}_*$, where $* = \mathrm{o}$ or e, and $\widetilde{\mathfrak{F}}$ are as depicted in Fig.* 6.

We denote by $\widetilde{\mathbf{0}}_*$ the equivalence classes corresponding to regular fibers. Note that they are also co-orientable. Then for the coboundary homomorphism, we get the following:

$$\delta_1(\widetilde{\mathrm{I}}^0_{\mathrm{o}}) = \widetilde{\mathrm{II}}^{01}_{\mathrm{o}} - \widetilde{\mathrm{II}}^{01}_{\mathrm{e}} - \widetilde{\mathrm{II}}^{a}_{\mathrm{e}} - \widetilde{\mathrm{II}}^{0\alpha}_{\mathrm{o}} + \widetilde{\mathrm{II}}^{0\alpha}_{\mathrm{e}} - \widetilde{\mathrm{II}}^{\gamma}_{\mathrm{e}},$$
$$\delta_1(\widetilde{\mathrm{I}}^0_{\mathrm{e}}) = \widetilde{\mathrm{II}}^{01}_{\mathrm{o}} - \widetilde{\mathrm{II}}^{01}_{\mathrm{e}} + \widetilde{\mathrm{II}}^{a}_{\mathrm{o}} - \widetilde{\mathrm{II}}^{0\alpha}_{\mathrm{o}} + \widetilde{\mathrm{II}}^{0\alpha}_{\mathrm{e}} + \widetilde{\mathrm{II}}^{\gamma}_{\mathrm{o}},$$
$$\delta_1(\widetilde{\mathrm{I}}^1_{\mathrm{o}}) = -\widetilde{\mathrm{II}}^{01}_{\mathrm{o}} + \widetilde{\mathrm{II}}^{01}_{\mathrm{e}} - \widetilde{\mathrm{II}}^{a}_{\mathrm{o}} - \widetilde{\mathrm{II}}^{1\alpha}_{\mathrm{o}} + \widetilde{\mathrm{II}}^{1\alpha}_{\mathrm{e}} - \widetilde{\mathrm{II}}^{\beta}_{\mathrm{e}},$$
$$\delta_1(\widetilde{\mathrm{I}}^1_{\mathrm{e}}) = -\widetilde{\mathrm{II}}^{01}_{\mathrm{o}} + \widetilde{\mathrm{II}}^{01}_{\mathrm{e}} + \widetilde{\mathrm{II}}^{a}_{\mathrm{e}} - \widetilde{\mathrm{II}}^{1\alpha}_{\mathrm{o}} + \widetilde{\mathrm{II}}^{1\alpha}_{\mathrm{e}} + \widetilde{\mathrm{II}}^{\beta}_{\mathrm{o}},$$
$$\delta_1(\widetilde{\mathrm{I}}^\alpha_{\mathrm{o}}) = \widetilde{\mathrm{II}}^{0\alpha}_{\mathrm{o}} - \widetilde{\mathrm{II}}^{0\alpha}_{\mathrm{e}} + \widetilde{\mathrm{II}}^{1\alpha}_{\mathrm{o}} - \widetilde{\mathrm{II}}^{1\alpha}_{\mathrm{e}} + \widetilde{\mathrm{II}}^{\beta}_{\mathrm{e}} - \widetilde{\mathrm{II}}^{\gamma}_{\mathrm{o}},$$
$$\delta_1(\widetilde{\mathrm{I}}^\alpha_{\mathrm{e}}) = \widetilde{\mathrm{II}}^{0\alpha}_{\mathrm{o}} - \widetilde{\mathrm{II}}^{0\alpha}_{\mathrm{e}} + \widetilde{\mathrm{II}}^{1\alpha}_{\mathrm{o}} - \widetilde{\mathrm{II}}^{1\alpha}_{\mathrm{e}} - \widetilde{\mathrm{II}}^{\beta}_{\mathrm{o}} + \widetilde{\mathrm{II}}^{\gamma}_{\mathrm{e}}.$$

