SINGULAR FIBERS OF STABLE MAPS OF 3-MANIFOLDS WITH BOUNDARY INTO SURFACES AND THEIR APPLICATIONS

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Abstract. In this paper, we first classify the singular fibers of proper C^∞ stable maps of 3-dimensional manifolds with boundary into surfaces. Then, we compute the cohomology groups of the associated universal complex of singular fibers, and obtain certain cobordism invariants for Morse functions on compact surfaces with boundary.

1. Introduction

Let M and N be smooth manifolds, where M may possibly have boundary, while N has no boundary. For a C^∞ map $f\colon M \to N$ and a point $q \in N$, the map germ along the inverse image

$$f\colon (M, f^{-1}(q)) \to (N, q)$$

is called the *fiber* over q (see [7]). In particular, if a point $q \in N$ is a regular value of both f and $f|_{\partial M}$, then we call the fiber over q a *regular fiber*; otherwise, a *singular fiber*.

Equivalences among the fibers are defined as follows. Let $f_i\colon M_i \to N_i$, $i = 0, 1$, be C^∞ maps. For $q_i \in N_i$, $i = 0, 1$, we say that the fibers over q_0 and q_1 are C^∞ *equivalent* (or C^0 *equivalent*) if for some open neighborhoods U_i of q_i in N_i, there exist diffeomorphisms (resp. homeomorphisms) $\Phi\colon f^{-1}(U_0) \to f^{-1}(U_1)$ and $\varphi\colon U_0 \to U_1$ with $\varphi(q_0) = q_1$ which make the following diagram commutative:

$$\begin{array}{ccc} (f_0^{-1}(U_0), f_0^{-1}(q_0)) & \xrightarrow{\Phi} & (f_1^{-1}(U_1), f_1^{-1}(q_1)) \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f_1} \\ (U_0, q_0) & \xrightarrow{\varphi} & (U_1, q_1). \end{array}$$

Denote by $C^\infty(M, N)$ the set of C^∞ maps $M \to N$ equipped with the Whitney C^∞ topology. A C^∞ map $f\colon M \to N$ is called a *stable map* (or more precisely, a C^∞ *stable map*) if there exists a neighborhood $N(f) \subset C^\infty(M, N)$ of f such that every map $g \in N(f)$ is C^∞ right-left equivalent to f [4]. Here, two maps f and $g \in C^\infty(M, N)$ are C^∞ *right-left equivalent* if there exist diffeomorphisms $\Psi\colon M \to M$ and $\psi\colon N \to N$ such that $f \circ \Psi = \psi \circ g$.

The notion of singular fibers of C^∞ maps between manifolds without boundary was first introduced in [7], where classifications of singular fibers for stable maps $M \to N$ with $(\dim M, \dim N) = (2, 1), (3, 2)$ and $(4, 3)$ were obtained. Later, singular fibers of stable maps of manifolds without boundary were studied in [7, 8, 11, 12, 15, 16, 17], especially in connection with cobordisms. The first author [7] established the theory of universal complex of singular fibers of C^∞ maps as an analogy of the Vassiliev complex for map germs [6, 14]. This can be used for

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getting certain cobordism invariants of singular maps. For example, the second author [17] studied the universal complex of singular fibers of two-colored C^∞ maps, and computed its cohomology groups. In these theories, for a certain set of singular fibers τ, cohomology classes of the universal complexes of singular fibers of τ-maps provide τ-cobordism invariants for τ-maps.

In this paper, we study the singular fibers of proper C^∞ stable maps of 3-dimensional manifolds with boundary into surfaces without boundary. By using the fibers, we obtain cobordism invariants for stable Morse functions on compact surfaces with boundary.

The paper is organized as follows. In §2, we classify the fibers of proper C^∞ stable maps of 3-dimensional manifolds with boundary into surfaces without boundary, with respect to the C^∞ equivalence relation. For this we use several known results on the classification of stable singularities of maps on manifolds with boundary [5, 13] together with the techniques developed by the first author in [7]. In §3, we obtain several co-existence formulae of singular fibers of C^∞ stable maps of compact 3-dimensional manifolds with boundary into surfaces without boundary. These formulae can be obtained by analyzing the adjacencies of the singular fibers. In §4, we construct the universal complex of singular fibers of proper C^∞ stable maps of 3-dimensional manifolds with boundary into surfaces without boundary. By carefully computing the cohomology groups of the universal complex, we obtain certain cobordism invariants for stable Morse functions on compact surfaces with boundary.

Throughout the paper, all manifolds and maps between them are smooth of class C^∞. For a smooth map $f\colon M \to N$ between manifolds, we denote by $S(f)$ the set of points in M where the differential of f does not have maximal rank $\min\{\dim M, \dim N\}$. For a space X, id_X denotes the identity map of X. For a (co)cycle c, we denote by $[c]$ the (co)homology class represented by c.

2. Classification of singular fibers

In this section, we classify the singular fibers of proper C^∞ stable maps of 3-dimensional manifolds with boundary into surfaces without boundary.

We can prove the following characterization of C^∞ stable maps $f\colon M \to N$ by using standard techniques in singularity theory together with the results on local normal forms obtained in [5, 13].

Proposition 2.1 (Shibata [13], Martins and Nabarro [5])**.** *Let M be a 3-manifold possibly with boundary and N a surface without boundary. A proper C^∞ map $f\colon M \to N$ is C^∞ stable if and only if it satisfies the following conditions.*

(1) (*Local conditions*) *In the following, for $p \in \partial M$, we use local coordinates (x, y, z) around p such that $\mathrm{Int}\, M$ and ∂M correspond to the sets $\{z > 0\}$ and $\{z = 0\}$, respectively.*

(1a) *For $p \in \mathrm{Int}\, M$, the germ of f at p is right-left equivalent to one of the following:*

$$(x,y,z) \mapsto \begin{cases} (x,y), & p\text{: regular point,} \\ (x, y^2+z^2), & p\text{: definite fold point,} \\ (x, y^2-z^2), & p\text{: indefinite fold point,} \\ (x, y^3+xy-z^2), & p\text{: cusp point.} \end{cases}$$

(1b) *For $p \in \partial M \setminus S(f)$, the germ of f at p is right-left equivalent to one of the following:*

$$(x,y,z) \mapsto \begin{cases} (x,y), & p\text{: regular point of } f|_{\partial M}, \\ (x, y^2+z), & p\text{: boundary definite fold point}, \\ (x, y^2-z), & p\text{: boundary indefinite fold point}, \\ (x, y^3+xy+z), & p\text{: boundary cusp point.} \end{cases}$$

(1c) *For $p \in \partial M \cap S(f)$, the germ of f at p is right-left equivalent to the map germ*

$$(x,y,z) \mapsto (x, y^2+xz \pm z^2).$$

(2) (*Global conditions*) *For each $q \in f(S(f)) \cup f(S(f|_{\partial M}))$, the multi-germ*

$$(f|_{S(f)\cup S(f|_{\partial M})}, f^{-1}(q) \cap (S(f) \cup S(f|_{\partial M})))$$

is right-left equivalent to one of the eight multi-germs as depicted in Figure 1, where the ordinary curves correspond to the singular value set $f(S(f))$ and the dotted curves to $f(S(f|_{\partial M}))$: (1) *corresponds to a single fold point,* (4) *corresponds to a single boundary fold point,* (3), (6) *and* (7) *represent normal crossings of two immersion germs, each of which corresponds to a fold point or a boundary fold point,* (2) *corresponds to a cusp point,* (5) *corresponds to a boundary cusp point, and* (8) *corresponds to a single point in $\partial M \cap S(f)$.*

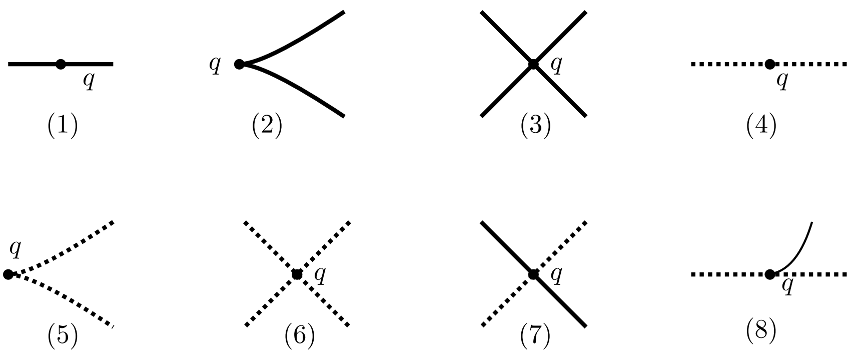

FIGURE 1. Multi-germs of $f|_{S(f)\cup S(f|_{\partial M})}$

Note that if a C^∞ map $f\colon M \to N$ is C^∞ stable, then so is $f|_{\partial M} : \partial M \to N$.

