CONTACT AS APPLIED TO THE GEOMETRY OF CURVES IN HOMOGENEOUS SPACES

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Abstract. Given a certain curve in a homogeneous space G/H we associate to it a curve in the Lie algebra $\mathfrak{g}$ of G, in such a way that geometric features of the original curve may be recovered by the associated curve in $\mathfrak{g}$. Indeed we use it to establish some results on congruence. Moreover we show that the associated curve yields an intrinsic way to introduce a notion of a natural parameter for the original curve, i.e., a G-invariant for curves to start with.

1. Introduction

According to Klein's Erlangen Program, Geometry is the study of equivalence of figures in a space M up to the action of some group of motions of the space. Having in mind the use of tools coming from the Differential Calculus, Differential Geometry considers particular figures in that study, for instance the ones that are images of differentiable embeddings $\alpha : I \to M$, where $I \subset \mathbb{R}$ is an open interval. Given such a mapping one can try to find the relevant geometric features of its image. However, many such smooth mappings have the same image so that the task seems to be not so easy. Concerning this, in Riemannian Geometry the notion of arc length parametrization is so helpful; it is easy to argue why. One can just say that when we are looking at the figure, it does not matter how fast we travel on it. However, for general groups of motions that argument is not applicable and presumably no physical one, either. As observed by Weyl [7], "Cartan goes here to the opposite extreme by normalizing the parameters in terms of the frames"; but as observed by Green [2] "he did not give a precise proof that a moving frame exists in general". Thus, speaking of a natural parameter needs further investigations.

In Section 2 of the present paper we recall some basic constructions appearing in [5]. In Section 3 we characterize the curves that are good enough to be dealt with by our methods. We also present, for these curves, an answer to a linearized version of the problem of congruence (see Theorem 8). In Section 4, we show that as far as we are interested in the images of the mappings, there exists a sort of appropriate parameter to be considered. Finally, in Section 5 we explore two examples. With the first one we would like

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to show that our method reveals itself to be a mixture of Cartan's method and a sort of generalization of Frenet's method. With the second example we discuss geometric aspects behind the obstruction to the applicability of the method.

2. Preliminaries

To begin with, let us consider $\Phi : G \times G/H \to G/H$, the natural action of the Lie group G on the quotient space G/H, where $H \subset G$ is a closed subgroup. For $g \in G$ and $x \in G/H$ we sometimes write $\Phi(g, x) = g \cdot x$. Let $\mathfrak{g}$ be the Lie algebra of G, G_x be the isotropy subgroup at $x \in G/H$ and $\mathfrak{g}_x$ its Lie algebra. Then every $\mathbf{v} \in \mathfrak{g}$, $\mathbf{v} \notin \mathfrak{g}_x$, determines in G/H the *exponential curve* $s \mapsto \exp(s\mathbf{v}) \cdot x$, $s \in \mathbb{R}$, which is an embedding in some neighborhood of $s = 0$.

Now, given an embedding $\alpha : I \to G/H$ of a real open interval I into G/H, for each $t \in I$ one may consider the contact between α and $\beta_{\mathbf{v}}$ at $\alpha(t) = \beta_{\mathbf{v}}(0)$, where $\beta_{\mathbf{v}}$ is the exponential curve $\beta_{\mathbf{v}}(s) = \exp(s\mathbf{v}) \cdot \alpha(t)$.

For the purpose of Geometry, as we are interested in the images, the nice notion of contact between mappings is as follows. We say that, for $k \in \mathbb{N}$, α and $\beta_{\mathbf{v}}$ are in *contact of order* k at $\alpha(t) = \beta_{\mathbf{v}}(0)$ if some reparameterizations of them have the same k-jet there.

Let $D_e\Phi_{\alpha(t)} : \mathfrak{g} \to T_{\alpha(t)}(G/H)$ be the differential at the neutral element $e \in G$ of $\Phi_{\alpha(t)} : G \to G/H$, $g \mapsto \Phi(g, \alpha(t))$. Then it is easy to see that the vector subspace $S_1^\alpha(t) \subset \mathfrak{g}$ given by

$$S_1^\alpha(t) = \left(D_e\Phi_{\alpha(t)}\right)^{-1}(\mathbb{R}\alpha'(t))$$

contains besides $\mathfrak{g}_{\alpha(t)}$ precisely those elements $\mathbf{v} \in \mathfrak{g}$ such that $\beta_{\mathbf{v}}$ and α are in contact of order 1 at $\alpha(t)$.

For each $t \in I$ let $\{\mathbf{N}_1(t), \dots, \mathbf{N}_r(t)\} \subset \mathfrak{g}^*$ be a linearly independent set such that

$$S_1^\alpha(t) = \cap_{i=1}^r \ker \mathbf{N}_i(t).$$

It follows that $r = \dim G - \dim S_1^\alpha(t)$, which does not depend on t.

From now on we suppose that $\mathfrak{g}$ is endowed with an inner product, $\langle \cdot, \cdot \rangle$, and thus we often identify $\mathfrak{g}^*$ with $\mathfrak{g}$. We may further assume that each $\mathbf{N}_i$, $i = 1, \dots, r$, varies differentiably with respect to t; then we call $\{\mathbf{N}_1, \dots, \mathbf{N}_r\}$ a *coframe* along α (see [5]). In this case we consider the $(k-1)$-th derivative $\mathbf{N}_i^{(k-1)}$ and define inductively $S_k^\alpha(t)$ by the rule

$$S_k^\alpha(t) = \{\mathbf{v} \in S_{k-1}^\alpha(t); \mathbf{v} \in \cap_{i=1}^r \ker \mathbf{N}_i^{(k-1)}(t)\}. \tag{1}$$

Theorem 2.5 of [5] shows that $\mathbf{v} \in S_k^\alpha(t)$ if and only if either $\mathbf{v} \in \mathfrak{g}_{\alpha(t)} \cap S_k^\alpha(t)$ or $\beta_{\mathbf{v}}$ and α are in contact of order k at $\alpha(t) = \beta_{\mathbf{v}}(0)$. In particular, $S_k^\alpha(t)$ does not depend on the choice of a coframe.

3. On congruence

Being invariant by diffeomorphisms of G/H, the contact order is a too strong invariant as far as we are interested only in the G-motions. However, in some situations it may help us to choose a specific exponential curve at each point of α and, since the exponential curves are completely determined by the G-action, we may expect to get some insight. As we shall see, under the conditions in the following definition, such an appropriate choice is possible. Roughly speaking, the conditions will be fulfilled whenever the one parameter subgroups are plenty enough and their orbits are distinguishable enough among themselves. As far as we know, it has been always the case that some non-degeneracy assumption for α is necessary (see the introductions in [2, 3]), and in this context, our approach may be revealing a role played by one parameter subgroups of G.

Definition 1. Let $\alpha : I \to G/H$ be an embedding. Take $S_0^\alpha(t)$ as being the Lie algebra $\mathfrak{g}$. Suppose that there exists a positive integer k such that $S_{k-1}^\alpha(t)$ has constant dimension, and, for all $t \in I$, $S_k^\alpha(t)$ is one dimensional, and $S_k^\alpha(t) \cap \mathfrak{g}_{\alpha(t)} = \{\mathbf{0}\}$. The smallest such k is called the *preferred contact order* for α, hereafter denoted by κ.