Then by the same argument as before, we see that

$$||\widetilde{\mathrm{I}}^\alpha_{\mathrm{o}}(f)|| - ||\widetilde{\mathrm{I}}^\alpha_{\mathrm{e}}(f)|| + ||\widetilde{\mathrm{I}}^1_{\mathrm{o}}(f)|| - ||\widetilde{\mathrm{I}}^1_{\mathrm{e}}(f)|| \quad \text{and} \quad -||\widetilde{\mathrm{I}}^0_{\mathrm{o}}(f)|| + ||\widetilde{\mathrm{I}}^0_{\mathrm{e}}(f)|| \tag{7.3}$$

are cobordism invariants of f in an appropriate sense. Furthermore, we see that the algebraic number of cusps of F given by

$$||\widetilde{\mathrm{II}}^a_{\mathrm{o}}(F)|| + ||\widetilde{\mathrm{II}}^a_{\mathrm{e}}(F)|| + ||\widetilde{\mathrm{II}}^\gamma_{\mathrm{o}}(F)|| + ||\widetilde{\mathrm{II}}^\gamma_{\mathrm{e}}(F)||$$

is equal to both of the integers (7.3).

Now, let $g : (\mathbf{R}^3, 0) \to (\mathbf{R}^2, 0)$ be a C^∞ map germ which is generic in the sense of [1]. Then for any sufficiently small positive real numbers ε and δ, the upper bound of δ depending on g and the upper bound of ε depending on δ and g, we have

(1) $D^3_\delta \cap g^{-1}(S^1_\varepsilon)$ is a C^∞ manifold, in general with boundary,
(2) $g_\partial = g|_{D^3_\delta \cap g^{-1}(S^1_\varepsilon)} : D^3_\delta \cap g^{-1}(S^1_\varepsilon) \to S^1_\varepsilon$ is C^∞ stable, and
(3) $g|_{\partial D^3_\delta \cap g^{-1}(D^2_\varepsilon)} : \partial D^3_\delta \cap g^{-1}(D^2_\varepsilon) \to D^2_\varepsilon$ is a submersion,

where D^3_δ (or D^2_ε) denotes the 3-dimensional ball in $\mathbf{R}^3$ (resp. 2-dimensional disk in $\mathbf{R}^2$) with radius δ (resp. ε) centered at the origin. Then by applying the above observations to the map

$$g|_{D^3_\delta \cap g^{-1}(D^2_\varepsilon)} : D^3_\delta \cap g^{-1}(D^2_\varepsilon) \to D^2,$$

we get the following.

Theorem 7.7. *Let $g : (\mathbf{R}^3, 0) \to (\mathbf{R}^2, 0)$ be a generic C^∞ map germ. Then the algebraic number of cusps of a C^∞ stable perturbation $\widetilde{g}$ of a representative of g is an invariant of the topological $\mathcal{A}_+$-equivalence class of g, and is equal to*

$$-||\widetilde{\mathrm{I}}^0_{\mathrm{o}}(g_\partial)|| + ||\widetilde{\mathrm{I}}^0_{\mathrm{e}}(g_\partial)|| = ||\widetilde{\mathrm{I}}^\alpha_{\mathrm{o}}(g_\partial)|| - ||\widetilde{\mathrm{I}}^\alpha_{\mathrm{e}}(g_\partial)|| + ||\widetilde{\mathrm{I}}^1_{\mathrm{o}}(g_\partial)|| - ||\widetilde{\mathrm{I}}^1_{\mathrm{e}}(g_\partial)||,$$

where $0 < \varepsilon << \delta$ are sufficiently small. In particular, the absolute value of the algebraic number of cusps of $\widetilde{g}$ is an invariant of the topological $\mathcal{A}$-equivalence class of g.

It would be an interesting problem to find a formula expressing the algebraic number of cusps of a C^∞ stable perturbation in algebraic terms.

8. Example 2 – Euler characteristic formula

Let f be a stable Morse function on a closed surface M. It is known that the Euler characteristic $\chi(M)$ has the same parity as the number of critical points of f.

By the formulae

$$\delta_0(\widetilde{\mathbf{0}}_{\mathrm{o}}) = \widetilde{\mathrm{I}}^0_{\mathrm{o}} + \widetilde{\mathrm{I}}^0_{\mathrm{e}} + \widetilde{\mathrm{I}}^1_{\mathrm{o}} + \widetilde{\mathrm{I}}^1_{\mathrm{e}} \quad \text{and} \quad \delta_0(\widetilde{\mathbf{0}}_{\mathrm{e}}) = \widetilde{\mathrm{I}}^0_{\mathrm{o}} + \widetilde{\mathrm{I}}^0_{\mathrm{e}} + \widetilde{\mathrm{I}}^1_{\mathrm{o}} + \widetilde{\mathrm{I}}^1_{\mathrm{e}}$$

(see (6.4)), we see that the total number of singular fibers of types $\widetilde{\mathrm{I}}^0$ and $\widetilde{\mathrm{I}}^1$ is always even. Thus we get the following.