In the following, a map germ at a point on the boundary right-left equivalent to the normal form

$$(x,y,z) \mapsto (x, y^2+xz+z^2) \quad \text{or} \quad (x,y,z) \mapsto (x, y^2+xz-z^2)$$

is called a *definite* $\Sigma^{2,0}_{1,0}$ *point* or an *indefinite* $\Sigma^{2,0}_{1,0}$ *point*, respectively. By Proposition 2.1, we have the following.

Lemma 2.2. *Let M be a 3-manifold possibly with boundary, N a surface without boundary, and $f\colon M \to N$ a proper C^∞ stable map. Then, every point $p \in S(f) \cup S(f|_{\partial M})$ has one of the following neighborhoods in its corresponding singular fiber (see Figures* 2 *and* 3 *for details):*

(1) *isolated point diffeomorphic to $\{(y,z) \in \mathbb{R}^2 \,|\, y^2+z^2=0\}$, if $p \in \operatorname{Int} M$ is a definite fold point,*
(2) *union of two transverse arcs diffeomorphic to $\{(y,z) \in \mathbb{R}^2 \,|\, y^2-z^2=0\}$, if $p \in \operatorname{Int} M$ is an indefinite fold point,*

(3) *cuspidal arc diffeomorphic to* $\{(y,z) \in \mathbb{R}^2 \,|\, y^3 - z^2 = 0\}$, *if* $p \in \operatorname{Int} M$ *is a cusp point,*
(4) *isolated point diffeomorphic to* $\{(y,z) \in \mathbb{R}^2 \,|\, y^2 + z = 0,\ z \geq 0\}$, *if* $p \in \partial M$ *is a boundary definite fold point,*
(5) *arc diffeomorphic to* $\{(y,z) \in \mathbb{R}^2 \,|\, y^2 - z = 0,\ z \geq 0\}$, *if* $p \in \partial M$ *is a boundary indefinite fold point,*
(6) *arc diffeomorphic to* $\{(y,z) \in \mathbb{R}^2 \,|\, y^3 + z = 0,\ z \geq 0\}$, *if* $p \in \partial M$ *is a boundary cusp point,*
(7) *isolated point diffeomorphic to* $\{(y,z) \in \mathbb{R}^2 \,|\, y^2 + z^2 = 0,\ z \geq 0\}$, *if* $p \in \partial M \cap S(f)$ *is a definite* $\Sigma^{2,0}_{1,0}$ *point,*
(8) *polygonal line diffeomorphic to* $\{(y,z) \in \mathbb{R}^2 \,|\, y^2 - z^2 = 0,\ z \geq 0\}$, *if* $p \in \partial M \cap S(f)$ *is an indefinite* $\Sigma^{2,0}_{1,0}$ *point.*

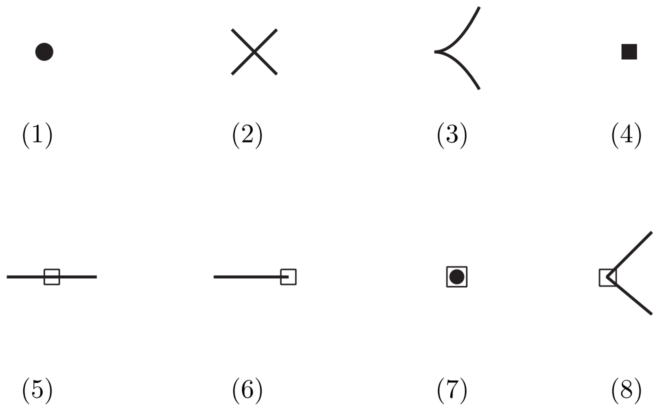

FIGURE 2. Neighborhoods of singular points in their singular fibers

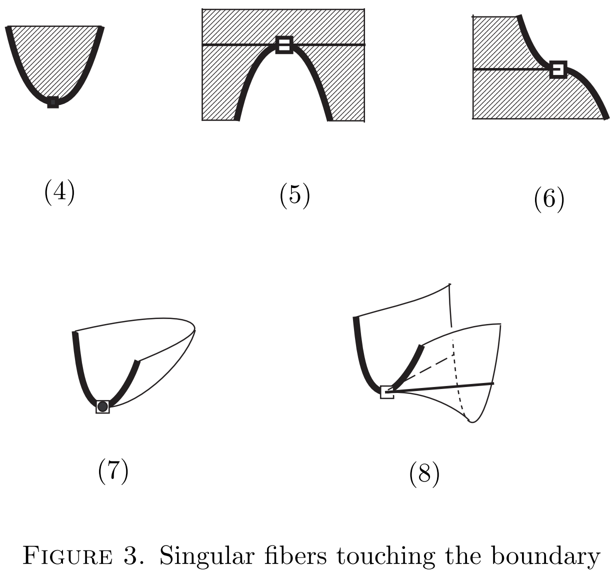

FIGURE 3. Singular fibers touching the boundary

Note that in Figure 2, the black dot (1), the black square (4), and the black dot surrounded by a square (7) all represent an isolated point; however, we use distinct symbols in order to distinguish them as map-germs. In the figures, the squares represent points on the boundary; more precisely, they are points in $S(f|_{\partial M})$. In Figure 3, singular fibers that intersect the boundary are depicted, where the surfaces appearing in the figures correspond to the hypersurface $x = 0$, the intersection of the hypersurface with the boundary is depicted by thick curves, and the fibers correspond to the level sets of the respective height functions.

For the local nearby fibers, we have the following, which can be proved by direct calculations using the corresponding normal forms.

Lemma 2.3. *Let M be a 3-manifold possibly with boundary, N a surface without boundary, $f\colon M \to N$ a proper C^∞ stable map, and $p \in S(f) \cup S(f|_{\partial M})$ a singular point such that $f^{-1}(f(p)) \cap (S(f) \cup S(f|_{\partial M})) = \{p\}$. Then, the fibers near p are as depicted in Figure 4:*

(1) *$p \in \operatorname{Int} M$ is a definite fold point,*
(2) *$p \in \operatorname{Int} M$ is an indefinite fold point,*
(3) *$p \in \operatorname{Int} M$ is a cusp point,*
(4) *$p \in \partial M$ is a boundary definite fold point,*
(5) *$p \in \partial M$ is a boundary indefinite fold point,*
(6) *$p \in \partial M$ is a boundary cusp point,*
(7) *$p \in S(f) \cap \partial M$ is a definite $\Sigma^{2,0}_{1,0}$ point,*
(8) *$p \in S(f) \cap \partial M$ is an indefinite $\Sigma^{2,0}_{1,0}$ point,*

where each of the 0-dimensional objects and the thin 1-dimensional objects represents a portion of the fiber over the corresponding point in the target, each of the thick curves represents $f(S(f))$, and each of the dotted curves represents $f(S(f|_{\partial M}))$ near $f(p)$. Furthemore, the dotted squares represent (transverse) intersections with ∂M.