Suppose that α has preferred contact order κ. Then, for each t there exists a unique $\mathbf{w}(t) \in S_\kappa^\alpha(t)$ such that

$$D_e\Phi_{\alpha(t)}(\mathbf{w}(t)) = \alpha'(t).$$

In fact, $(D_e\Phi_{\alpha(t)})^{-1}(\alpha'(t))$ is a hyperplane of $S_1^\alpha(t)$ parallel to $\mathfrak{g}_{\alpha(t)}$. We let $\mathbf{w}(t)$ be the intersection of that hyperplane with $S_\kappa^\alpha(t)$. Under the hypothesis on the dimension of $S_\kappa^\alpha(t)$, the uniqueness follows immediately from the linearity of $D_e\Phi_{\alpha(t)}$. Thus $\mathbf{w}_\alpha : t \mapsto \mathbf{w}(t)$ is a map into $\mathfrak{g}$ which we shall call the *linearized osculatrix* of α.

Remark 2. In the case of a curve α in $\mathbb{R}^2$ under rigid motions, for each t the orbit through $\alpha(t)$ by the exponential curve $\beta_{\mathbf{w}_\alpha}(s) = \exp(s\mathbf{w}_\alpha(t)) \cdot \alpha(t)$ is precisely either the osculating circle of α at $\alpha(t)$ if the curvature of α does not vanish there or its tangent line otherwise. This was what suggested us the name for $\mathbf{w}_\alpha$. In this case, α always satisfies the condition in Definition 1 with preferred contact order 2 (for details, see [4]).

Proposition 3. *The linearized osculatrix of α is differentiable.*

Proof. Let $\{\mathbf{N}_1, \ldots, \mathbf{N}_r\}$ be a coframe along α and let κ be the preferred contact order for α. Once an inner product on $\mathfrak{g}$ is fixed, we may identify each $\mathbf{N}_i \in \mathfrak{g}^*$ with a well defined element in $\mathfrak{g}$, as it is well known. In what follows we shall use the same symbol $\mathbf{N}_i$ for that element. Then, it follows from equation (1) that $S_\kappa^\alpha(t)$ is the orthogonal complement of the subspace $V(t)$ spanned by

$$\{\mathbf{N}_1(t), \ldots, \mathbf{N}_r(t), \mathbf{N}_1'(t), \ldots, \mathbf{N}_r'(t), \ldots, \mathbf{N}_1^{(\kappa-1)}(t), \ldots, \mathbf{N}_r^{(\kappa-1)}(t)\}. \tag{2}$$

Fix $t_0 \in I$ and choose a basis $\{\mathbf{n}_1(t_0), \ldots, \mathbf{n}_{\dim G-1}(t_0)\}$ for $V(t_0)$, from the set of generators of $V(t_0)$ given in (2). By the continuity of the elements of $\{\mathbf{n}_1(t), \ldots, \mathbf{n}_{\dim G-1}(t)\}$ we have that this set is a basis, for each t in some neighborhood I_0 of t_0. Thus the product

$$\mathbf{u}(t) = \mathbf{n}_1(t) \times \cdots \times \mathbf{n}_{\dim G-1}(t)$$

gives a generator for $S_\kappa^\alpha(t)$ on I_0. Now it is clear that $\mathbf{u}$ is differentiable and that we can write the linearized osculatrix $\mathbf{w}_\alpha$ in the form $\mathbf{w}_\alpha(t) = \lambda(t)\mathbf{u}(t)$, for some nowhere zero real-valued function λ. We shall show that λ is differentiable, thus providing the differentiability of $\mathbf{w}_\alpha$.

By definition we have

$$D_e\Phi_{\alpha(t)}(\mathbf{w}_\alpha(t)) = D_e\Phi_{\alpha(t)}(\lambda(t)\mathbf{u}(t)) = \alpha'(t)$$

and choosing an arbitrary Riemannian metric on G/H, $p \mapsto \langle\langle\cdot,\cdot\rangle\rangle_p$, we can write

$$\lambda^2(t)\langle\langle D_e\Phi_{\alpha(t)}(\mathbf{u}(t)), D_e\Phi_{\alpha(t)}(\mathbf{u}(t))\rangle\rangle_{\alpha(t)} = \langle\langle \alpha'(t), \alpha'(t)\rangle\rangle_{\alpha(t)}. \tag{3}$$

Since λ is nowhere zero it is sufficient to show that λ^2 is differentiable, and considering the differentiability of the Riemannian metric we only need to show that $t \mapsto D_e\Phi_{\alpha(t)}(\mathbf{u}(t))$ is differentiable.

Let $\Psi : G \times I \to G/H$ be given by $\Psi(g,t) = g \cdot \alpha(t)$, $g \in G$, $t \in I$. We identify the tangent bundle

$$T(G \times I) \approx TG \times TI \approx (G \times \mathfrak{g}) \times (I \times \mathbb{R})$$

and we consider the map $\Theta : I \to (G \times \mathfrak{g}) \times (I \times \mathbb{R})$, $\Theta(t) = ((e, \mathbf{u}(t)), (t, 0))$, which is differentiable. Taking the differential $D\Psi : T(G \times I) \to T(G/H)$ of Ψ we obtain

$$(D\Psi \circ \Theta)(t) = D_{(e,t)}\Psi(\mathbf{u}(t), 0) = D_e\Phi_{\alpha(t)}(\mathbf{u}(t)),$$

which shows that $t \mapsto D_e\Phi_{\alpha(t)}(\mathbf{u}(t))$ is differentiable. $\square$

Whenever our early constructions work well, the following result is to be expected.

Proposition 4. *Suppose that α is itself an exponential curve, i.e., $\alpha(s) = \exp(s\mathbf{v}) \cdot p$ for some $p \in G/H$, $\mathbf{v} \notin \mathfrak{g}_p$, and has preferred contact order κ. Then $\mathbf{w}_\alpha(s) = \mathbf{v}$, for every s.*

Proof. Since $S^\alpha_\kappa(s)$ is one dimensional, it follows from [5, Corollary 2.7] that $S^\alpha_\kappa(s)$ is generated by $\mathbf{v}$. On the other hand it is clear that $D_e\Phi_{\alpha(s)}(\mathbf{v}) = \alpha'(s)$, and the result follows from the definition of the linearized osculatrix. $\square$

Remark 5. Even if α is an exponential curve it may have no preferred contact order. This means that there may exist linearly independent elements in the Lie algebra giving rise to the same orbit (cf. Case 1 in Example 2).

From now on we assume that to each curve we consider we can associate its linearized osculatrix.

We also have the converse to Proposition 4. It will follow as a corollary to the next lemma.

