Topology of Definite Fold Singularities

Osamu Saeki
Kyushu University

February 16, 2011

§1. Special Generic Maps

Morse function

M^m : compact C^∞ manifold without boundary

Morse function

M^m : compact C^∞ manifold without boundary

Definition 1.1 A **Morse function** $M^m \to \mathbf{R}$ is a C^∞ function with each critical point being of the form

$$(x_1, x_2, \ldots, x_m) \mapsto \pm x_1^2 \pm x_2^2 \pm \cdots \pm x_m^2 + c.$$

Number of negative signs is called the **index** of a critical point.

M^m : compact C^∞ manifold without boundary

Definition 1.1 A **Morse function** $M^m \to \mathbf{R}$ is a C^∞ function with each critical point being of the form

$$(x_1, x_2, \ldots, x_m) \mapsto \pm x_1^2 \pm x_2^2 \pm \cdots \pm x_m^2 + c.$$

Number of negative signs is called the **index** of a critical point.

$$\begin{cases} \text{local minimum} & \Longleftrightarrow \text{index } 0 \\ \text{local maximum} & \Longleftrightarrow \text{index } m \end{cases}$$

Morse function

M^m : compact C^∞ manifold without boundary

Definition 1.1 A **Morse function** $M^m \to \mathbf{R}$ is a C^∞ function with each critical point being of the form

$$(x_1, x_2, \ldots, x_m) \mapsto \pm x_1^2 \pm x_2^2 \pm \cdots \pm x_m^2 + c.$$

Number of negative signs is called the **index** of a critical point.

$$\begin{cases} \text{local minimum} & \iff \text{index } 0 \\ \text{local maximum} & \iff \text{index } m \end{cases}$$

They always appear if M^m is compact.

Reeb's theorem

Theorem 1.2 (Reeb, Smale, Cerf et al)

M^m: *compact* C^∞ *manifold without boundary*

$\exists$*Morse function* $M^m \to \mathbf{R}$ *with only critical points of index* 0 *or* m

$\Longleftrightarrow$

(1) $M^m \approx S^m$ *(homeomorphic)* $(m \neq 4)$

(2) $M^m \cong S^m$ *(diffeomorphic)* $(m = 4)$

Reeb's theorem

Theorem 1.2 (Reeb, Smale, Cerf et al)

M^m: compact C^∞ manifold without boundary

$\exists$Morse function $M^m \to \mathbf{R}$ with only critical points of index 0 or m

$\Longleftrightarrow$

(1) $M^m \approx S^m$ *(homeomorphic)* $(m \neq 4)$

(2) $M^m \cong S^m$ *(diffeomorphic)* $(m = 4)$

Remark 1.3

Generalized Poincaré conjecture is still open in dimension 4 in the C^∞ category.

Fold singularity

Definition 1.4 A singularity of a C^∞ map $M^m \to N^n$, $m \geq n$, that has the normal form

$$(x_1, x_2, \ldots, x_m) \mapsto (x_1, x_2, \ldots, x_{n-1}, \pm x_n^2 \pm x_{n+1}^2 \pm \cdots \pm x_m^2)$$

is called a **fold singularity**.

Fold singularity

Definition 1.4 A singularity of a C^∞ map $M^m \to N^n$, $m \geq n$, that has the normal form

$$(x_1, x_2, \ldots, x_m) \mapsto (x_1, x_2, \ldots, x_{n-1}, \pm x_n^2 \pm x_{n+1}^2 \pm \cdots \pm x_m^2)$$

is called a **fold singularity**.

It is a **definite fold singularity** if all the signs are the same.

Fold singularity

Definition 1.4 A singularity of a C^∞ map $M^m \to N^n$, $m \geq n$, that has the normal form

$$(x_1, x_2, \ldots, x_m) \mapsto (x_1, x_2, \ldots, x_{n-1}, \pm x_n^2 \pm x_{n+1}^2 \pm \cdots \pm x_m^2)$$

is called a **fold singularity**.

It is a **definite fold singularity** if all the signs are the same.

Definition 1.5 $f : M^m \to N^n$ is a **special generic map** (**SGM**, for short) if it has **only definite fold singularities**.

Fold singularity

Definition 1.4 A singularity of a C^∞ map $M^m \to N^n$, $m \geq n$, that has the normal form

$$(x_1, x_2, \dots, x_m) \mapsto (x_1, x_2, \dots, x_{n-1}, \pm x_n^2 \pm x_{n+1}^2 \pm \cdots \pm x_m^2)$$

is called a **fold singularity**.

It is a **definite fold singularity** if all the signs are the same.

Definition 1.5 $f : M^m \to N^n$ is a **special generic map** (**SGM**, for short) if it has **only definite fold singularities**.