Then, by using the method developed in [7], we get the following list of singular fibers.

Proposition 2.4. *Let $f\colon M \to N$ be a proper C^∞ stable map of a 3-manifold M with boundary into a surface N without boundary. Then, every fiber of f is equivalent to the disjoint union of one of the fibers in the following list, a finite number of copies of a fiber of the trivial circle bundle, and a finite number of copies of a fiber of the trivial I-bundle, where $I = [-1, 1]$:*

(1) *fibers as depicted in Figure 5, i.e. $\widetilde{\mathrm{b0}}^0$, $\widetilde{\mathrm{b0}}^1$, and $\widetilde{\mathrm{bI}}^\mu$ with $2 \le \mu \le 10$,*
(2) *disconnected fibers $\widetilde{\mathrm{bII}}^{\mu,\nu}$ with $2 \le \mu \le \nu \le 10$, where $\widetilde{\mathrm{bII}}^{\mu,\nu}$ means the disjoint union of $\widetilde{\mathrm{bI}}^\mu$ and $\widetilde{\mathrm{bI}}^\nu$,*
(3) *the connected fibers as depicted in Figure 6, i.e. $\widetilde{\mathrm{bII}}^\mu$ with $11 \le \mu \le 39$, $\widetilde{\mathrm{bII}}^a$, $\widetilde{\mathrm{bII}}^b$, $\widetilde{\mathrm{bII}}^c$, $\widetilde{\mathrm{bII}}^d$, $\widetilde{\mathrm{bII}}^e$ and $\widetilde{\mathrm{bII}}^f$.*

In Figures 5 and 6, κ denotes the *codimension* of the set of points in the target N whose corresponding fibers are C^∞ equivalent to the relevant one (see [7] for details). Furthermore, the symbols $\widetilde{\mathrm{b0}}^*$, $\widetilde{\mathrm{bI}}^*$, and $\widetilde{\mathrm{bII}}^*$ mean the names of the corresponding singular fibers. Note that we have named the fibers so that each connected fiber has its own number or letter, and a disconnected fiber has the name consisting of the numbers of its connected components. Note also that each figure represents a map germ along the corresponding fiber and not just the inverse image of a point.

Note that the fibers that do not intersect $S(f|_{\partial M})$ have been essentially obtained in [8, §6].

Theorem 6 is proved by using the relative version of Ehresmann's fibration Theorem. See [7, Theorem 1.4] for details.

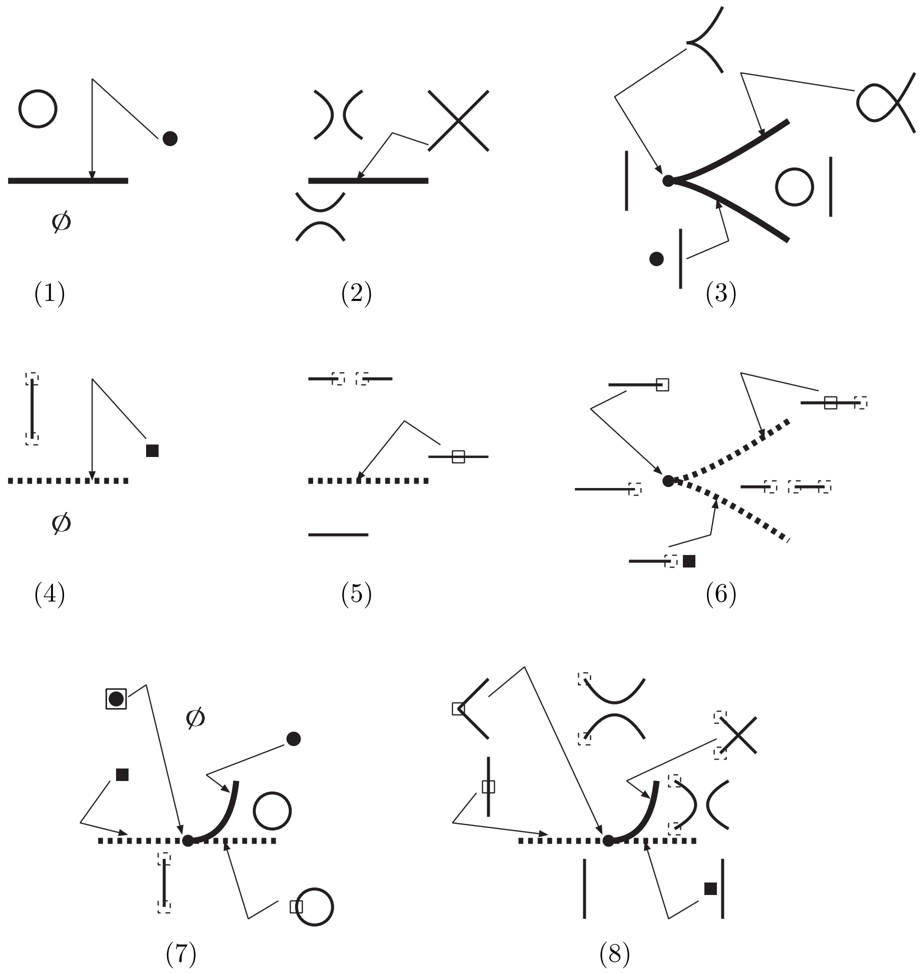

FIGURE 4. Local degenerations of fibers

Then, we immediately obtain the following corollary. (For details, see [7, Proof of Corollary 3.9].)

Corollary 2.5. *Two fibers of proper C^∞ stable maps of 3-manifolds with boundary into surfaces without boundary are C^∞ equivalent if and only if they are C^0 equivalent.*

Remark 2.6. If the source 3-manifold is orientable, then the singular fibers of types $\widetilde{\mathrm{bI}}^9$, $\widetilde{\mathrm{bI}}^{10}$, $\widetilde{\mathrm{bII}}^{26}$, $\widetilde{\mathrm{bII}}^{27}$, $\widetilde{\mathrm{bII}}^{28}$, $\widetilde{\mathrm{bII}}^{29}$, $\widetilde{\mathrm{bII}}^{30}$, $\widetilde{\mathrm{bII}}^{31}$, $\widetilde{\mathrm{bII}}^{32}$, $\widetilde{\mathrm{bII}}^{33}$, $\widetilde{\mathrm{bII}}^{34}$, $\widetilde{\mathrm{bII}}^{35}$, $\widetilde{\mathrm{bII}}^{36}$, $\widetilde{\mathrm{bII}}^{37}$, $\widetilde{\mathrm{bII}}^{38}$, and $\widetilde{\mathrm{bII}}^{39}$ never appear.

Remark 2.7. Our classification result of singular fibers of stable maps of compact orientable 3-dimensional manifolds with boundary into surfaces has already been applied in computer science. More precisely, it helps to visualize characteristic features of certain multi-variate data. For details, see [9, 10].

Remark 2.8. Let V be a surface with boundary and W be the real line $\mathbb{R}$ or the circle S^1. A proper C^∞ function $f\colon V \to W$ is C^∞ stable (i.e. it is a *stable Morse function*) if and only if it satisfies the following conditions.