Let $\pi : G \to G/H$ be the canonical projection and $R_g : G \to G$ be the right translation by $g \in G$. Take z a lifting of α to G and consider the linearized osculatrix $\mathbf{w}_\alpha$. It is well known ([6, p. 29]) that, for a fixed $t_0 \in I$, there exists a unique $g : I \to G$ such that $g(t_0) = z(t_0)$ and $D_gR_{g^{-1}}(g') = \mathbf{w}_\alpha$. Under these conditions we have:

Lemma 6. *The curve g is a lifting of α.*

Proof. From $D_e\Phi_{\alpha(t)} = D_{z(t)}\pi \circ D_eR_{z(t)}$ it follows that $D_eR_z(\mathbf{w}_\alpha) - z' \in \ker D_z\pi$, where $z = z(t)$. Since $\ker D_z\pi = D_eL_z(\mathfrak{h})$, we obtain

$$(D_eL_z)^{-1}(z' - D_eR_z(\mathbf{w}_\alpha)) = D_zL_{z^{-1}}(z') - \mathrm{Ad}(z^{-1})(\mathbf{w}_\alpha) \in \mathfrak{h},$$

where $\mathfrak{h}$ is the Lie algebra of $H \subset G$. Then, again by [6, p. 29] we get $h : I \to H$ with $h(t_0) = e$ and $D_h L_{h^{-1}}(h') = D_z L_{z^{-1}}(z') - \mathrm{Ad}(z^{-1})(\mathbf{w}_\alpha)$. Now consider the curve $x(t) = g(t)^{-1} z(t)$, $t \in I$. We shall show that $x = h$. Treating h as a curve in G, from [3, (1.3)] $x = h$ if and only if $D_x L_{x^{-1}}(x') = D_h L_{h^{-1}}(h')$, since $x(t_0) = h(t_0) = e$. But this is exactly what we have: by the Leibniz rule the derivative of $gx = z$ is equal to

$$D_z L_{z^{-1}}(z') = \mathrm{Ad}(x^{-1}) \circ D_g L_{g^{-1}}(g') + D_x L_{x^{-1}}(x'), \tag{4}$$

and putting $x^{-1} = z^{-1} g$ in equation (4) we obtain

$$\begin{aligned} D_x L_{x^{-1}}(x') &= D_z L_{z^{-1}}(z') - \mathrm{Ad}(z^{-1}) \circ D_g R_{g^{-1}}(g') \\ &= D_z L_{z^{-1}}(z') - \mathrm{Ad}(z^{-1})(\mathbf{w}_\alpha) \\ &= D_h L_{h^{-1}}(h'). \end{aligned}$$

As h is a curve in H, $\pi(g) = \pi(gh) = \pi(z) = \alpha$. □

Corollary 7. *Suppose that the linearized osculatrix of α is defined over an interval I containing 0, and $\mathbf{w}_\alpha(s) = \mathbf{v}$, for every $s \in I$. Then $\alpha(s) = \exp(s\mathbf{v}) \cdot \alpha(0)$ for every $s \in I$.*

Proof. Let z be a lifting of α with $\pi(z(0)) = \alpha(0) \in G/H$. By Lemma 6, the curve in G satisfying $D_g R_{g^{-1}}(g') = \mathbf{v}$ and $g(0) = z(0)$ is a lifting of α. From the very definition of $\exp(s\mathbf{v})$ it follows that $g(s) = \exp(s\mathbf{v}) \cdot z(0)$ is such a curve. □

Now we are ready to prove the following theorem which characterizes the congruence between two curves.

Theorem 8. *Let $\alpha, \gamma : I \to G/H$ be curves with $\mathbf{w}_\alpha, \mathbf{w}_\gamma : I \to \mathfrak{g}$ the corresponding linearized osculatrices. If $g_0 \in G$ is fixed, then $\gamma = g_0 \cdot \alpha$ if and only if $\mathbf{w}_\gamma = \mathrm{Ad}(g_0)(\mathbf{w}_\alpha)$ and $\gamma(t_0) = g_0 \cdot \alpha(t_0)$ for some $t_0 \in I$.*

Proof. First suppose $\gamma = g_0 \cdot \alpha$ and let κ be the preferred contact order for α (also for γ by [5, Proposition 4.1]). Fix z a lifting of α to G. By definition $\mathbf{w}_\alpha$ is the unique element $\mathbf{w} \in S^\alpha_\kappa$ such that $D_z\pi \circ D_e R_z(\mathbf{w}) = \alpha'$. Since $g_0 z$ is a lifting of γ to G, $\mathbf{w}_\gamma$ is the unique element $\mathbf{w} \in S^\gamma_\kappa$ such that $D_{g_0 z}\pi \circ D_e R_{g_0 z}(\mathbf{w}) = \gamma'$. Now we have that

$$\begin{aligned} & D_{g_0 z}\pi \circ D_e R_{g_0 z} \circ \mathrm{Ad}(g_0)(\mathbf{w}_\alpha) \\ &= D_{g_0 z}\pi \circ D_{g_0} R_z \circ D_e R_{g_0} \circ \mathrm{Ad}(g_0)(\mathbf{w}_\alpha) \\ &= D_{g_0 z}\pi \circ D_{g_0} R_z \circ D_e R_{g_0} \circ D_{g_0} R_{g_0^{-1}} \circ D_e L_{g_0}(\mathbf{w}_\alpha) \\ &= D_{g_0 z}\pi \circ D_{g_0} R_z \circ D_e L_{g_0}(\mathbf{w}_\alpha) \\ &= D_{g_0 z}\pi \circ D_z L_{g_0} \circ D_e R_z(\mathbf{w}_\alpha) \\ &= D_{g_0 z}\pi \circ D_z L_{g_0}(z') \end{aligned}$$

and the last term coincides with γ', since $\gamma = \pi(g_0 z)$. By [5, Proposition 4.1] $S^\gamma_\kappa = \mathrm{Ad}(g_0) S^\alpha_\kappa$ so that the uniqueness yields $\mathrm{Ad}(g_0)(\mathbf{w}_\alpha) = \mathbf{w}_\gamma$.

Conversely, suppose that $\mathrm{Ad}(g_0)(\mathbf{w}_\alpha) = \mathbf{w}_\gamma$ and $\gamma(t_0) = g_0 \cdot \alpha(t_0)$. If we take a lifting g of α to G satisfying $D_g R_{g^{-1}}(g') = \mathbf{w}_\alpha$ as in Lemma 6, then it is easy to see that $r = g_0 g$ satisfies $D_r R_{r^{-1}}(r') = \mathrm{Ad}(g_0)(\mathbf{w}_\alpha) = \mathbf{w}_\gamma$. Since $\gamma(t_0) = g_0 \cdot \alpha(t_0)$, we know that there exists a lifting of γ passing through

$g_0 g(t_0) = r(t_0)$, and then again by Lemma 6, r is such a lifting of γ. Thus $\gamma = \pi(r) = \pi(g_0 g) = g_0 \cdot \alpha$. □

4. On parameters

In this section we introduce a notion of natural parameters for curves in G/H. Theorem 14 is the main result of this section. It tells us that if we consider curves with such appropriate parameters, then the reparameterizations we need work out in the study of congruence are very restricted.

Let $\bar{\alpha} : I_1 \to G/H$ and $\bar{\gamma} : I_2 \to G/H$ be embeddings. For $g_0 \in G$, $\bar{\gamma}(I_2) = g_0 \cdot \bar{\alpha}(I_1)$ if and only if there exist reparameterizations $\alpha = \bar{\alpha} \circ \varphi_1 : I \to G/H$ and $\gamma = \bar{\gamma} \circ \varphi_2 : I \to G/H$ such that $\gamma = g_0 \cdot \alpha$.

Proposition 9. *For $k \geq 2$, if $\mathbf{v} : I \to \mathfrak{g}$ is a curve such that $\mathbf{v}(t) \in S_k^\alpha(t)$ for all $t \in I$, then $\mathbf{v}'(t) \in S_{k-1}^\alpha(t)$ for all $t \in I$.*

Proof. The proof is by induction on k. Let $\{\mathbf{N}_1, \ldots, \mathbf{N}_r\}$ be a coframe along α. Suppose $k = 2$. For each $i = 1, \ldots, r$ we have

$$\mathbf{N}_i(\mathbf{v}) = 0, \tag{5}$$

$$\mathbf{N}_i'(\mathbf{v}) = 0. \tag{6}$$

Taking the derivative of (5) we get $\mathbf{N}_i'(\mathbf{v}) + \mathbf{N}_i(\mathbf{v}') = 0$ and by (6) we have $\mathbf{N}_i(\mathbf{v}') = 0$, for $i = 1, \ldots, r$, which proves that $\mathbf{v}' \in S_1^\alpha$. So the result is true for $k = 2$.