Example 1.6 A function $f : M^m \to \mathbf{R}$ is a SGM iff it is a Morse function with only critical points of index 0 or m

Examples

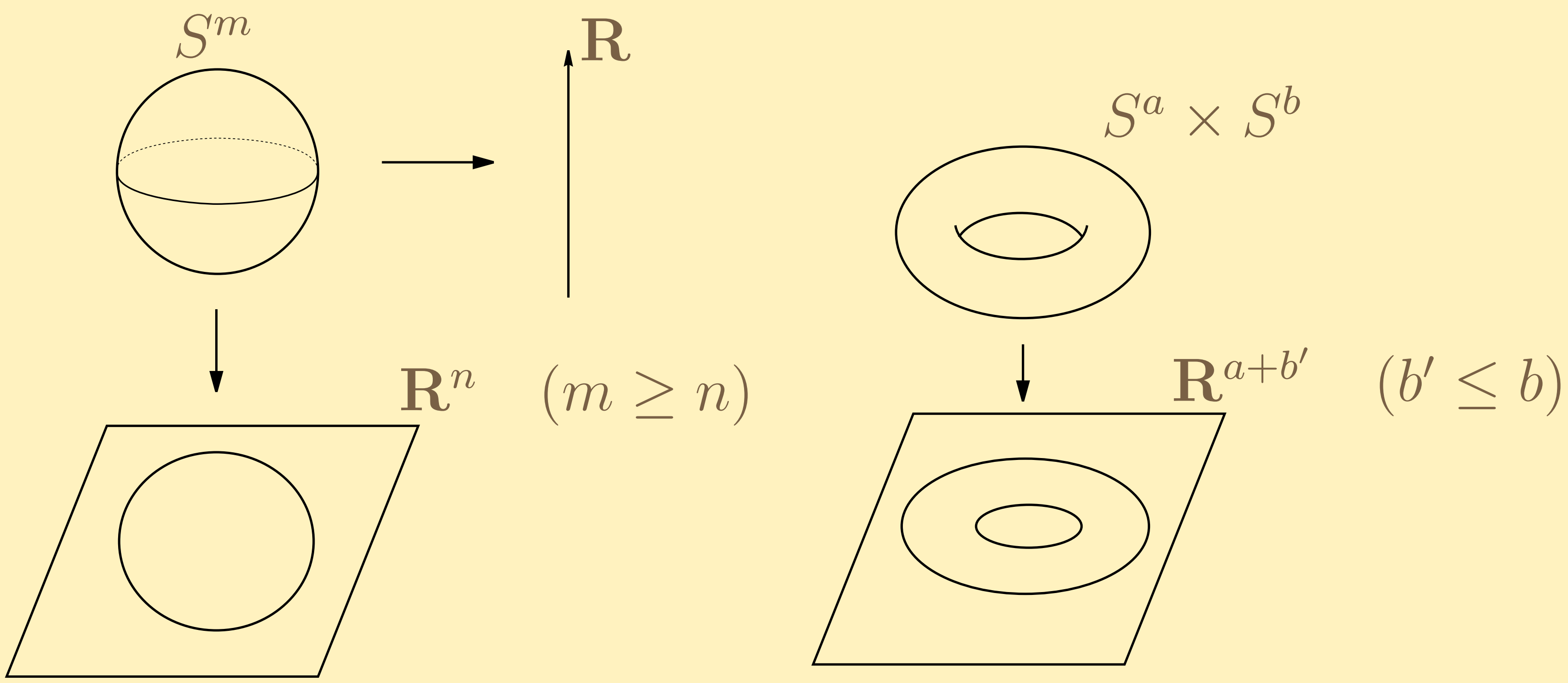


Figure 1: Examples of special generic maps

Definition 1.7 M^m: compact

$$\mathcal{S}(M^m) = \{n \in \mathbf{Z} \,|\, 1 \leq n \leq m,\ \exists f : M^m \to \mathbf{R}^n \ \ \textbf{SGM}\}$$

Definition 1.7 M^m: compact

$$\mathcal{S}(M^m) = \{n \in \mathbf{Z} \,|\, 1 \leq n \leq m, \, \exists f : M^m \to \mathbf{R}^n \;\; \textbf{SGM}\}$$

This is a diffeomorphism invariant of M^m.

$$M_0 \cong M_1 \quad \text{(diffeomorphic)} \Longrightarrow \mathcal{S}(M_0) = \mathcal{S}(M_1)$$

An Invariant

Definition 1.7 M^m: compact

$$\mathcal{S}(M^m) = \{n \in \mathbf{Z} \,|\, 1 \le n \le m,\ \exists f : M^m \to \mathbf{R}^n \ \ \textbf{SGM}\}$$

This is a diffeomorphism invariant of M^m.

$$M_0 \cong M_1 \quad \text{(diffeomorphic)} \Longrightarrow \mathcal{S}(M_0) = \mathcal{S}(M_1)$$

Example 1.8

(1) $\mathcal{S}(S^m) = \{1, 2, \ldots, m\}$

An Invariant

Definition 1.7 M^m: compact

$$\mathcal{S}(M^m) = \{n \in \mathbf{Z} \,|\, 1 \leq n \leq m,\ \exists f : M^m \to \mathbf{R}^n \ \ \textbf{SGM}\}$$

This is a diffeomorphism invariant of M^m.

$$M_0 \cong M_1 \quad \text{(diffeomorphic)} \Longrightarrow \mathcal{S}(M_0) = \mathcal{S}(M_1)$$

Example 1.8

(1) $\mathcal{S}(S^m) = \{1, 2, \ldots, m\}$

(2) $\mathcal{S}(S^a \times S^b) = \{a+1, a+2, \ldots, a+b\} \quad (a \leq b)$

Characterization of the sphere

Theorem 1.9 (S., 1993)
M^m: compact C^∞ manifold

$$\mathcal{S}(M^m) = \{1, 2, \ldots, m\} \Longleftrightarrow M^m \cong S^m \quad \text{(diffeomorphic)}$$

Characterization of the sphere

Theorem 1.9 (S., 1993)
M^m: compact C^∞ manifold

$$\mathcal{S}(M^m) = \{1, 2, \ldots, m\} \Longleftrightarrow M^m \cong S^m \quad \text{(diffeomorphic)}$$

SGMs can detect the standard differentiable structure on a sphere!