Theorem 8.1. *Let $f : M \to \mathbf{R}$ be a stable Morse function on a closed surface M. Then the Euler characteristic $\chi(M)$ of M has the same parity as the number of singular fibers of type $\widetilde{\mathrm{I}}^2$.*

From the viewpoint of the theory of universal complex of singular fibers, the above result can be interpreted as follows.

Let $\tau(2, 1)$ (or $\tau(3, 2)$) be the set of singular fibers appearing for proper stable Morse functions on surfaces (resp. for proper C^∞ stable maps of 3-manifolds into surfaces).

Furthermore, let $\varrho_{2,1}(2)$ (or $\varrho_{3,2}(2)$) be the equivalence relation modulo two circle components. Then for the universal complex $\mathcal{C}^*(\tau(3,2),\varrho_{3,2})$, we have

$$\begin{aligned}
\delta_0(\widetilde{\mathbf{0}}_{\mathrm{o}}) &= \widetilde{\mathrm{I}}^0_{\mathrm{o}} + \widetilde{\mathrm{I}}^0_{\mathrm{e}} + \widetilde{\mathrm{I}}^1_{\mathrm{o}} + \widetilde{\mathrm{I}}^1_{\mathrm{e}},\\
\delta_0(\widetilde{\mathbf{0}}_{\mathrm{e}}) &= \widetilde{\mathrm{I}}^0_{\mathrm{o}} + \widetilde{\mathrm{I}}^0_{\mathrm{e}} + \widetilde{\mathrm{I}}^1_{\mathrm{o}} + \widetilde{\mathrm{I}}^1_{\mathrm{e}},\\
\delta_1(\widetilde{\mathrm{I}}^0_{\mathrm{o}}) &= \widetilde{\mathrm{II}}^{01}_{\mathrm{o}} + \widetilde{\mathrm{II}}^{01}_{\mathrm{e}} + \widetilde{\mathrm{II}}^{a}_{\mathrm{e}},\\
\delta_1(\widetilde{\mathrm{I}}^0_{\mathrm{e}}) &= \widetilde{\mathrm{II}}^{01}_{\mathrm{o}} + \widetilde{\mathrm{II}}^{01}_{\mathrm{e}} + \widetilde{\mathrm{II}}^{a}_{\mathrm{o}},\\
\delta_1(\widetilde{\mathrm{I}}^1_{\mathrm{o}}) &= \widetilde{\mathrm{II}}^{01}_{\mathrm{o}} + \widetilde{\mathrm{II}}^{01}_{\mathrm{e}} + \widetilde{\mathrm{II}}^{a}_{\mathrm{o}},\\
\delta_1(\widetilde{\mathrm{I}}^1_{\mathrm{e}}) &= \widetilde{\mathrm{II}}^{01}_{\mathrm{o}} + \widetilde{\mathrm{II}}^{01}_{\mathrm{e}} + \widetilde{\mathrm{II}}^{a}_{\mathrm{e}},\\
\delta_1(\widetilde{\mathrm{I}}^2_{\mathrm{o}}) &= \widetilde{\mathrm{II}}^{02}_{\mathrm{o}} + \widetilde{\mathrm{II}}^{02}_{\mathrm{e}} + \widetilde{\mathrm{II}}^{12}_{\mathrm{o}} + \widetilde{\mathrm{II}}^{12}_{\mathrm{e}} + \widetilde{\mathrm{II}}^{6}_{\mathrm{o}} + \widetilde{\mathrm{II}}^{6}_{\mathrm{e}},\\
\delta_1(\widetilde{\mathrm{I}}^2_{\mathrm{e}}) &= \widetilde{\mathrm{II}}^{02}_{\mathrm{o}} + \widetilde{\mathrm{II}}^{02}_{\mathrm{e}} + \widetilde{\mathrm{II}}^{12}_{\mathrm{o}} + \widetilde{\mathrm{II}}^{12}_{\mathrm{e}} + \widetilde{\mathrm{II}}^{6}_{\mathrm{o}} + \widetilde{\mathrm{II}}^{6}_{\mathrm{e}}.
\end{aligned}$$

(Compare this with (6.4).)