Now, suppose that it is true for $k = 2, \ldots, m$ and let $\mathbf{v} \in S_{m+1}^\alpha$ so that $\mathbf{v} \in S_j^\alpha$, $j = 1, \ldots, m$. By hypothesis $\mathbf{v}' \in S_{m-1}^\alpha$ and we shall show that $\mathbf{N}_i^{(m-1)}(\mathbf{v}') = 0$, for $i = 1, \ldots, r$. In fact, since $\mathbf{v} \in S_{m+1}^\alpha \subset S_m^\alpha$ it follows that $\mathbf{N}_i^{(m-1)}(\mathbf{v}) = 0$, for $i = 1, \ldots, r$, and taking the derivative we obtain

$$\mathbf{N}_i^{(m)}(\mathbf{v}) + \mathbf{N}_i^{(m-1)}(\mathbf{v}') = 0.$$

Finally $\mathbf{v} \in S_{m+1}^\alpha$ implies that $\mathbf{N}_i^{(m-1)}(\mathbf{v}') = 0$, for $i = 1, \ldots, r$. □

Note that the above proposition holds true even if α does not have preferred contact order k.

Proposition 10. *Let $\alpha : I \to G/H$ be an embedding and $\mathbf{w}_\alpha : I \to \mathfrak{g}$ its linearized osculatrix. If $\gamma = \alpha \circ \varphi : \varphi^{-1}(I) \to G/H$ is a reparameterization of α, then γ has linearized osculatrix given by*

$$\mathbf{w}_\gamma(s) = \varphi'(s)\mathbf{w}_\alpha(\varphi(s))$$

for all $s \in \varphi^{-1}(I)$.

Note that by [5, Proposition 2.8] we have $S_k^\gamma(s) = S_k^\alpha(\varphi(s))$ for every k. In particular, α and γ have the same preferred contact order.

Proof of Proposition 10. Let κ be the preferred contact order for α and γ. We recall that $\mathbf{w}_\alpha$ is simply the intersection of S_κ^α with $(D_e\Phi_\alpha)^{-1}(\alpha')$, where $\Phi_{\alpha(t)} : G \to G/H$ is given by $\Phi_{\alpha(t)}(g) = g \cdot \alpha(t)$, for $g \in G$. Thus

$$\mathbf{w}_\gamma(s) = \left((D_e\Phi_{\gamma(s)})^{-1}(\gamma'(s))\right) \cap S_\kappa^\gamma(s) = \left((D_e\Phi_{\gamma(s)})^{-1}(\gamma'(s))\right) \cap S_\kappa^\alpha(\varphi(s)),$$

and since φ' does not vanish we see that

$$(D_e\Phi_{\gamma(s)})^{-1}(\gamma'(s)) = \varphi'(s)(D_e\Phi_{\alpha(\varphi(s))})^{-1}(\alpha'(\varphi(s)))$$

and the result follows. □

Lemma 11. *Let* $\mathbf{w}_\alpha$ *be the linearized osculatrix of* $\alpha : I \to G/H$ *with preferred contact order* $\kappa \geq 2$. *Then, for* $t_0 \in I$, *there exist a neighborhood* I_0 *of* t_0 *and a reparameterization* $\gamma = \alpha \circ \varphi$ *such that*

$$\mathbf{w}'_\gamma(s) \in S^\gamma_{\kappa-1}(s) \cap \mathfrak{g}_{\gamma(s)}$$

for $s \in \varphi^{-1}(I_0)$.

Proof. Since $\mathbf{w}_\alpha(t) \in S^\alpha_\kappa(t)$ for all $t \in I$, by Proposition 9 we have $\mathbf{w}'_\alpha(t) \in S^\alpha_{\kappa-1}(t)$, for all $t \in I$. Thus we can write

$$\mathbf{w}'_\alpha(t) = \lambda(t)\mathbf{w}_\alpha(t) + \mathbf{v}(t), \tag{7}$$

with $\mathbf{v}(t) \in S^\alpha_{\kappa-1}(t) \cap \mathfrak{g}_{\alpha(t)}$ for all $t \in I$ and some real-valued function $\lambda(t)$. We shall show that λ is differentiable. Since the dimension of the subspace $S^\alpha_{\kappa-1}(t) \cap \mathfrak{g}_{\alpha(t)}$ is constant for all $t \in I$ by Definition 1, we can choose a basis, say $\{\mathbf{v}_1(t), \ldots, \mathbf{v}_n(t)\}$, for it which varies smoothly. Then $\{\mathbf{w}_\alpha(t), \mathbf{v}_1(t), \ldots, \mathbf{v}_n(t)\}$ is a basis of $S^\alpha_{\kappa-1}(t)$ varying smoothly. The differentiability of $\lambda(t)$ follows immediately, since it is one of the coefficients of the expansion of $\mathbf{w}'_\alpha(t)$ in that basis.

Fix $t_0 \in I$ and consider the solution ψ of the differential equation $\psi''(t) - \lambda(t)\psi'(t) = 0$ with initial conditions $\psi(t_0) = 0$ and $\psi'(t_0) = c \neq 0$. Then ψ is a diffeomorphism from a neighborhood I_0 of t_0 to some neighborhood of 0. Taking $\gamma = \alpha \circ \psi^{-1}$ we obtain $\alpha = \gamma \circ \psi$ and by Proposition 10 we have for all $t \in I_0$,

$$\mathbf{w}_\alpha(t) = \psi'(t)\mathbf{w}_\gamma(\psi(t)), \tag{8}$$

$$\mathbf{w}'_\alpha(t) = (\psi'(t))^2\mathbf{w}'_\gamma(\psi(t)) + \psi''(t)\mathbf{w}_\gamma(\psi(t)). \tag{9}$$

Making use of (7) we rewrite (9) as

$$\lambda(t)\mathbf{w}_\alpha(t) + \mathbf{v}(t) = (\psi'(t))^2\mathbf{w}'_\gamma(\psi(t)) + \psi''(t)\mathbf{w}_\gamma(\psi(t)).$$

From (8) it follows that

$$\mathbf{v}(t) = (\psi''(t) - \lambda(t)\psi'(t))\mathbf{w}_\gamma(\psi(t)) + (\psi'(t))^2\mathbf{w}'_\gamma(\psi(t)),$$

and by the hypotheses on ψ we obtain

$$\mathbf{w}'_\gamma(\psi(t)) = \frac{\mathbf{v}(t)}{(\psi'(t))^2} \in S^\alpha_{\kappa-1}(t) \cap \mathfrak{g}_{\alpha(t)} = S^\gamma_{\kappa-1}(\psi(t)) \cap \mathfrak{g}_{\gamma(\psi(t))}.$$

We complete the proof by taking $\varphi = \psi^{-1}$. □

Definition 12. Let $\mathbf{w}_\alpha : I \to \mathfrak{g}$ be the linearized osculatrix of the embedding $\alpha : I \to G/H$. If $\mathbf{w}'_\alpha(t) \in \mathfrak{g}_{\alpha(t)}$ for all $t \in I$, then we shall say that α carries a *natural parameter*.