Characterization of the sphere

Theorem 1.9 (S., 1993)
M^m: compact C^∞ manifold

$$\mathcal{S}(M^m) = \{1, 2, \ldots, m\} \iff M^m \cong S^m \quad \text{(diffeomorphic)}$$

SGMs can detect the standard differentiable structure on a sphere!

Example 1.10
Σ^7 : Milnor's exotic 7-sphere
$\{1, 2, 7\} \subset \mathcal{S}(\Sigma^7) \subset \{1, 2, 3, 7\}$

Definition 1.11 $f_i : M_i^m \to \mathbf{R}^n$ SGMs, $i = 0, 1$, are **cobordant** if

Cobordism

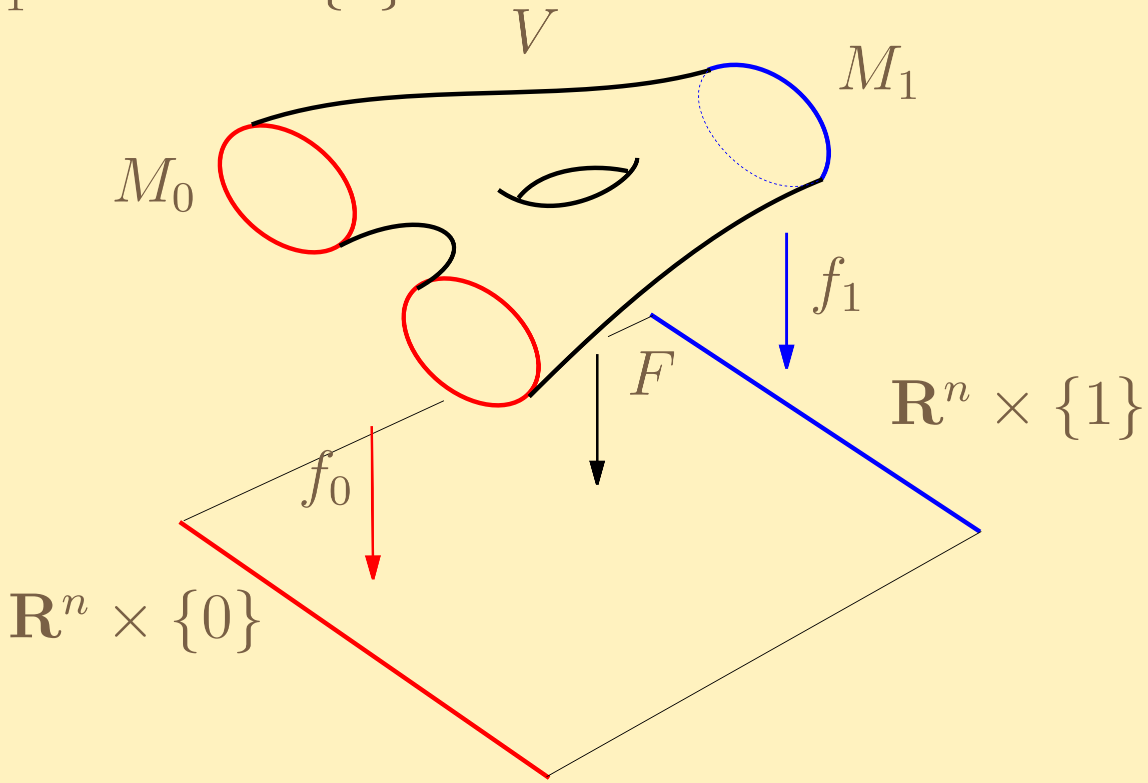
Definition 1.11 $f_i : M_i^m \to \mathbf{R}^n$ SGMs, $i = 0, 1$, are **cobordant** if
$\exists V^{m+1}$: compact manifold with $\partial V^{m+1} = M_0 \cup M_1$,
$\exists F : V^{m+1} \to \mathbf{R}^n \times [0, 1]$ SGM such that

Definition 1.11 $f_i : M_i^m \to \mathbf{R}^n$ SGMs, $i = 0, 1$, are **cobordant** if
$\exists V^{m+1}$: compact manifold with $\partial V^{m+1} = M_0 \cup M_1$,
$\exists F : V^{m+1} \to \mathbf{R}^n \times [0, 1]$ SGM such that

$$\begin{cases} F|_{M_0} = f_0 : M_0^m \to \mathbf{R}^n \times \{0\} \\ F|_{M_1} = f_1 : M_1^m \to \mathbf{R}^n \times \{1\} \end{cases}$$

Definition 1.11 $f_i : M_i^m \to \mathbf{R}^n$ SGMs, $i = 0, 1$, are **cobordant** if
$\exists V^{m+1}$: compact manifold with $\partial V^{m+1} = M_0 \cup M_1$,
$\exists F : V^{m+1} \to \mathbf{R}^n \times [0,1]$ SGM such that

$$\begin{cases} F|_{M_0} = f_0 : M_0^m \to \mathbf{R}^n \times \{0\} \\ F|_{M_1} = f_1 : M_1^m \to \mathbf{R}^n \times \{1\} \end{cases}$$



Set of cobordism classes of all SGMs of m-dim. compact manifolds into $\mathbf{R}^n$ forms an **abelian group**, denoted by $\Gamma(m, n)$.