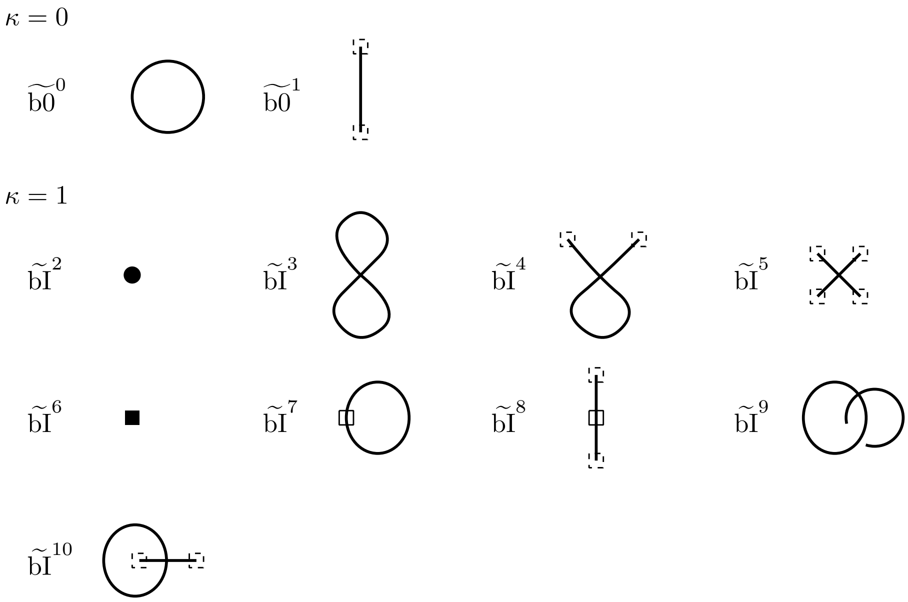

FIGURE 5. List of the fibers of proper C^∞ stable maps of 3-manifolds with boundary into surfaces without boundary; 1

(1) (Local conditions) In the following, for $p \in \partial V$, we use local coordinates (x, y) around p such that $\operatorname{Int} V$ and ∂V correspond to the sets $\{y > 0\}$ and $\{y = 0\}$, respectively.

(1a) For $p \in \operatorname{Int} V$, the germ of f at p is right-left equivalent to one of the following:

$$(x, y) \mapsto \begin{cases} x & \text{if } p \text{ is a regular point,} \\ x^2 \pm y^2 & \text{if } p \text{ is a definite fold point,} \end{cases}$$

(1b) For $p \in \partial V$, the germ of f at p is right-left equivalent to one of the following:

$$(x, y) \mapsto \begin{cases} x & \text{if } p \text{ is a regular point of } f|_{\partial V}, \\ x^2 \pm y & \text{if } p \text{ is a boundary definite fold point,} \end{cases}$$

(2) (Global conditions) $f(p_1) \neq f(p_2)$ if $p_1 \neq p_2 \in S(f) \cup S(f|_{\partial V})$.

The C^∞ equivalence classes of singular fibers of proper stable Morse functions on surfaces with boundary are nothing but those appearing in Figure 5 with $\kappa = 0, 1$.

3. Co-existence of singular fibers

Let M be a compact 3-dimensional manifold with boundary, N a surface without boundary, $f\colon M \to N$ a C^∞ stable map, and $\widetilde{\mathcal{F}}$ an equivalence class of singular fibers of codimension ≥ 1. Define $\widetilde{\mathcal{F}}(f)$ to be the set of points $q \in N$ such that the fiber over q is equivalent to the union of $\widetilde{\mathcal{F}}$ and some copies of a fiber of a trivial circle bundle and some copies of a fiber of a trivial I-bundle. Furthermore, define $\widetilde{\mathcal{F}}_o(f)$ (or $\widetilde{\mathcal{F}}_e(f)$) to be the subset of $\widetilde{\mathcal{F}}(f)$ which consists of the points $q \in N$ such that the number of regular fiber components, namely the total number of $\widetilde{\mathrm{bI}}^0$ components and $\widetilde{\mathrm{bI}}^1$ components in the fiber, is odd (resp. even). For codimension zero fibers, by convention, we denote by $\widetilde{\mathrm{b0}}_o(f)$ (or $\widetilde{\mathrm{b0}}_e(f)$) the set of points $q \in N$ over which lies a regular fiber consisting of an odd (resp. even) number of components.

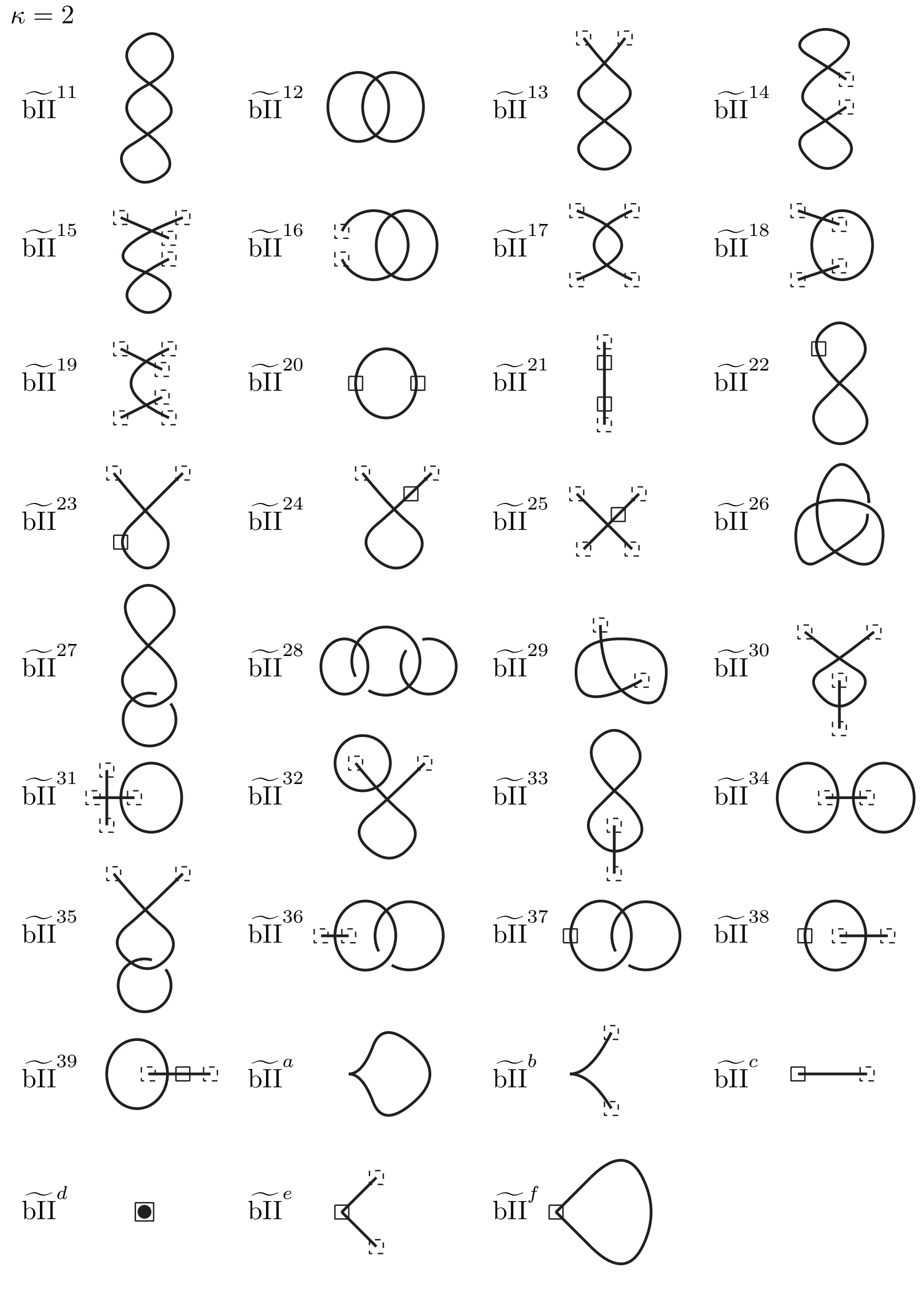

FIGURE 6. List of the fibers of proper C^∞ stable maps of 3-manifolds with boundary into surfaces without boundary; 2

If $\widetilde{\mathcal{F}}$ is of codimension 1, then the closure of $\widetilde{\mathcal{F}}_o(f)$ (or $\widetilde{\mathcal{F}}_e(f)$) is a finite graph embedded in N. Its vertices correspond to points over which lies a singular fiber of codimension 2.