We remark that *i*) in the above definition we are not supposing the preferred contact order for α greater than one; *ii*) Proposition 4 guarantees that every exponential curve having a preferred contact order carries a natural parameter; *iii*) Theorem 8 shows that α carrying a natural parameter is a G-invariant property; *iv*) applying our definition to the euclidean plane geometry, we see that α carries a natural parameter if and only if $\|\alpha'\|$ is constant (see [4, p. 72]).

Proposition 13. *If $\alpha : I \to G/H$ carries a natural parameter and if $\gamma = \alpha \circ \varphi : \varphi^{-1}(I) \to G/H$ is a reparameterization of α, then γ also carries a natural parameter if and only if $\varphi(s) = as + b$, for some $a, b \in \mathbb{R}$ with $a \neq 0$.*

Proof. Recalling that κ is the preferred contact order for α, we will split the proof into two cases.

Case 1: $\kappa \geq 2$. We know that $\mathbf{w}_\gamma(s) = \varphi'(s)\mathbf{w}_\alpha(\varphi(s))$ by Proposition 10 and then

$$\mathbf{w}'_\gamma(s) = (\varphi'(s))^2 \mathbf{w}'_\alpha(\varphi(s)) + \varphi''(s)\mathbf{w}_\alpha(\varphi(s)). \tag{10}$$

If γ has a natural parameter, then

$$\mathbf{w}'_\gamma(s) \in S^\gamma_{\kappa-1}(s) \cap \mathfrak{g}_{\gamma(s)} = S^\alpha_{\kappa-1}(\varphi(s)) \cap \mathfrak{g}_{\alpha(\varphi(s))}.$$

Since α has a natural parameter, we obtain

$$\varphi''(s)\mathbf{w}_\alpha(\varphi(s)) = \mathbf{w}'_\gamma(s) - (\varphi'(s))^2 \mathbf{w}'_\alpha(\varphi(s)) \in S^\alpha_{\kappa-1}(\varphi(s)) \cap \mathfrak{g}_{\alpha(\varphi(s))}.$$

Consequently $\varphi''(s)$ must vanish for all s, since otherwise we would have $\mathbf{w}_\alpha(\varphi(s_0)) \in \mathfrak{g}_{\alpha(\varphi(s_0))}$ for some s_0, which is impossible by the definition of linearized osculatrix. So $\varphi(s) = as+b$ with $a \neq 0$ (for it is a diffeomorphism).

Case 2: $\kappa = 1$. Observe that having preferred contact order equal to 1 is equivalent to $\dim S^\alpha_1(t) = 1$ for all $t \in I$, and that then, by the very definition of $S^\alpha_1(t)$ one has $\mathfrak{g}_{\alpha(t)} = \{\mathbf{0}\}$. So if α carries a natural parameter, then $\mathbf{w}'_\alpha(t) = \mathbf{0}$ for all $t \in I$, that is, $\mathbf{w}_\alpha$ is constant. In view of (10) we obtain $\mathbf{w}'_\gamma(s) = \varphi''(s)\mathbf{w}_\alpha(\varphi(s))$ and, as in Case 1, if γ also carries a natural parameter, then we have $\varphi''(s) = 0$ for all s.

The converse follows from Proposition 10 in both cases. □

Let $\bar{\alpha} : \bar{I}_1 \to G/H$ and $\bar{\gamma} : \bar{I}_2 \to G/H$ be embeddings with the same preferred contact order $\kappa \geq 2$. Suppose that there exist reparameterizations $\alpha : I_1 \to G/H$ and $\gamma : I_2 \to G/H$ of $\bar{\alpha}$ and $\bar{\gamma}$, respectively, both carrying natural parameters (by Lemma 11 such reparameterizations do exist at least locally). Given $g_0 \in G$ it is clear that $\bar{\gamma}(\bar{I}_2) = g_0 \cdot \bar{\alpha}(\bar{I}_1)$ if and only if $\gamma(I_2) = g_0 \cdot \alpha(I_1)$, which leads to:

Theorem 14. *For $g_0 \in G$, the equality $\gamma(I_2) = g_0 \cdot \alpha(I_1)$ holds if and only if there exists a bijection $\varphi : I_1 \to I_2$ of the form $\varphi(s) = as + b$ for some $a, b \in \mathbb{R}$ such that $\gamma \circ \varphi = g_0 \cdot \alpha$.*

Proof. Clearly the condition is sufficient. Now suppose that $\gamma(I_2) = g_0 \cdot \alpha(I_1)$ and then as observed at the beginning of Section 4, there exists a diffeomorphism $\varphi : I_1 \to I_2$ satisfying $\gamma \circ \varphi = g_0 \cdot \alpha$, that is, $g_0 \cdot \alpha$ is a reparameterization of γ. Since α carries a natural parameter we know by Theorem 8 that $g_0 \cdot \alpha$ also carries a natural parameter. By hypotheses γ also carries a natural parameter and then the result follows from Corollary 13. □

Applying Theorems 8 and 14 and Proposition 10 we obtain:

Corollary 15. *Let $\alpha : I_1 \to G/H$ and $\gamma : I_2 \to G/H$ be embeddings having natural parameters and let $\mathbf{w}_\alpha$ and $\mathbf{w}_\gamma$ their linearized osculatrices, respectively. Then for $g_0 \in G$, we have $\gamma(I_2) = g_0 \cdot \alpha(I_1)$ if and only if there* *exists a bijection* $\varphi : I_1 \to I_2$ *of the form* $\varphi(s) = as+b$ *for some* $a, b \in \mathbb{R}$ *such that* $a\mathbf{w}_\gamma(\varphi(s)) = \mathrm{Ad}(g_0)(\mathbf{w}_\alpha(s))$ *for all* $s \in I_1$*, and* $\gamma(\varphi(s_0)) = g_0 \cdot \alpha(s_0)$ *for some* $s_0 \in I_1$*.*

5. Examples

Let H be a closed subgroup of $GL(n, \mathbb{R})$ and $G = H \rtimes \mathbb{R}^n$ be the semidirect product with the multiplication given by

$$(A, \mathbf{a})(B, \mathbf{b}) = (AB, A\mathbf{b} + \mathbf{a})$$

for $A, B \in H$ and $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$. Let $\mathfrak{g} = \mathfrak{h} \rtimes \mathbb{R}^n$ be its Lie algebra. It is well known that each $(\mathbf{X}, \mathbf{v}) \in \mathfrak{g}$ determines the one-parameter subgroup of G,

$$\phi(s) = \exp(s(\mathbf{X}, \mathbf{v})) = \left(\mathrm{e}^{s\mathbf{X}}, \left(\int_0^s \mathrm{e}^{x\mathbf{X}} dx\right)\mathbf{v}\right),$$

where $\mathrm{e}^{s\mathbf{X}}$ denotes the one-parameter subgroup of H determined by $\mathbf{X} \in \mathfrak{h}$.

Example 1 (Plane curves under similarities). As an enlightening example, let us consider the similarities as motions of the plane. This is simple enough to exemplify our constructions and to show how Theorem 8 opens up the possibility of getting invariants for curves other than their natural parameters.