Groups of SGMs

Set of cobordism classes of all SGMs of m-dim. compact manifolds into $\mathbf{R}^n$ forms an **abelian group**, denoted by $\Gamma(m, n)$.
Addition $\longleftrightarrow$ Disjoint union

Groups of SGMs

Set of cobordism classes of all SGMs of m-dim. compact manifolds into $\mathbf{R}^n$ forms an **abelian group**, denoted by $\Gamma(m,n)$.
Addition $\longleftrightarrow$ Disjoint union

$\widetilde{\Gamma}(m,n)$: oriented version

Groups of SGMs

Set of cobordism classes of all SGMs of m-dim. compact manifolds into $\mathbf{R}^n$ forms an **abelian group**, denoted by $\Gamma(m, n)$.
Addition $\longleftrightarrow$ Disjoint union

$\widetilde{\Gamma}(m, n)$: oriented version

Θ_m : group of differentiable structures on an oriented m-sphere
Addition $\longleftrightarrow$ Connected sum

Groups of SGMs

Set of cobordism classes of all SGMs of m-dim. compact manifolds into $\mathbf{R}^n$ forms an **abelian group**, denoted by $\Gamma(m, n)$.
Addition $\longleftrightarrow$ Disjoint union

$\widetilde{\Gamma}(m, n)$: oriented version

Θ_m : group of differentiable structures on an oriented m-sphere
Addition $\longleftrightarrow$ Connected sum

Theorem 1.12 (S., 2002) $m \geq 6$
$\widetilde{\Gamma}(m, 1) \cong \Theta_m$
$\Gamma(m, 1) \cong \Theta_m \otimes \mathbf{Z}_2$

Set of cobordism classes of all SGMs of m-dim. compact manifolds into $\mathbf{R}^n$ forms an **abelian group**, denoted by $\Gamma(m, n)$.
Addition $\longleftrightarrow$ Disjoint union

$\widetilde{\Gamma}(m, n)$: oriented version

Θ_m : group of differentiable structures on an oriented m-sphere
Addition $\longleftrightarrow$ Connected sum

Theorem 1.12 (S., 2002) $m \geq 6$
$\widetilde{\Gamma}(m, 1) \cong \Theta_m$
$\Gamma(m, 1) \cong \Theta_m \otimes \mathbf{Z}_2$

Special generic maps
$\updownarrow$ closely related !
Differentiable structures

§2. 4-Dimensional Case

Exotic 4-manifolds

Theorem 2.1 (Sakuma-S., 1990's)
$\exists(M_1^4, M_2^4)$: *pair of compact* C^∞ 4*-manifolds such that*
$M_1^4 \approx M_2^4$ *(homeomorphic),*
$\exists f_1 : M_1^4 \to \mathbf{R}^3$ *SGM,*
$\not\exists f_2 : M_2^4 \to \mathbf{R}^3$ *SGM.*
In fact, there are infinitely many such pairs.

Exotic 4-manifolds

Theorem 2.1 (Sakuma-S., 1990's)
$\exists(M_1^4, M_2^4)$: *pair of compact* C^∞ 4*-manifolds such that*
$M_1^4 \approx M_2^4$ *(homeomorphic),*
$\exists f_1 : M_1^4 \to \mathbf{R}^3$ *SGM,*
$\nexists f_2 : M_2^4 \to \mathbf{R}^3$ *SGM.*
In fact, there are infinitely many such pairs.

$M_1^4 \not\cong M_2^4$ non-diffeomorphic

Theorem 2.1 (Sakuma-S., 1990's)
$\exists (M_1^4, M_2^4)$: *pair of compact* C^∞ 4*-manifolds such that*
$M_1^4 \approx M_2^4$ *(homeomorphic),*
$\exists f_1 : M_1^4 \to \mathbf{R}^3$ *SGM,*
$\nexists f_2 : M_2^4 \to \mathbf{R}^3$ *SGM.*
In fact, there are infinitely many such pairs.

$$M_1^4 \not\cong M_2^4 \quad \text{non-diffeomorphic}$$

SGMs can detect distinct differentiable structures on a given topological manifold.

Compact 1-connected 4-manifolds

Theorem 2.2 (S. (1993) + $\mathbf{3}$-dim. Poincaré Conj.)
M^4: *compact simply connected* C^∞ 4*-manifold*
$\exists f : M^4 \to \mathbf{R}^3$ *special generic map*
$\Longleftrightarrow M^4 \cong \sharp^k(S^2 \times S^2)$ *or* $\sharp^k(S^2 \tilde{\times} S^2)$ *(diffeomorphic)*

Theorem 2.2 (S. (1993) + 3-dim. Poincaré Conj.)
M^4: compact simply connected C^∞ 4-manifold
$\exists f : M^4 \to \mathbf{R}^3$ *special generic map*
$\Longleftrightarrow M^4 \cong \sharp^k(S^2 \times S^2)$ *or* $\sharp^k(S^2 \mathbin{\widetilde{\times}} S^2)$ *(diffeomorphic)*

Corollary 2.3
M^4: C^∞ 4-manifold
$M^4 \approx \sharp^k(S^2 \times S^2)$ *or* $\sharp^k(S^2 \mathbin{\widetilde{\times}} S^2)$ *(homeomorphic)*
$\exists f : M^4 \to \mathbf{R}^3$ *special generic map*
$\Longleftrightarrow M^4 \cong \sharp^k(S^2 \times S^2)$ *or* $\sharp^k(S^2 \mathbin{\widetilde{\times}} S^2)$ *(diffeomorphic)*

Remarks

Remark 2.4 Smooth structures on $\sharp^k(S^2 \times S^2)$ are not unique.
In fact, there are *infinitely many* such structures if k is a sufficiently big odd integer (Jongil Park, 2002).