The handshake lemma of the classical graph theory implies the following formula. In the following, for a finite set S, $|S|$ denotes its cardinality.

Proposition 3.1. *Let $f\colon M \to N$ be a C^∞ stable map of a compact 3-manifold M with boundary into a surface N without boundary. Then, the following numbers are always even:*

(1) $|\widetilde{\mathrm{bII}}^{2,3}(f)| + |\widetilde{\mathrm{bII}}^{2,4}(f)| + |\widetilde{\mathrm{bII}}^{2,6}(f)| + |\widetilde{\mathrm{bII}}^{2,8}(f)| + |\widetilde{\mathrm{bII}}^{a}_{e}(f)| + |\widetilde{\mathrm{bII}}^{b}_{e}(f)| + |\widetilde{\mathrm{bII}}^{d}_{o}(f)|$,

(2) $|\widetilde{\mathrm{bII}}^{2,3}(f)| + |\widetilde{\mathrm{bII}}^{2,4}(f)| + |\widetilde{\mathrm{bII}}^{2,6}(f)| + |\widetilde{\mathrm{bII}}^{2,8}(f)| + |\widetilde{\mathrm{bII}}^{a}_{o}(f)| + |\widetilde{\mathrm{bII}}^{b}_{o}(f)| + |\widetilde{\mathrm{bII}}^{d}_{e}(f)|$,

(3) $|\widetilde{\mathrm{bII}}^{2,3}(f)| + |\widetilde{\mathrm{bII}}^{3,4}(f)| + |\widetilde{\mathrm{bII}}^{3,6}(f)| + |\widetilde{\mathrm{bII}}^{3,8}(f)| + |\widetilde{\mathrm{bII}}^{13}_{e}(f)| + |\widetilde{\mathrm{bII}}^{22}_{o}(f)| + |\widetilde{\mathrm{bII}}^{a}_{o}(f)|$,

(4) $|\widetilde{\mathrm{bII}}^{2,3}(f)| + |\widetilde{\mathrm{bII}}^{3,4}(f)| + |\widetilde{\mathrm{bII}}^{3,6}(f)| + |\widetilde{\mathrm{bII}}^{3,8}(f)| + |\widetilde{\mathrm{bII}}^{13}_{o}(f)| + |\widetilde{\mathrm{bII}}^{22}_{e}(f)| + |\widetilde{\mathrm{bII}}^{a}_{e}(f)|$,

(5) $|\widetilde{\mathrm{bII}}^{2,4}(f)| + |\widetilde{\mathrm{bII}}^{3,4}(f)| + |\widetilde{\mathrm{bII}}^{4,6}(f)| + |\widetilde{\mathrm{bII}}^{4,8}(f)| + |\widetilde{\mathrm{bII}}^{13}_{e}(f)| + |\widetilde{\mathrm{bII}}^{22}_{o}(f)| + |\widetilde{\mathrm{bII}}^{23}_{o}(f)| + |\widetilde{\mathrm{bII}}^{24}(f)| + |\widetilde{\mathrm{bII}}^{b}_{o}(f)| + |\widetilde{\mathrm{bII}}^{f}_{o}(f)|$,

(6) $|\widetilde{\mathrm{bII}}^{2,4}(f)| + |\widetilde{\mathrm{bII}}^{3,4}(f)| + |\widetilde{\mathrm{bII}}^{4,6}(f)| + |\widetilde{\mathrm{bII}}^{4,8}(f)| + |\widetilde{\mathrm{bII}}^{13}_{o}(f)| + |\widetilde{\mathrm{bII}}^{22}_{e}(f)| + |\widetilde{\mathrm{bII}}^{23}_{e}(f)| + |\widetilde{\mathrm{bII}}^{24}(f)| + |\widetilde{\mathrm{bII}}^{b}_{e}(f)| + |\widetilde{\mathrm{bII}}^{f}_{e}(f)|$,

(7) $|\widetilde{\mathrm{bII}}^{2,5}(f)| + |\widetilde{\mathrm{bII}}^{3,5}(f)| + |\widetilde{\mathrm{bII}}^{4,5}(f)| + |\widetilde{\mathrm{bII}}^{5,6}(f)| + |\widetilde{\mathrm{bII}}^{5,8}(f)| + |\widetilde{\mathrm{bII}}^{15}(f)| + |\widetilde{\mathrm{bII}}^{23}_{o}(f)| + |\widetilde{\mathrm{bII}}^{25}(f)| + |\widetilde{\mathrm{bII}}^{30}_{o}(f)| + |\widetilde{\mathrm{bII}}^{38}_{o}(f)| + |\widetilde{\mathrm{bII}}^{e}_{o}(f)|$,

(8) $|\widetilde{\mathrm{bII}}^{2,5}(f)| + |\widetilde{\mathrm{bII}}^{3,5}(f)| + |\widetilde{\mathrm{bII}}^{4,5}(f)| + |\widetilde{\mathrm{bII}}^{5,6}(f)| + |\widetilde{\mathrm{bII}}^{5,8}(f)| + |\widetilde{\mathrm{bII}}^{15}(f)| + |\widetilde{\mathrm{bII}}^{23}_{e}(f)| + |\widetilde{\mathrm{bII}}^{25}(f)| + |\widetilde{\mathrm{bII}}^{30}_{e}(f)| + |\widetilde{\mathrm{bII}}^{38}_{e}(f)| + |\widetilde{\mathrm{bII}}^{e}_{e}(f)|$,

(9) $|\widetilde{\mathrm{bII}}^{2,6}(f)| + |\widetilde{\mathrm{bII}}^{3,6}(f)| + |\widetilde{\mathrm{bII}}^{4,6}(f)| + |\widetilde{\mathrm{bII}}^{6,8}(f)| + |\widetilde{\mathrm{bII}}^{c}_{e}(f)| + |\widetilde{\mathrm{bII}}^{d}_{o}(f)| + |\widetilde{\mathrm{bII}}^{e}_{e}(f)| + |\widetilde{\mathrm{bII}}^{f}_{e}(f)|$,

(10) $|\widetilde{\mathrm{bII}}^{2,6}(f)| + |\widetilde{\mathrm{bII}}^{3,6}(f)| + |\widetilde{\mathrm{bII}}^{4,6}(f)| + |\widetilde{\mathrm{bII}}^{6,8}(f)| + |\widetilde{\mathrm{bII}}^{c}_{o}(f)| + |\widetilde{\mathrm{bII}}^{d}_{e}(f)| + |\widetilde{\mathrm{bII}}^{e}_{o}(f)| + |\widetilde{\mathrm{bII}}^{f}_{o}(f)|$,

(11) $|\widetilde{\mathrm{bII}}^{2,7}(f)| + |\widetilde{\mathrm{bII}}^{3,7}(f)| + |\widetilde{\mathrm{bII}}^{4,7}(f)| + |\widetilde{\mathrm{bII}}^{6,7}(f)| + |\widetilde{\mathrm{bII}}^{7,8}(f)| + |\widetilde{\mathrm{bII}}^{22}(f)| + |\widetilde{\mathrm{bII}}^{23}_{e}(f)| + |\widetilde{\mathrm{bII}}^{d}_{o}(f)| + |\widetilde{\mathrm{bII}}^{f}_{o}(f)|$,