With the above notation now we take $H = \{\lambda A; A \in SO(2), \lambda \in \mathbb{R}, \lambda > 0\}$ which has Lie algebra

$$\mathfrak{h} = \left\{ \begin{pmatrix} \delta & x \\ -x & \delta \end{pmatrix};\ x, \delta \in \mathbb{R} \right\}.$$

Thus for each $\mathbf{X} \in \mathfrak{h}$, we have the decomposition

$$\mathbf{X} = \begin{pmatrix} \delta & x \\ -x & \delta \end{pmatrix} = \delta \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & x \\ -x & 0 \end{pmatrix},$$

so that $\mathrm{e}^{s\mathbf{X}}$ can easily be calculated. Moreover, the orbit through $p \in \mathbb{R}^2$ determined by $\mathbf{u} = (\mathbf{X}, \mathbf{v})$ is parameterized as

$$s \mapsto \beta_{\mathbf{u}}(s) = \phi(s) \cdot p = \mathrm{e}^{s\mathbf{X}} p + \left(\int_0^s \mathrm{e}^{r\mathbf{X}} dr\right)\mathbf{v}.$$

If $\mathbf{u} \notin \mathfrak{g}_p$, then the orbit is either a straight line, a circle, or a logarithmic spiral.

To simplify the notation we identify $\mathbf{u} = (\mathbf{X}, \mathbf{v}) \in \mathfrak{g}$ with $(\delta, x, v_1, v_2) \in \mathbb{R}^4$, where $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathbb{R}^2$.

Let $I \subset \mathbb{R}$ be an open interval and let $\alpha = (\alpha_1, \alpha_2) : I \to \mathbb{R}^2$ be an embedding. Given $t \in I$ we consider $\beta_{\mathbf{u}}(s) = \exp(s\mathbf{u}) \cdot \alpha(t)$, the orbit through $\alpha(t)$ determined by $\mathbf{u}$. We have by definition

$$S_1^\alpha(t) = \{\mathbf{u} \in \mathfrak{g};\ \beta'_{\mathbf{u}}(0) = \lambda\alpha'(t),\ \text{for some } \lambda \in \mathbb{R}\}. \tag{11}$$

Since $\beta'_{\mathbf{u}}(0) = \mathbf{X}\alpha(t) + \mathbf{v}$, we obtain that $(\delta, x, v_1, v_2) \in S_1^\alpha(t)$ if and only if $v_1 = \lambda\alpha_1'(t) - \delta\alpha_1(t) - x\alpha_2(t)$ and $v_2 = \lambda\alpha_2'(t) - \delta\alpha_2(t) + x\alpha_1(t)$, for some real number λ.

Let $\langle \cdot,\cdot \rangle$ be the usual inner product on $\mathbb{R}^4 = \mathfrak{g}$ and let $\mathbf{u}\cdot\mathbf{v}$ denote the usual inner product on $\mathbb{R}^2$ for $\mathbf{u},\mathbf{v}\in\mathbb{R}^2$. It is easy to verify that if

$$\mathbf{N} = (\alpha_2\alpha_1' - \alpha_1\alpha_2', -\alpha\cdot\alpha', -\alpha_2', \alpha_1'),$$

then $\{\langle \mathbf{N}, -\rangle\}$ is a coframe along α. Now taking $\lambda \equiv 1$ in equation (11), the map

$$\mathbf{w}(t) = (\delta, x, \alpha_1'(t) - \delta\alpha_1(t) - x\alpha_2(t), \alpha_2'(t) - \delta\alpha_2(t) + x\alpha_1(t))$$

is the linearized osculatrix of α with preferred contact order $\kappa = 3$ if and only if the system of linear equations

$$\begin{cases} \langle \mathbf{w}(t), \mathbf{N}'(t)\rangle = 0, \\ \langle \mathbf{w}(t), \mathbf{N}''(t)\rangle = 0, \end{cases}$$

has unique solution. In fact, this happens if and only if

$$\mu := \frac{\alpha_1'\alpha_2'' - \alpha_1''\alpha_2'}{\alpha'\cdot\alpha'} \neq 0,$$

in which case the solution is

$$x = -\mu, \quad \delta = \frac{\alpha_2'\alpha_1''' - \alpha_2'''\alpha_1'}{\mu\alpha'\cdot\alpha'} + 3\frac{\alpha'\cdot\alpha''}{\alpha'\cdot\alpha'}.$$

Remark 16. It is clear from $\dim\mathfrak{g} = 4$ that the preferred contact order cannot be less than 3. Also it cannot be greater than 3 as it follows from [5, Theorem 3.2].

From now on we assume that $\mu \neq 0$. Let us write

$$\mathbf{w}_\alpha = (0, -\mu, \alpha_1' + \mu\alpha_2, \alpha_2' - \mu\alpha_1) + \delta(1, 0, -\alpha_1, -\alpha_2). \tag{12}$$

It is easy to see that, for $\bar{g} = (A, \mathbf{v}) \in G$ and $\mathbf{u} = (\mathbf{X}, \mathbf{w}) \in \mathfrak{g}$,

$$\mathrm{Ad}(\bar{g})(\mathbf{u}) = (A\mathbf{X}A^{-1}, A\mathbf{w} - A\mathbf{X}A^{-1}\mathbf{v}).$$

If $\gamma = (\gamma_1, \gamma_2) : I \to \mathbb{R}^2$ is another embedding with preferred contact order $\kappa = 3$, then we can write the linearized osculatrices as

$$\begin{aligned} \mathbf{w}_\alpha &= (0, -\mu_\alpha, \alpha_1' + \mu_\alpha\alpha_2, \alpha_2' - \mu_\alpha\alpha_1) + \delta_\alpha(1, 0, -\alpha_1, -\alpha_2), \\ \mathbf{w}_\gamma &= (0, -\mu_\gamma, \gamma_1' + \mu_\gamma\gamma_2, \gamma_2' - \mu_\gamma\gamma_1) + \delta_\gamma(1, 0, -\gamma_1, -\gamma_2), \end{aligned}$$

where

$$\begin{aligned} \mu_\alpha &= \frac{\alpha_1'\alpha_2'' - \alpha_1''\alpha_2'}{\alpha'\cdot\alpha'}, \quad \delta_\alpha = \frac{\alpha_2'\alpha_1''' - \alpha_2'''\alpha_1'}{\mu_\alpha\alpha'\cdot\alpha'} + 3\frac{\alpha'\cdot\alpha''}{\alpha'\cdot\alpha'}, \\ \mu_\gamma &= \frac{\gamma_1'\gamma_2'' - \gamma_1''\gamma_2'}{\gamma'\cdot\gamma'}, \quad \delta_\gamma = \frac{\gamma_2'\gamma_1''' - \gamma_2'''\gamma_1'}{\mu_\gamma\gamma'\cdot\gamma'} + 3\frac{\gamma'\cdot\gamma''}{\gamma'\cdot\gamma'}. \end{aligned}$$

Suppose α and γ are congruent, that is, $\gamma = g_0\cdot\alpha$ for some $g_0 = (A, \mathbf{v})$. By Theorem 8, $\mathbf{w}_\gamma = \mathrm{Ad}(g_0)(\mathbf{w}_\alpha)$ or, equivalently,

$$\begin{pmatrix} \delta_\gamma & -\mu_\gamma \\ \mu_\gamma & \delta_\gamma \end{pmatrix} = A \begin{pmatrix} \delta_\alpha & -\mu_\alpha \\ \mu_\alpha & \delta_\alpha \end{pmatrix} A^{-1}$$

and

$$\begin{pmatrix} \gamma_1' + \mu_\gamma\gamma_2 - \delta_\gamma\gamma_1 \\ \gamma_2' - \mu_\gamma\gamma_1 - \delta_\gamma\gamma_2 \end{pmatrix} = A \begin{pmatrix} \alpha_1' + \mu_\alpha\alpha_2 - \delta_\alpha\alpha_1 \\ \alpha_2' - \mu_\alpha\alpha_1 - \delta_\alpha\alpha_2 \end{pmatrix} - A \begin{pmatrix} \delta_\alpha & -\mu_\alpha \\ \mu_\alpha & \delta_\alpha \end{pmatrix} A^{-1}\mathbf{v}.$$

Since H is an abelian group the first of these equalities gives us $\mu_\alpha = \mu_\gamma$ and $\delta_\alpha = \delta_\gamma$: in other words, we obtain two congruence invariants.