Remarks

Remark 2.4 Smooth structures on $\sharp^k(S^2 \times S^2)$ are not unique.
In fact, there are *infinitely many* such structures if k is a sufficiently big odd integer (Jongil Park, 2002).

Remark 2.5 M_1^4, M_2^4: compact orientable C^∞ 4-manifolds
If $M_1^4 \approx M_2^4$ (homeomorphic), then
$\exists f_1 : M_1^4 \to \mathbf{R}^3$ smooth map with only fold singularities ($=$ **fold map**)
$\Longleftrightarrow \exists f_2 : M_2^4 \to \mathbf{R}^3$ fold map

Theorem 2.6 (S., 2010)

M^4: open simply connected C^∞ 4-manifold of "finite type"

$\exists f : M^4 \to N^3$ proper special generic map

for some 3-manifold N^3 with $S(f) \neq \emptyset$

$\Longleftrightarrow$ M^4 is diffeomorphic to the connected sum

of a finite number of copies of the following manifolds:

$\mathbf{R}^4(= S^4 \setminus \{\text{point}\})$,

$\mathrm{Int}\left(\natural^k(S^2 \times D^2)\right) = S^4 \setminus (\vee^k S^1)$,

$S^2 \times S^2$,

$S^2 \mathbin{\widetilde{\times}} S^2$,

$\mathbf{R}^2$*-bundle over* S^2

Standard $\mathbb{R}^4$

Corollary 2.7

M^4: C^∞ 4-manifold with $M^4 \approx \mathbf{R}^4$ (homeomorphic)

$\exists f : M^4 \to \mathbf{R}^p$ proper SGM with $S(f) \neq \emptyset$ for $1 \leq \exists p \leq 3$

$\Longleftrightarrow M^4 \cong \mathbf{R}^4$ (diffeomorphic)

Standard $\mathbb{R}^4$

Corollary 2.7

M^4: C^∞ 4-manifold with $M^4 \approx \mathbf{R}^4$ (homeomorphic)
$\exists f : M^4 \to \mathbf{R}^p$ proper SGM with $S(f) \neq \emptyset$ for $1 \leq \exists p \leq 3$
$\Longleftrightarrow M^4 \cong \mathbf{R}^4$ (diffeomorphic)

Remark 2.8

It is known that $\mathbf{R}^n$, $n \neq 4$, has a unique differentiable structure (Munkres, Stallings, $\sim$ 60's).

Corollary 2.7

M^4: C^∞ 4-manifold with $M^4 \approx \mathbf{R}^4$ (homeomorphic)

$\exists f : M^4 \to \mathbf{R}^p$ proper SGM with $S(f) \neq \emptyset$ for $1 \leq \exists p \leq 3$

$\Longleftrightarrow M^4 \cong \mathbf{R}^4$ (diffeomorphic)

Remark 2.8

It is known that $\mathbf{R}^n$, $n \neq 4$, has a unique differentiable structure (Munkres, Stallings, $\sim$ 60's).

However, $\mathbf{R}^4$ admits **uncountably many** differentiable structures (Donaldson, Freedman, Taubes, $\sim$ 80's).

Remark & Conjecture

Remark 2.9 Every 4-manifold as in Theorem 2.6 admits infinitely many (or uncountably many) distinct differentiable structures.

Remark 2.9 Every 4-manifold as in Theorem 2.6 admits infinitely many (or uncountably many) distinct differentiable structures.

Theorem 2.6 implies that among them there is exactly one structure that allows the existence of a proper SGM into a 3-manifold.

Remark & Conjecture

Remark 2.9 Every 4-manifold as in Theorem 2.6 admits infinitely many (or uncountably many) distinct differentiable structures.

Theorem 2.6 implies that among them there is exactly one structure that allows the existence of a proper SGM into a 3-manifold.

Conjecture 2.10
M^4: topological 4-manifold
$\Longrightarrow$ *There exists at most one differentiable structure on M^4 that allows the existence of a proper SGM into $\mathbf{R}^3$.*

§3. Broken Lefschetz Fibrations

Broken Lefschetz fibration

M, Σ: compact connected oriented manifolds
$\dim M = 4$, $\dim \Sigma = 2$

M, Σ: compact connected oriented manifolds
$\dim M = 4$, $\dim \Sigma = 2$

Definition 3.1

A singularity of a smooth map $M \to \Sigma$ that has the normal form

$$(z, w) \mapsto zw$$

w.r.t. complex coordinates compatible with the orientations, is called a **Lefschetz singularity**.

Broken Lefschetz fibration

M, Σ: compact connected oriented manifolds
$\dim M = 4$, $\dim \Sigma = 2$

Definition 3.1
A singularity of a smooth map $M \longrightarrow \Sigma$ that has the normal form

$$(z, w) \longmapsto zw$$

w.r.t. complex coordinates compatible with the orientations, is called a **Lefschetz singularity**.