(12) $|\widetilde{\mathrm{bII}}^{2,7}(f)| + |\widetilde{\mathrm{bII}}^{3,7}(f)| + |\widetilde{\mathrm{bII}}^{4,7}(f)| + |\widetilde{\mathrm{bII}}^{6,7}(f)| + |\widetilde{\mathrm{bII}}^{7,8}(f)| + |\widetilde{\mathrm{bII}}^{22}(f)| + |\widetilde{\mathrm{bII}}^{23}_{o}(f)| + |\widetilde{\mathrm{bII}}^{d}_{e}(f)| + |\widetilde{\mathrm{bII}}^{f}_{e}(f)|$,

(13) $|\widetilde{\mathrm{bII}}^{2,8}(f)| + |\widetilde{\mathrm{bII}}^{3,8}(f)| + |\widetilde{\mathrm{bII}}^{4,8}(f)| + |\widetilde{\mathrm{bII}}^{6,8}(f)| + |\widetilde{\mathrm{bII}}^{23}_{o}(f)| + |\widetilde{\mathrm{bII}}^{24}(f)| + |\widetilde{\mathrm{bII}}^{c}_{o}(f)| + |\widetilde{\mathrm{bII}}^{e}_{o}(f)|$,

(14) $|\widetilde{\mathrm{bII}}^{2,8}(f)| + |\widetilde{\mathrm{bII}}^{3,8}(f)| + |\widetilde{\mathrm{bII}}^{4,8}(f)| + |\widetilde{\mathrm{bII}}^{6,8}(f)| + |\widetilde{\mathrm{bII}}^{23}_{e}(f)| + |\widetilde{\mathrm{bII}}^{24}(f)| + |\widetilde{\mathrm{bII}}^{c}_{e}(f)| + |\widetilde{\mathrm{bII}}^{e}_{e}(f)|$,

(15) $|\widetilde{\mathrm{bII}}^{2,9}(f)| + |\widetilde{\mathrm{bII}}^{3,9}(f)| + |\widetilde{\mathrm{bII}}^{4,9}(f)| + |\widetilde{\mathrm{bII}}^{6,9}(f)| + |\widetilde{\mathrm{bII}}^{8,9}(f)| + |\widetilde{\mathrm{bII}}^{27}(f)| + |\widetilde{\mathrm{bII}}^{35}_{e}(f)| + |\widetilde{\mathrm{bII}}^{37}_{o}(f)|$,

(16) $|\widetilde{\mathrm{bII}}^{2,9}(f)| + |\widetilde{\mathrm{bII}}^{3,9}(f)| + |\widetilde{\mathrm{bII}}^{4,9}(f)| + |\widetilde{\mathrm{bII}}^{6,9}(f)| + |\widetilde{\mathrm{bII}}^{8,9}(f)| + |\widetilde{\mathrm{bII}}^{27}(f)| + |\widetilde{\mathrm{bII}}^{35}_{o}(f)| + |\widetilde{\mathrm{bII}}^{37}_{e}(f)|$,

(17) $|\widetilde{\mathrm{bII}}^{2,10}(f)| + |\widetilde{\mathrm{bII}}^{3,10}(f)| + |\widetilde{\mathrm{bII}}^{4,10}(f)| + |\widetilde{\mathrm{bII}}^{6,10}(f)| + |\widetilde{\mathrm{bII}}^{8,10}(f)| + |\widetilde{\mathrm{bII}}^{30}_{e}(f)| + |\widetilde{\mathrm{bII}}^{32}(f)| + |\widetilde{\mathrm{bII}}^{33}(f)| + |\widetilde{\mathrm{bII}}^{35}_{o}(f)| + |\widetilde{\mathrm{bII}}^{37}_{o}(f)| + |\widetilde{\mathrm{bII}}^{38}_{o}(f)| + |\widetilde{\mathrm{bII}}^{39}(f)|$,

(18) $|\widetilde{\mathrm{bII}}^{2,10}(f)| + |\widetilde{\mathrm{bII}}^{3,10}(f)| + |\widetilde{\mathrm{bII}}^{4,10}(f)| + |\widetilde{\mathrm{bII}}^{6,10}(f)| + |\widetilde{\mathrm{bII}}^{8,10}(f)| + |\widetilde{\mathrm{bII}}^{30}_{o}(f)| + |\widetilde{\mathrm{bII}}^{32}(f)| + |\widetilde{\mathrm{bII}}^{33}(f)| + |\widetilde{\mathrm{bII}}^{35}_{e}(f)| + |\widetilde{\mathrm{bII}}^{37}_{e}(f)| + |\widetilde{\mathrm{bII}}^{38}_{e}(f)| + |\widetilde{\mathrm{bII}}^{39}(f)|$.

By eliminating the terms of the forms $\mathcal{F}_o(f)$ and $\mathcal{F}_e(f)$, we obtain the following.

Corollary 3.2. *Let $f\colon M \to N$ be a C^∞ stable map of a compact 3-manifold M with boundary into a surface N without boundary. Then, the following numbers are always even:*

(1) $|\widetilde{\mathrm{bII}}^{a}(f)| + |\widetilde{\mathrm{bII}}^{b}(f)| + |\widetilde{\mathrm{bII}}^{d}(f)|$,

(2) $|\widetilde{\mathrm{bII}}^{13}(f)| + |\widetilde{\mathrm{bII}}^{22}(f)| + |\widetilde{\mathrm{bII}}^{a}(f)|$,
(3) $|\widetilde{\mathrm{bII}}^{13}(f)| + |\widetilde{\mathrm{bII}}^{22}(f)| + |\widetilde{\mathrm{bII}}^{23}(f)| + |\widetilde{\mathrm{bII}}^{b}(f)| + |\widetilde{\mathrm{bII}}^{f}(f)|$,
(4) $|\widetilde{\mathrm{bII}}^{23}(f)| + |\widetilde{\mathrm{bII}}^{30}(f)| + |\widetilde{\mathrm{bII}}^{38}(f)| + |\widetilde{\mathrm{bII}}^{e}(f)|$,
(5) $|\widetilde{\mathrm{bII}}^{c}(f)| + |\widetilde{\mathrm{bII}}^{d}(f)| + |\widetilde{\mathrm{bII}}^{e}(f)| + |\widetilde{\mathrm{bII}}^{f}(f)|$,
(6) $|\widetilde{\mathrm{bII}}^{23}(f)| + |\widetilde{\mathrm{bII}}^{d}(f)| + |\widetilde{\mathrm{bII}}^{f}(f)|$,
(7) $|\widetilde{\mathrm{bII}}^{23}(f)| + |\widetilde{\mathrm{bII}}^{c}(f)| + |\widetilde{\mathrm{bII}}^{e}(f)|$,
(8) $|\widetilde{\mathrm{bII}}^{35}(f)| + |\widetilde{\mathrm{bII}}^{37}(f)|$,
(9) $|\widetilde{\mathrm{bII}}^{30}(f)| + |\widetilde{\mathrm{bII}}^{35}(f)| + |\widetilde{\mathrm{bII}}^{37}(f)| + |\widetilde{\mathrm{bII}}^{38}(f)|$.