By a straightforward calculation, we see that $H^1(\tau(3,2),\varrho_{3,2}) \cong \mathbf{Z}_2 \oplus \mathbf{Z}_2$, which is generated by

$$\widehat{\alpha}'_1 = [\widetilde{\mathrm{I}}^0_{\mathrm{o}} + \widetilde{\mathrm{I}}^1_{\mathrm{e}}] = [\widetilde{\mathrm{I}}^0_{\mathrm{e}} + \widetilde{\mathrm{I}}^1_{\mathrm{o}}] \quad \text{and} \quad \widehat{\alpha}'_2 = [\widetilde{\mathrm{I}}^2_{\mathrm{o}} + \widetilde{\mathrm{I}}^2_{\mathrm{e}}].$$

After the suspension, $\widehat{\alpha}'_1$ gives a trivial cobordism invariant, while $\widehat{\alpha}'_2$ gives the cobordism invariant which counts the number of singular fibers of type $\widetilde{\mathrm{I}}^2$ modulo 2. In other words, this shows the cobordism invariance of the parity of the number of $\widetilde{\mathrm{I}}^2$-type singular fibers.

Let $\mathfrak{N}_2$ denotes the unoriented cobordism group of 2-dimensional manifolds. Let us define the map

$$\Phi : \mathfrak{N}_2 \to \mathbf{Z}_2$$

by associating to each cobordism class of a closed 2-manifold M to the parity of the number of singular fibers of $\widetilde{\mathrm{I}}^2$-type of an arbitrary stable Morse function on M. By the above argument, this is a well-defined homomorphism.

Furthermore, let

$$\Phi' : \mathfrak{N}_2 \to \mathbf{Z}_2$$

be the map which sends a cobordism class of a closed 2-manifold M to the parity of the Euler characteristic $\chi(M)$. It is known that this defines an isomorphism of groups.

Now, it is easy to construct a stable Morse function on $\mathbf{R}P^2$ which has exactly three critical points and which has two singular fibers of $\widetilde{\mathrm{I}}^0$-type and a singular fiber of $\widetilde{\mathrm{I}}^2$-type. Therefore, the homomorphism Φ and Φ' coincides on the generator of $\mathfrak{N}_2 \cong \mathbf{Z}_2$. Hence, we have $\Phi = \Phi'$. In other words, we have the following.

Proposition 8.2. *Let $f : M \to \mathbf{R}$ be a stable Morse function on a closed surface M. Then we have*

$$\varphi_f(s_1^*\widehat{\alpha}'_2) = f_! w_2(M) \in H^1_c(\mathbf{R};\mathbf{Z}_2) = \mathbf{Z}_2,$$

where $w_2(M) \in H^2(M;\mathbf{Z}_2)$ is the second Stiefel-Whitney class of M, $f_! : H^2(M;\mathbf{Z}_2) = H^2_c(M;\mathbf{Z}_2) \to H^1_c(\mathbf{R};\mathbf{Z}_2)$ is the Gysin homomorphism induced by f, and H^*_c denotes the cohomology of compact support.

Corollary 8.3. *Let $f : M \to N$ be a proper C^∞ map of an n-dimensional manifold into an $(n-1)$-dimensional manifold which is Thom-Boardman generic. Then the* *closure of* $\widetilde{\mathrm{I}}^2(f)$ *forms a* $\mathbf{Z}_2$*-cycle of closed support in* N*, and the Poincaré dual to the* $\mathbf{Z}_2$*-homology class represented by it coincides with*

$$f_! w_2(M) + (f_! w_1(M)) w_1(N)$$

in $H^1(N; \mathbf{Z}_2)$ *(when* M *is closed, in* $H^1_c(N; \mathbf{Z}_2)$*).*

9. Example 3 – Signature formula

Let us consider the singular fibers of proper C^∞ stable maps of oriented 5-manifolds into 4-manifolds. For κ with $3 \leq \kappa \leq 4$, let C^κ be the free $\mathbf{Z}$-module generated by the C^0 equivalence classes modulo regular circle components of *chiral* singular fibers of codimension κ. Note that $\operatorname{rank} C^3 = 3$ and $\operatorname{rank} C^4 = 14$ according to [15, Proposition 6.1]. Since there exist no chiral singular fibers of codimension $\kappa \neq 3, 4$, we put $C^\kappa = 0$ for $\kappa \neq 3, 4$.