Definition 3.2 (Auroux–Donaldson–Katzarkov 2005, etc.)
Let $f : M \longrightarrow \Sigma$ be a C^∞ map.
f is a **broken Lefschetz fibration** (**BLF**, for short) if
it has at most Lefschetz and **indefinite** fold singularities.

Symplectic structures

A usual Lefschetz fibration is a special case of a BLF.

Symplectic structures

A usual Lefschetz fibration is a special case of a BLF.

Donaldson, Gompf, 1990's
Lefschetz fibrations $\Longleftrightarrow$ **symplectic structures** (up to blow-up)

Symplectic structures

A usual Lefschetz fibration is a special case of a BLF.

Donaldson, Gompf, 1990's
Lefschetz fibrations $\Longleftrightarrow$ **symplectic structures** (up to blow-up)

Auroux–Donaldson–Katzarkov, 2005
broken Lefschetz fibrations $\Longleftrightarrow$ **near-symplectic structures**
(admitting 1-dim. sing. locus)

Symplectic structures

A usual Lefschetz fibration is a special case of a BLF.

Donaldson, Gompf, 1990's
Lefschetz fibrations $\Longleftrightarrow$ **symplectic structures** (up to blow-up)

Auroux–Donaldson–Katzarkov, 2005
broken Lefschetz fibrations $\Longleftrightarrow$ **near-symplectic structures**
(admitting 1-dim. sing. locus)

Symplectic structure: $\omega \in \Omega^2(M^4)$, $d\omega = 0$, non-degenerate $(\omega^2 > 0)$

Symplectic structures

A usual Lefschetz fibration is a special case of a BLF.

Donaldson, Gompf, 1990's
Lefschetz fibrations $\Longleftrightarrow$ **symplectic structures** (up to blow-up)

Auroux–Donaldson–Katzarkov, 2005
broken Lefschetz fibrations $\Longleftrightarrow$ **near-symplectic structures**
(admitting 1-dim. sing. locus)

Symplectic structure: $\omega \in \Omega^2(M^4)$, $d\omega = 0$, non-degenerate ($\omega^2 > 0$)

Kähler $\Longrightarrow$ **symplectic** $\Longrightarrow$ almost complex

A usual Lefschetz fibration is a special case of a BLF.

Donaldson, Gompf, 1990's
Lefschetz fibrations $\Longleftrightarrow$ **symplectic structures** (up to blow-up)

Auroux–Donaldson–Katzarkov, 2005
broken Lefschetz fibrations $\Longleftrightarrow$ **near-symplectic structures**
(admitting 1-dim. sing. locus)

Symplectic structure: $\omega \in \Omega^2(M^4)$, $d\omega = 0$, non-degenerate ($\omega^2 > 0$)

Kähler $\Longrightarrow$ **symplectic** $\Longrightarrow$ almost complex

$\Downarrow$

Gauge theoretic invariants can be defined.

Regular fibers

Remark 3.3

Regular fibers of a BLF may not be connected.
Even if they are connected, their genera may not be constant.

Remark 3.3

Regular fibers of a BLF may not be connected.
Even if they are connected, their genera may not be constant.

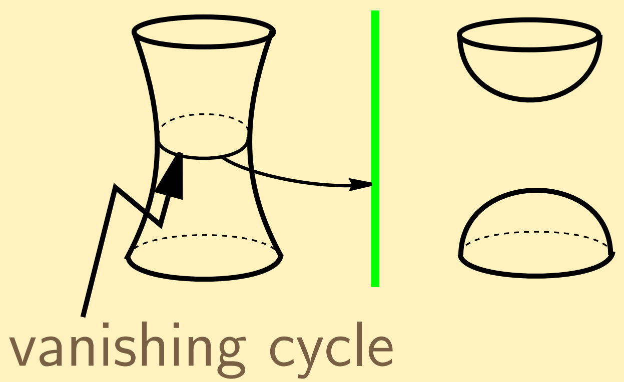


Figure 2: Regular fibers near **indefinite fold**

Cusps

Definition 3.4 A singularity that has the normal form

$$(x_1, x_2, x_3, x_4) \mapsto (x_1, x_2^3 - 3x_1x_2 + x_3^2 \pm x_4^2)$$

is called a **cusp**.

Cusps

Definition 3.4 A singularity that has the normal form

$$(x_1, x_2, x_3, x_4) \mapsto (x_1, x_2^3 - 3x_1x_2 + x_3^2 \pm x_4^2)$$

is called a **cusp**.

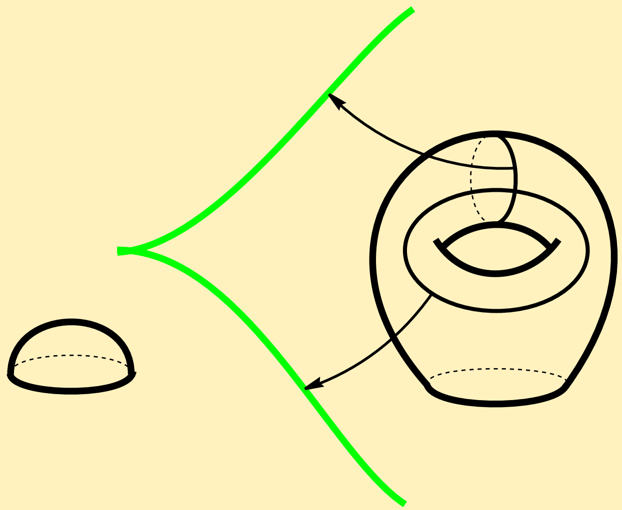


Figure 3: Indefinite cusp

Definition 3.4 A singularity that has the normal form

$$(x_1, x_2, x_3, x_4) \mapsto (x_1, x_2^3 - 3x_1x_2 + x_3^2 \pm x_4^2)$$

is called a **cusp**.