Remark 3.3. The numbers in Corollary 3.2 are all even if and only if the following seven are all even:

(1) $|\widetilde{\mathrm{bII}}^{a}(f)| + |\widetilde{\mathrm{bII}}^{b}(f)| + |\widetilde{\mathrm{bII}}^{d}(f)|$,
(2) $|\widetilde{\mathrm{bII}}^{13}(f)| + |\widetilde{\mathrm{bII}}^{22}(f)| + |\widetilde{\mathrm{bII}}^{a}(f)|$,
(3) $|\widetilde{\mathrm{bII}}^{23}(f)| + |\widetilde{\mathrm{bII}}^{e}(f)|$,
(4) $|\widetilde{\mathrm{bII}}^{c}(f)|$,
(5) $|\widetilde{\mathrm{bII}}^{d}(f)| + |\widetilde{\mathrm{bII}}^{e}(f)| + |\widetilde{\mathrm{bII}}^{f}(f)|$,
(6) $|\widetilde{\mathrm{bII}}^{35}(f)| + |\widetilde{\mathrm{bII}}^{37}(f)|$,
(7) $|\widetilde{\mathrm{bII}}^{30}(f)| + |\widetilde{\mathrm{bII}}^{38}(f)|$.

Remark 3.4. By using the same method, we obtain similar co-existence results for singular fibers of a stable Morse function $f\colon V \to W$ of a compact surface V with boundary into $W = \mathbb{R}$ or S^1:

$$|\widetilde{\mathrm{bI}}^{2}(f)| + |\widetilde{\mathrm{bI}}^{3}(f)| + |\widetilde{\mathrm{bI}}^{4}(f)| + |\widetilde{\mathrm{bI}}^{6}(f)| + |\widetilde{\mathrm{bI}}^{8}(f)| \equiv 0 \mod 2.$$

For a stable Morse function $f\colon V \to W$ as above and a C^∞ equivalence class $\widetilde{\mathcal{F}}$ of singular fibers, denote by $\widetilde{\mathcal{F}}_{o,o}(f)$ (or $\widetilde{\mathcal{F}}_{o,e}(f)$, $\widetilde{\mathcal{F}}_{e,o}(f)$, $\widetilde{\mathcal{F}}_{e,e}(f)$) the subset of $\widetilde{\mathcal{F}}(f)$ which consists of the points $q \in W$ such that the numbers of fibers of types $\widetilde{\mathrm{b0}}^{0}$ and $\widetilde{\mathrm{b0}}^{1}$ are both odd (resp. odd and even, even and odd, or both even). Then, by applying the same method to the graphs $\widetilde{\mathcal{F}}_{*,\star}$ ($*, \star = o$ or e), we obtain the following.

Lemma 3.5. *Let $f\colon V \to W$ be a stable Morse function on a compact surface V with boundary into $W = \mathbb{R}$ or S^1. Then, the following numbers are always even:*

(1) $|\widetilde{\mathrm{bI}}^{2}(f)| + |\widetilde{\mathrm{bI}}^{3}(f)| + |\widetilde{\mathrm{bI}}^{4}(f)| + |\widetilde{\mathrm{bI}}^{7}(f)|$,
(2) $|\widetilde{\mathrm{bI}}^{6}(f)| + |\widetilde{\mathrm{bI}}^{7}(f)| + |\widetilde{\mathrm{bI}}^{8}(f)|$.

4. Universal complex

In this section we review the theory of universal complex of singular fibers and study the universal complex of stable maps of manifolds with boundary. Throughout this section, for a C^∞ map $M \to N$, we assume that M is an m-dimensional manifold which is not necessarily closed, and that N is an n-dimensional manifold without boundary.

In what follows, a codimension of a map $f\colon M \to N$ refers to the difference $\dim N - \dim M \in \mathbb{Z}$. To construct the universal complex of singular fibers of C^∞ maps, we fix an integer $\ell \in \mathbb{Z}$ for the codimension of the maps, and consider the following:

(1) a set τ of fibers of Thom maps[1] of codimension ℓ, and
(2) an equivalence relation ρ among fibers in τ.

We further assume that the set τ and the relation ρ satisfy the following conditions.

(a) The set τ is closed under adjacency relations.
(b) The relation ρ is weaker than the C^0 equivalence: each ρ-class is a union of C^0 equivalence classes.

In particular, the condition (b) above implies that for each proper Thom map f and each ρ-class $\mathcal{F}$, $\mathcal{F}(f)$ is a C^0 submanifold of constant codimension unless it is not empty, where

$$\mathcal{F}(f) = \{q \in N \mid f^{-1}(q) \text{ belongs to the class } \mathcal{F}\}.$$

The *codimension* of $\mathcal{F}$ is defined to be that of $\mathcal{F}(f)$ in N, and is denoted by $\kappa(\mathcal{F})$.

Furthermore, we assume that ρ satisfies the following condition.

(c) Let $f_i\colon M_i \to N_i$ be proper Thom maps and $q_i \in N_i$, $i = 0, 1$. Suppose that the fibers over q_0 and q_1 are in τ and that they are equivalent with respect to ρ. Then, there exist open neighborhoods U_i of q_i in N_i, $i = 0, 1$, and a homeomorphism $\varphi\colon U_0 \to U_1$ satisfying $\varphi(q_0) = q_1$ and $\varphi(U_0 \cap \mathcal{F}(f_0)) = U_1 \cap \mathcal{F}(f_1)$, for each ρ-class $\mathcal{F}$.

We call a proper Thom map $f\colon M \to N$ a *τ-map* if all of its fibers are in τ.

For each $\kappa \in \mathbb{Z}$, let $C^\kappa(\tau, \rho)$ be the formal $\mathbb{Z}_2$-vector space spanned by the ρ-classes of codimension κ in τ. If there are no such fibers, then set $C^\kappa(\tau, \rho) = 0$. Note that $C^\kappa(\tau, \rho)$ may possibly be of infinite dimension in general.

Define the $\mathbb{Z}_2$-linear map $\delta_\kappa\colon C^\kappa(\tau, \rho) \to C^{\kappa+1}(\tau, \rho)$ by

$$\delta_\kappa(\mathcal{F}) = \sum_{\kappa(\mathcal{G})=\kappa+1} n_{\mathcal{F}}(\mathcal{G})\, \mathcal{G},$$

where $n_{\mathcal{F}}(\mathcal{G}) \in \mathbb{Z}_2$ is the number modulo two of the local components of $\mathcal{F}(f)$ which are locally adjacent to a component of $\mathcal{G}(f)$ for a τ-map f satisfying $\mathcal{G}(f) \neq \emptyset$ (for details, see [7]). Note that the coefficient $n_{\mathcal{F}}(\mathcal{G}) \in \mathbb{Z}_2$ is well-defined by the condition (c) for ρ.

Since we see easily that $\delta_{\kappa+1} \circ \delta_\kappa = 0$ holds, we obtain the cochain complex

$$\mathcal{C}(\tau, \rho) = (C^\kappa(\tau, \rho), \delta_\kappa)_\kappa.$$

We call the resulting cochain complex the *universal complex of singular fibers for proper τ-maps with respect to the equivalence relation* ρ, and denote its cohomology group of dimension κ by $H^\kappa(\tau, \rho)$.

Let

$$c = \sum_{\kappa(\mathcal{F})=\kappa} n_{\mathcal{F}}\, \mathcal{F}$$

be a κ-dimensional cochain of $\mathcal{C}(\tau, \rho)$. For a τ-map $f\colon M \to N$, denote by $c(f)$ the set of points $q \in N$ such that the fiber over q is in $\mathcal{F}$ with $n_{\mathcal{F}} \neq 0$.

If c is a cocycle, then we can show that $c(f)$ is a $\mathbb{Z}_2$-cycle of closed support of codimension κ in N. In addition, if M is compact and $\kappa > 0$, $c(f)$ is a $\mathbb{Z}_2$-cycle in the usual sense.

Lemma 4.1. *Let c and c' be κ-dimensional cocycles of $\mathcal{C}(\tau, \rho)$. If they are cohomologous, then the $\mathbb{Z}_2$-cycles $c(f)$ and $c'(f)$ are $\mathbb{Z}_2$-homologous in N for each τ-map $f\colon M \to N$.*

---

[1] A *Thom map* $M \to N$ is a C^∞ stratified map with respect to Whitney regular stratifications of M and N such that it is a submersion on each stratum and satisfies certain regularity conditions. See [1, §2], [2, §2.5], [3], [7, Part I, §1], etc. for more details.