We call the resulting cochain complex $(C^\kappa, \delta_\kappa)_\kappa$ the *universal complex of chiral singular fibers* for proper C^∞ stable maps of oriented 5-manifolds into 4-manifolds. Note that its unique cohomology group that makes sense is its third cohomology group, and is nothing but the kernel of the coboundary homomorphism δ_3. Then we get the following.

Proposition 9.1. *The* 3*-dimensional cohomology group of the universal complex of chiral singular fibers for proper* C^∞ *stable maps of oriented* 5*-manifolds into* 4*-manifolds is an infinite cyclic group generated by the* C^0 *equivalence class modulo regular circle components of* III^8*-type fibers as depicted in Fig.* 2*.*

According to the above proposition, the 3-dimensional cohomology class represented by the cocycle III^8 gives an oriented cobordism invariant of the source closed oriented 4-manifold. Then by an argument similar to that in the previous section, we can prove the following [15].

Theorem 9.2. *Let* M *be a closed oriented* 4*-manifold and* N *a* 3*-manifold. Then, for any* C^∞ *stable map* $f : M \to N$*, the algebraic number of* III^8*-type fibers of* f *coincides with the signature of* M*.*

In other words, the corresponding cohomology class of the universal complex gives a complete invariant of the oriented bordism class of a C^∞ stable map of a closed oriented 4-manifold into $\mathbf{R}^3$.

We also see that the fiber which satisfies the property as in Theorem 9.2 should necessarily be the fiber of type III^8.

As in Corollary 8.3, we can prove the following.

Corollary 9.3. *Let* $f : M \to N$ *be a proper* C^∞ *map of an* n*-dimensional manifold into an* $(n-1)$*-dimensional manifold which is Thom-Boardman generic. Furthermore, we assume that* f *is an* <u>*oriented map*</u>*. Then the closure of* $\mathrm{III}^8(f)$ *forms a cycle of closed support in* <u>*twisted coefficients*</u> *in* N*, and three times the Poincaré dual to the* $\mathbf{Z}$*-homology class represented by it coincides with* $f_! p_1(M)$ *modulo torsion in* $H^3(N; \mathbf{Z})$ *(when* M *is closed, in* $H^3_c(N; \mathbf{Z})$*), where* $p_1(M) \in H^4(M; \mathbf{Z})$ *denotes the first Pontrjagin class of* M*.*

Note that the homology class represented by the closure of $\mathrm{III}^8(f)$ lies in the $(n-4)$-th homology group of N of closed support <u>with twisted coefficients</u>.

In the following, for a multi-index $I = (i_1, i_2, \ldots, i_n)$, we put

$$w_I = w_1^{i_1} w_2^{i_2} \cdots w_n^{i_n} \quad \text{and} \quad p_I = p_1^{i_1} p_2^{i_2} \cdots p_n^{i_n}.$$

Corollaries 8.3 and 9.3 suggest the following.

Conjecture 9.4. (1) *For any cohomology class α of the universal complex*

$$\mathcal{C}^*(\tau(n+1,p+1),\varrho_{n+1,p+1})$$

of singular fibers, there exists a universal polynomial $P_\alpha(\widetilde{w}_I, w_j)$ such that for any proper τ-map $f : M \to N$ of an n-dimensional manifold into a p-dimensional manifold, $\varphi_f(s^*\alpha)$ coincides with $P_\alpha(f_! w_I(M), w_j(N))$ in $H^*(N;\mathbf{Z}_2)$ (when M is closed, in $H^*_c(N;\mathbf{Z}_2)$).