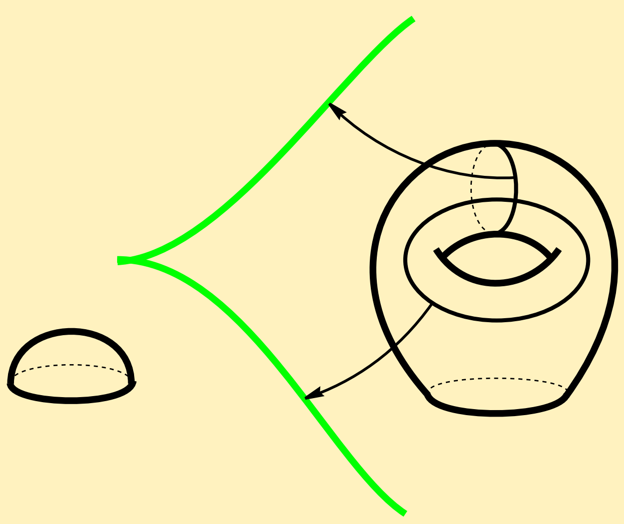


Figure 3: Indefinite cusp

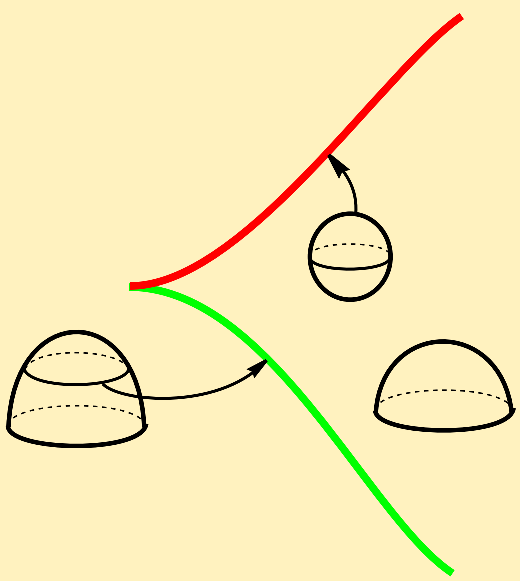


Figure 4: Definite cusp

Facts.

Whitney (1955)

Every smooth map $M \to \Sigma$ is homotopic to a map with at most definite fold, indefinite fold, and cusp singularities.

Facts.

Whitney (1955)

Every smooth map $M \to \Sigma$ is homotopic to a map with at most definite fold, indefinite fold, and cusp singularities.
Such a map is called an **excellent map**.

Excellent maps

Facts.

Whitney (1955)

Every smooth map $M \to \Sigma$ is homotopic to a map with at most definite fold, indefinite fold, and cusp singularities.
Such a map is called an **excellent map**.

Levine (1965)

Every smooth map $M \to \Sigma$ is homotopic to an excellent map without a cusp if $\chi(M)$ is even, and with exactly one cusp if $\chi(M)$ is odd.

Elimination of definite fold

Theorem 3.5 (S., 2006)

Every smooth map $g : M \to S^2$ is homotopic to an excellent map without definite fold singularities, and possibly with a cusp.

Elimination of definite fold

Theorem 3.5 (S., 2006)
Every smooth map $g : M \to S^2$ is homotopic to an excellent map without definite fold singularities, and possibly with a cusp.

In other words, we can eliminate **definite fold singularities** by homotopy.

Existence of BLF

Corollary 3.6 (Baykur, 2008)

Every closed oriented 4-manifold admits a BLF over S^2.

Corollary 3.6 (Baykur, 2008)

Every closed oriented 4-manifold admits a BLF over S^2.