Next we will look for the natural parameter. We want $\mathbf{w}'_\alpha \in S_2^\alpha \cap \mathfrak{g}_\alpha$. We claim that $(1, 0, -\alpha_1, -\alpha_2)$ generates $S_2^\alpha \cap \mathfrak{g}_\alpha$. Since $g = (E, \alpha) \in G$ satisfies $g \cdot 0 = \alpha$, where E is the identity matrix, we have $\mathfrak{g}_\alpha = \mathrm{Ad}(g)\mathfrak{g}_0 = \mathrm{Ad}(g)\mathfrak{h}$. As the elements of $\mathfrak{h}$ are of the form $(\mathbf{X}, \mathbf{0})$, we conclude that the elements of $\mathfrak{g}_\alpha$ are of the form

$$\mathbf{u} = \left(\begin{pmatrix} \epsilon & x \\ -x & \epsilon \end{pmatrix}, \begin{pmatrix} -\epsilon\alpha_1 - x\alpha_2 \\ -\epsilon\alpha_2 + x\alpha_1 \end{pmatrix} \right),$$

for some $x, \epsilon \in \mathbb{R}$, or better, by the identification above, $\mathbf{u} = (\epsilon, x, -\epsilon\alpha_1 - x\alpha_2, -\epsilon\alpha_2 + x\alpha_1)$. Such an element also belongs to S_2^α if and only if $\langle \mathbf{u}, \mathbf{N}' \rangle = 0$, i.e. $x = 0$. So $S_2^\alpha \cap \mathfrak{g}_\alpha$ is spanned by $(1, 0, -\alpha_1, -\alpha_2)$ as claimed.

Now

$$\begin{aligned} \mathbf{w}'_\alpha = (0, -\mu', \alpha''_1 + \mu'\alpha_2 + \mu\alpha'_2, \alpha''_2 - \mu'\alpha_1 - \mu\alpha'_1) \\ + \delta(0, 0, -\alpha'_1, -\alpha'_2) + \delta'(1, 0, -\alpha_1, -\alpha_2) \end{aligned}$$

and we see that if $\mathbf{w}'_\alpha \in S_2^\alpha \cap \mathfrak{g}_\alpha$, then $\mu' = 0$. This is also a sufficient condition, that is, if μ is constant, then

$$\alpha''_1 + \mu\alpha'_2 - \delta\alpha'_1 = \alpha''_2 - \mu\alpha'_1 - \delta\alpha'_2 = 0.$$

To see this, we first note that if $\mu = (\alpha'_1\alpha''_2 - \alpha''_1\alpha'_2)/\alpha' \cdot \alpha'$ is constant, then taking the derivative we obtain $2\mu\alpha' \cdot \alpha'' = \alpha'_1\alpha'''_2 - \alpha'''_1\alpha'_2$ and then we get $\delta = \alpha' \cdot \alpha''/\alpha' \cdot \alpha'$. Now it is easy to check that both equalities above hold. Summarizing we have the following:

Proposition 17. *Let $\alpha = (\alpha_1, \alpha_2) : I \to \mathbb{R}^2$ be an embedding with preferred contact order $\kappa = 3$. Then α carries a natural parameter if and only if $\mu = (\alpha'_1\alpha''_2 - \alpha''_1\alpha'_2)/\alpha' \cdot \alpha'$ is a non-zero constant.*

Even if α does not carry a natural parameter, making use of Lemma 11 it is possible to get a reparameterization, namely $\bar{\alpha} = \alpha \circ \varphi$, which does carry. Following the proof of that lemma we know that if ψ is a solution of $\psi'' - (\mu'/\mu)\psi' = 0$, then $\varphi = \psi^{-1}$ does work. So it is enough to fix $t_0 \in I$ and take

$$\psi(t) = \int_{t_0}^{t} \mu(r)dr.$$

As μ is G-invariant, we have obtained a G-invariant "arc element". More precisely, taking $\psi(t)$ as above, we have $\mu_{\bar{\alpha}} = (\bar{\alpha}'_1\bar{\alpha}''_2 - \bar{\alpha}''_1\bar{\alpha}'_2)/\bar{\alpha}' \cdot \bar{\alpha}' \equiv 1$.

Finally, we observe that if $\mathbf{w}_\alpha$ is the linearized osculatrix of α which carries a natural parameter, then we can write $\mathbf{w}'_\alpha = \delta'(1, 0, -\alpha_1, -\alpha_2)$. Thus, we have

$$\mathbf{w}'_\alpha(t_0) = \mathbf{0} \iff \delta'(t_0) = 0. \tag{13}$$

It is worth noting that if we carry out the above calculations in the euclidean case, then we get a result similar to (13) with δ being the curvature function (see [4]). So if in the present case we think of δ as a curvature function, then we may think of (13) as an analogue of the relationship between critical points of the curvature function and cuspidal points of the evolute of a curve in euclidean geometry. This generalizes what we observed in Remark 2.

Example 2 (Space curves under rigid motions). By Remark 5, one may fear that the non-degeneracy conditions in Definition 1 can be, generally speaking, too strong. In this example we investigate how the strongness shows up. As a byproduct we show that Remark 5 is true by showing that a straight line in $\mathbb{R}^3$ is not determined by a unique 1-dimensional subspace of the Lie algebra.

Set $H = SO(3)$ which has Lie algebra $\mathfrak{h}$ consisting of the anti-symmetric matrices of order 3. It is well known ([1, p. 198]) that the one-parameter subgroups of H give rise to rotations around some straight line in $\mathbb{R}^3$. Thus by choosing an appropriate coordinate system we can recognize the orbits: given $p \in \mathbb{R}^3$ and $\mathbf{u} = (\mathbf{X}, \mathbf{v}) \in \mathfrak{g} = \mathfrak{h} \rtimes \mathbb{R}^3$, the orbit of p by $\beta_{\mathbf{u}}$,

$$\beta_{\mathbf{u}}(s) = \phi(s) \cdot p = \mathrm{e}^{s\mathbf{X}} p + \left(\int_0^s \mathrm{e}^{r\mathbf{X}} dr \right) \mathbf{v},$$

is p itself if $(\mathbf{X}, \mathbf{v}) \in \mathfrak{g}_p$, otherwise it is either a straight line, a circle or a circular helix. Now for

$$\mathbf{A} = \begin{pmatrix} 0 & x & y \\ -x & 0 & z \\ -y & -z & 0 \end{pmatrix} \in \mathfrak{h}, \quad \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \in \mathbb{R}^3,$$

we identify the pair $(\mathbf{A}, \mathbf{v}) \in \mathfrak{g}$ with $(x, y, z, v_1, v_2, v_3) \in \mathbb{R}^6$.