(2) *For any cohomology class α of the universal complex of chiral singular fibers for proper oriented τ-maps of $(n+1)$-dimensional manifolds into $(p+1)$-dimensional manifolds, there exists a universal polynomial $P_\alpha(\widetilde{p}_I, p_j)$ such that for any proper oriented τ-map $f : M \to N$ of an n-dimensional manifold into a p-dimensional manifold, $\varphi_f(s^*\alpha)$ coincides with $P_\alpha(f_! p_I(M), p_j(N))$ in $H^*(N;\mathbf{Z})$ modulo torsion (when M is closed, in $H^*_c(N;\mathbf{Z})$).*

Note that the cohomology classes of the forms

$$P_\alpha(f_! w_I(M), w_j(N)) \quad \text{or} \quad P_\alpha(f_! p_I(M), w_j(N))$$

are easily seen to be bordism invariants (or oriented bordism invariants for the latter case) for (oriented) maps f into a fixed target manifold N.

10. Application to surface bundles

In order to consider characteristic classes of surface bundles,[9] let us first consider the class Γ of Morse functions $h : S \to \mathbf{R}$ with the following properties, where S is a closed connected and orientable surface.

1. Exactly three of the critical points of h have the same value c, and the values of the other critical points are different from each other and are not equal to c.
2. The singular fiber of h over c is a III^8-type fiber (possibly with several regular circle components), where a singular fiber of type III^8 refers to a singular fiber as depicted in Fig. 7.

Let S be as above and $\pi : E \to B$ a C^∞ fiber bundle with fiber S. For a generic function $f : E \to \mathbf{R}$ and a point $y \in B$, let us put $f_y = f|_{\pi^{-1}(y)} : \pi^{-1}(y) \cong S \to \mathbf{R}$. Then, let us denote by $\Gamma(f)$ the set of points y in B such that f_y belongs to Γ. Note that this does not depend on a particular diffeomorphism $\pi^{-1}(y) \cong S$. Note also that if f is generic enough, then $\Gamma(f)$ is a codimension two submanifold of B.

By using the chirality of the singular fiber of type III^8, we can give a co-orientation to $\Gamma(f)$, provided that the surface bundle $\pi : E \to B$ is *oriented*. Furthermore, by using the theory of universal complex of singular fibers, we can show that the closure $\overline{\Gamma(f)}$ defines a codimension two cycle of closed support in twisted coefficients in B. Let us denote by $[\overline{\Gamma(f)}]^* \in H^2(B;\mathbf{Z})$ the Poincaré dual to this homology class.

Theorem 10.1. *The cohomology class $[\overline{\Gamma(f)}]^* \in H^2(B;\mathbf{Z})$ coincides with the first Miller-Morita-Mumford characteristic class of the surface bundle $\pi : E \to B$ modulo torsion.*

For the Miller-Morita-Mumford characteristic classes of surface bundles, the reader is referred to [10], for example.

Note that the above theorem is an analogy of the results obtained by Kazarian [7, 8] about the Euler classes of oriented circle bundles.

[9] I hope that the details of this section will be presented by Takahiro Yamamoto.

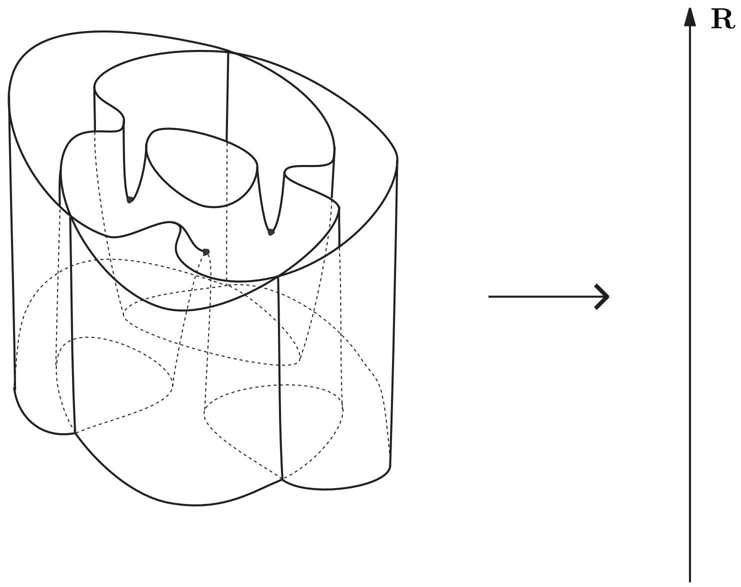

FIGURE 7. A III^8-type fiber of a Morse function on a surface

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