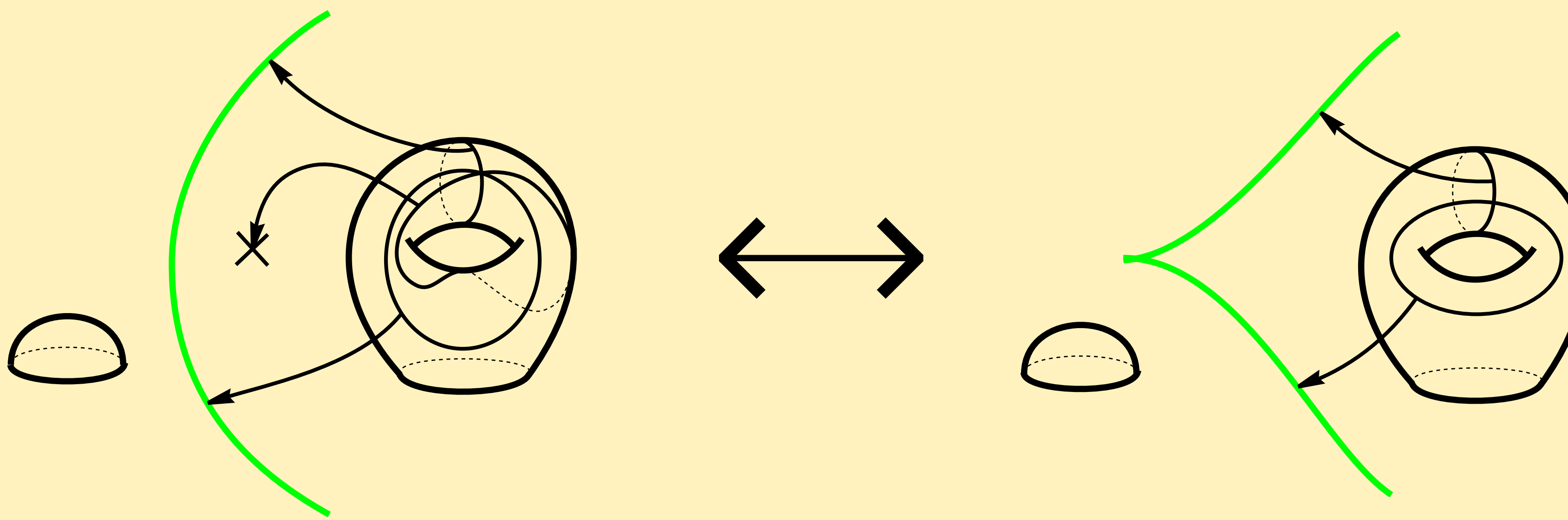


Figure 5: Sinking and Unsinking (Lekili 2009)

Moves for BLFs

Remark 3.7 For the existence of a BLF, several proofs have been known (Auroux–Donaldson–Katzarkov, Gay–Kirby, Baykur, Lekili, Akbulut–Karakurt).

Remark 3.7 For the existence of a BLF, several proofs have been known (Auroux–Donaldson–Katzarkov, Gay–Kirby, Baykur, Lekili, Akbulut–Karakurt).

Not every 4-manifold admits a Lefschetz fibration.

Moves for BLFs

Remark 3.7 For the existence of a BLF, several proofs have been known (Auroux–Donaldson–Katzarkov, Gay–Kirby, Baykur, Lekili, Akbulut–Karakurt).

Not every 4-manifold admits a Lefschetz fibration.
However, **every 4-manifold admits a BLF !**

Moves for BLFs

Remark 3.7 For the existence of a BLF, several proofs have been known (Auroux–Donaldson–Katzarkov, Gay–Kirby, Baykur, Lekili, Akbulut–Karakurt).

Not every 4-manifold admits a Lefschetz fibration.
However, **every 4-manifold admits a BLF !**

Furthermore, there are plenty of BLFs on a given 4-manifold.
Several **moves for BLFs** have been invented.
(Lekili, 2009; Williams, 2010; Gay–Kirby, 2011).

Moves for BLFs

Remark 3.7 For the existence of a BLF, several proofs have been known (Auroux–Donaldson–Katzarkov, Gay–Kirby, Baykur, Lekili, Akbulut–Karakurt).

Not every 4-manifold admits a Lefschetz fibration.
However, **every 4-manifold admits a BLF !**

Furthermore, there are plenty of BLFs on a given 4-manifold.
Several **moves for BLFs** have been invented.
(Lekili, 2009; Williams, 2010; Gay–Kirby, 2011).

> Two BLFs on a given 4-manifold are homotopic iff one can be obtained from the other by a finite iteration of Likili's moves.

Moves for BLFs

Remark 3.7 For the existence of a BLF, several proofs have been known (Auroux–Donaldson–Katzarkov, Gay–Kirby, Baykur, Lekili, Akbulut–Karakurt).

Not every 4-manifold admits a Lefschetz fibration.
However, **every 4-manifold admits a BLF !**

Furthermore, there are plenty of BLFs on a given 4-manifold.
Several **moves for BLFs** have been invented.
(Lekili, 2009; Williams, 2010; Gay–Kirby, 2011).

> Two BLFs on a given 4-manifold are homotopic iff one can be obtained from the other by a finite iteration of Likili's moves.

Elimination of definite fold for generic homotopy is possible.

Remark 3.8 To a BLF is associated a deformation class of **near-symplectic forms** (Lekili).

Concluding remarks

Remark 3.8 To a BLF is associated a deformation class of **near-symplectic forms** (Lekili).

Perutz (2007) defines **Lagrangian matching invariants** for BLFs.

Concluding remarks

Remark 3.8 To a BLF is associated a deformation class of **near-symplectic forms** (Lekili).

Perutz (2007) defines **Lagrangian matching invariants** for BLFs.

We do not know if they are invariant under Lekili's moves.

Concluding remarks

Remark 3.8 To a BLF is associated a deformation class of **near-symplectic forms** (Lekili).

Perutz (2007) defines **Lagrangian matching invariants** for BLFs.

We do not know if they are invariant under Lekili's moves.

It is conjectured that **Lagrangian matching invariants equal the Seiberg–Witten invariants**.

Concluding remarks

Remark 3.8 To a BLF is associated a deformation class of **near-symplectic forms** (Lekili).

Perutz (2007) defines **Lagrangian matching invariants** for BLFs.

We do not know if they are invariant under Lekili's moves.

It is conjectured that **Lagrangian matching invariants equal the Seiberg–Witten invariants**.

Singularities of C^∞ maps are closely related to **differentiable structures** of manifolds!

Thank you!