Let $\alpha = (\alpha_1, \alpha_2, \alpha_3) : \mathbb{R} \to \mathbb{R}^3$ be an embedding. Computations similar to those in Example 1 allow us to conclude that the subspace S_1^α of $\mathfrak{g}$ is spanned by

$$\begin{aligned} \mathbf{n}_1 &= (1, 0, 0, -\alpha_2, \alpha_1, 0), \quad & \mathbf{n}_2 &= (0, 1, 0, -\alpha_3, 0, \alpha_1), \\ \mathbf{n}_3 &= (0, 0, 1, 0, -\alpha_3, \alpha_2), \quad & \mathbf{n}_4 &= (0, 0, 0, \alpha_1', \alpha_2', \alpha_3'). \end{aligned}$$

Let us assume that, say, $\alpha_1' \neq 0$. Then one can check that

$$\begin{aligned} \mathbf{N}_1 &= (\alpha_2\alpha_2' + \alpha_1\alpha_1', \alpha_3\alpha_2', -\alpha_3\alpha_1', \alpha_2', -\alpha_1', 0), \\ \mathbf{N}_2 &= (\alpha_2\alpha_3', \alpha_3\alpha_3' + \alpha_1\alpha_1', \alpha_2\alpha_1', \alpha_3', 0, -\alpha_1') \end{aligned}$$

give rise to a coframe $\{\langle \mathbf{N}_1, - \rangle, \langle \mathbf{N}_2, - \rangle\}$ along α.

Now we look for the subspaces S_i^α at a point. Without loss of generality for our purposes, we may assume that this point is the origin of $\mathbb{R}^3$ and that the curve α is tangent to the first coordinate axis there. Then we take α in the form $\alpha(t) = (t, f(t), g(t))$ with $f(0) = g(0) = f'(0) = g'(0) = 0$.

Under these conditions we obtain the following equivalences:

$$(x, y, z, v_1, v_2, v_3) \in S_1^\alpha(0) \Leftrightarrow v_2 = v_3 = 0,$$

$$(x, y, z, v_1, 0, 0) \in S_2^\alpha(0) \Leftrightarrow \begin{cases} x = -f''(0)v_1, \\ y = -g''(0)v_1, \end{cases}$$

$$(-f''(0)v_1, -g''(0)v_1, z, v_1, 0, 0) \in S_3^\alpha(0) \Leftrightarrow \begin{cases} f''(0)z = -g'''(0)v_1, \\ g''(0)z = f'''(0)v_1. \end{cases}$$

We shall consider two cases, depending on the curvature K of α,

$$K = \frac{\|\alpha' \times \alpha''\|}{\|\alpha'\|^3} = \frac{\sqrt{(f'g'' - g'f'')^2 + f''^2 + g''^2}}{\left(\sqrt{1 + f'^2 + g'^2}\right)^3},$$

at the origin.

Case 1: $f''(0) = g''(0) = 0$. This is equivalent to the vanishing of the curvature of α at 0. In this case, there exists an orbit having contact of order 3 with α at the origin if and only if $g'''(0) = f'''(0) = 0$. This happens if and only if α has contact of order 3 with its tangent line there. Note that in this case the straight line tangent to α is not determined by a unique 1-dimensional subspace of the Lie algebra $\mathfrak{g}$, since $\dim S_3^\alpha(0) = 2$, $\dim(\mathfrak{g}_{\alpha(0)} \cap S_3^\alpha(0)) = 1$, and no other kind of orbit has zero curvature. This shows the truth of Remark 5.

Case 2: α has non-zero curvature at 0. In this case, there exists an orbit having contact of order 3 with α at the origin if and only if $f'''(0)f''(0) + g'''(0)g''(0) = 0$, which is equivalent to $\dim S_3^\alpha(0) = 1$. As one can easily check, these conditions are equivalent to $K'(0) = 0$. Since this property is G-invariant and is also invariant under reparameterizations of α, we see that a necessary and sufficient condition for the existence of the linearized osculatrix of α is the constancy of its curvature.

From Classical Differential Geometry we know that if the curvature of α is not zero at some point, then its osculating circle as well as any helix having this same circle as its osculating circle should be orbits having contact of order 2 with α there. In addition, that circle is unique but there are many such helices. We can see the latter statement by just computing the torsion of the orbit determined by an element of $S_2^\alpha(0)$. Set

$$\mathbf{X} = \begin{pmatrix} 0 & -f''(0)v_1 & -g''(0)v_1 \\ f''(0)v_1 & 0 & z \\ g''(0)v_1 & -z & 0 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} v_1 \\ 0 \\ 0 \end{pmatrix},$$

where $\mathbf{u} = (\mathbf{X}, \mathbf{v})$ is an arbitrary element of $S_2^\alpha(0)$. The orbit $\beta_{\mathbf{u}}(s) = \exp(s\mathbf{u}) \cdot \alpha(0)$ satisfies

$$\beta'_{\mathbf{u}}(0) = \begin{pmatrix} v_1 \\ 0 \\ 0 \end{pmatrix}, \; \beta''_{\mathbf{u}}(0) = \begin{pmatrix} 0 \\ f''(0)v_1 \\ g''(0)v_2 \end{pmatrix}, \; \beta'''_{\mathbf{u}}(0) = \begin{pmatrix} -(f''(0)^2 + g''(0)^2)v_1 \\ zg''(0)v_1 \\ -zf''(0)v_1 \end{pmatrix}.$$

Now take $v_1 \neq 0$ so that $\beta_{\mathbf{u}}$ is locally an embedding. Then it is easy to compute the torsion of $\beta_{\mathbf{u}}$,

$$\tau = \frac{(\beta'_{\mathbf{u}}(0) \times \beta''_{\mathbf{u}}(0)) \cdot \beta'''_{\mathbf{u}}(0)}{\|\beta'_{\mathbf{u}}(0) \times \beta''_{\mathbf{u}}(0)\|^2} = -\frac{z}{v_1}.$$

It follows that two helices coincide if and only if the ratios corresponding to z/v_1 coincide (of course, when $z = 0$, a helix degenerates to a circle).

It is clear that if α has contact of order 3 with some orbit, then α has the same curvature and torsion as the orbit. What the above computation shows is that having contact of order 3 is a too strong condition, in the sense that among all helices with contact of order 2 with α there is only one with the same torsion as α, but it may not necessarily have contact of order 3 with α. So the natural question is how to choose the helix having contact of order 2 with α and having the same torsion as α, without having previously mentioned about the torsion? Presumably it would be unavoidable to consider contact of curves with planes, but also we believe that it would be possible to enlarge the class of model curves in a reasonable way. This discussion suggests us the inclusion of curves with constant curvature but we do not see how naturally they emerge from the group G.

References

[1] Crampin, M. and Pirani, F. A. E., "Applicable Differential Geometry", London Mathematical Society Lecture Note Series, Vol. 59, Cambridge Univ. Press, Cambridge, 1986.

[2] Green, M. L., *The moving frame, differential invariants and rigidity theorems for curves in homogeneous spaces*, Duke Math. J. **45** (1978), 735–779.

[3] Griffiths, P., *On Cartan's method of Lie groups and moving frames as applied to uniqueness and existence questions in differential geometry*, Duke Math. J. **41** (1974), 775–814.

[4] do Nascimento, V. M., "Curvas em Espaços Homogêneos" (in Portuguese), PhD Thesis, ICMC, University of São Paulo, 1998.

[5] do Nascimento, V. M. and Saeki, O., *Curves in homogeneous spaces and their contact with* 1*-dimensional orbits*, Geom. Dedicata **154** (2011), 117–131.

[6] Nomizu, K., "Lie Groups and Differential Geometry", The Mathematical Society of Japan, 1956.

[7] Weyl, H., *Cartan on groups and differential geometry* [Review of the book *"La Théorie des Groupes Finis et Continus et la Géométrie Différentielle traitées par la Méthode du Repère Mobile"* by Élie Cartan], Bull. Amer. Math. Soc. **44** (1938), 598–601.

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