Proof. There exists a $(\kappa-1)$-dimensional cochain d of $\mathcal{C}(\tau,\rho)$ such that $c - c' = \delta_{\kappa-1}d$. Then, we see easily that $c(f) - c'(f) = \partial d(f)$ holds, and the result follows immediately. □

Let $[c]$ be a κ-dimensional cohomology class of $\mathcal{C}(\tau,\rho)$ represented by a cochain c. For a τ-map $f\colon M \to N$, define $[c(f)] \in H^c_{n-\kappa}(N;\mathbb{Z}_2)$ to be the $\mathbb{Z}_2$-homology class represented by the $\mathbb{Z}_2$-cycle $c(f)$ of closed support.

Define the $\mathbb{Z}_2$-linear map

$$\varphi_f\colon\ H^\kappa(\tau,\rho) \to H^\kappa(N;\mathbb{Z}_2)$$

by $\varphi_f([c]) = [c(f)]^*$, where $[c(f)]^* \in H^\kappa(N;\mathbb{Z}_2)$ is the Poincaré dual of $[c(f)] \in H^c_{n-\kappa}(N;\mathbb{Z}_2)$.

The suspension of a Thom map is introduced as follows.

Definition 4.2. For a proper Thom map $f\colon M \to N$, consider the Thom map

$$f \times \mathrm{id}_{\mathbb{R}}\colon\ M \times \mathbb{R} \to N \times \mathbb{R}.$$

Note that $S(f \times \mathrm{id}_{\mathbb{R}}) = S(f) \times \mathbb{R}$ and $(f \times \mathrm{id}_{\mathbb{R}})(S(f) \times \mathbb{R}) = f(S(f)) \times \mathbb{R}$. We call $f \times \mathrm{id}_{\mathbb{R}}$ and the fiber of $f \times \mathrm{id}_{\mathbb{R}}$ over a point $(q,0) \in N \times \mathbb{R}$ the *suspension* of f and the *suspension* of the fiber of f over $q \in N$, respectively.

Let τ be a set of fibers for proper Thom maps of codimension ℓ. For a dimension pair (m,n) with $n-m=\ell$, let $\tau(m,n)$ denote the set of fibers in τ for proper Thom maps of manifolds of dimension m into those of dimension n. The equivalence relation on $\tau(m,n)$ induced by ρ is denoted by $\rho_{m,n}$.

In the following, in addition to the conditions (a)–(c) above, we assume the following two additional conditions.

- (d) The suspension of each fiber in $\tau(m,n)$ is also in $\tau(m+1,n+1)$.
- (e) If two fibers in $\tau(m,n)$ are equivalent with respect to $\rho_{m,n}$, then their suspensions are also equivalent with respect to $\rho_{m+1,n+1}$.

For each $\kappa \in \mathbb{Z}$, the suspension induces the $\mathbb{Z}_2$-linear map

$$s_\kappa\colon\ C^\kappa(\tau(m+1,n+1),\rho_{m+1,n+1}) \to C^\kappa(\tau(m,n),\rho_{m,n}),$$

where $s_\kappa(\mathcal{F})$ is the (possibly infinite) sum of all $\rho_{m,n}$-classes of codimension κ whose suspensions are in $\mathcal{F}$. Note that s_κ is well-defined.

Lemma 4.3. *The system of $\mathbb{Z}_2$-linear maps $\{s_\kappa\}$ defines the cochain map*

$$\{s_\kappa\}\colon\ \mathcal{C}(\tau(m+1,n+1),\rho_{m+1,n+1}) \to \mathcal{C}(\tau(m,n),\rho_{m,n}).$$

We omit the proof of the above lemma. For details, see [7].

Let us introduce a geometric equivalence relation for τ-maps.

Definition 4.4. Two τ-maps $f_i\colon M_i \to N$, $i=0,1$, of compact manifolds with boundary into a manifold without boundary are *τ-cobordant* if there exist a compact manifold X with corners and a τ-map $F\colon X \to N \times [0,1]$ that satisfy the following conditions:

1. $\partial X = M_0 \cup Q \cup M_1$, where M_0, M_1 and Q are codimension 0 smooth submanifolds of ∂X, $M_0 \cap M_1 = \emptyset$, and $\partial Q = (M_0 \cap Q) \cup (M_1 \cap Q)$,
2. X has corners along ∂Q,
3. $F|_{M_0 \times [0,\varepsilon)} = f_0 \times \mathrm{id}_{[0,\varepsilon)}$ and $F|_{M_1 \times (1-\varepsilon,1]} = f_1 \times \mathrm{id}_{(1-\varepsilon,1]}$, where $M_0 \times [0,\varepsilon)$ and $M_1 \times (1-\varepsilon,1]$ denote the collar neighborhoods (with corners) of M_0 and M_1 in X, respectively.

In this case, we call the map F a *τ-cobordism* between f_0 and f_1.

Note that the τ-cobordism relation is an equivalence relation among the τ-maps into a fixed manifold N. For a manifold N, we denote by $Cob_\tau(N)$ the set of all equivalence classes of τ-maps of compact manifolds into N with respect to the τ-cobordism.

Proposition 4.5. *If two τ-maps $f_i\colon M_i \to N$ of compact manifolds into N, $i = 0, 1$, are τ-cobordant, then for each κ we have*

$$\varphi_{f_0}|_{\mathrm{Im}\, s_{\kappa *}} = \varphi_{f_1}|_{\mathrm{Im}\, s_{\kappa *}}\colon\ \mathrm{Im}\, s_{\kappa *} \to H^\kappa(N; \mathbb{Z}_2),$$

where $s_{\kappa *}\colon H^\kappa(\tau(m+1, n+1), \rho_{m+1,n+1}) \to H^\kappa(\tau(m,n), \rho_{m,n})$ denotes the homomorphism induced by the suspension.

Proof. Let $F\colon X \to N \times [0,1]$ be a τ-cobordism between f_0 and f_1. For each κ-dimensional cocycle c of

$$\mathcal{C}(\tau(m+1, n+1), \rho_{m+1,n+1}),$$

put $\overline{c} = s_\kappa(c) \in C^\kappa(\tau(m,n), \rho_{m,n})$. Then, we have

$$\partial c(F) = \overline{c}(f_1) \times \{1\} - \overline{c}(f_0) \times \{0\},$$

since c is a cocycle. Then the result follows immediately. □

Thus, for each cohomology class $[c] \in H^\kappa(\tau(m+1, n+1), \rho_{m+1,n+1})$ and an n-dimensional manifold N without boundary, we obtain the map

$$I_{[c]}\colon\ Cob_\tau(N) \to H^\kappa(N; \mathbb{Z}_2)$$

defined by $I_{[c]}(f) = \varphi_f([s_{\kappa *}c])$, which does not depend on the choice of a representative f of a given τ-cobordism class. Namely, each element in

$$H^\kappa(\tau(m+1, n+1), \rho_{m+1,n+1})$$

induces a τ-cobordism invariant for τ-maps into an n-dimensional manifold N.

4.1. **Universal complex for stable maps of n-dimensional manifolds with boundary into $(n-1)$-dimensional manifolds.** For a positive integer n, let $\tau(n, n-1) = b\mathcal{S}_{\mathrm{pr}}(n, n-1)$ be the set of fibers for proper C^0 stable Thom maps of n-dimensional manifolds with boundary into $(n-1)$-dimensional manifolds without boundary.

Remark 4.6. If the dimension pair $(n, n-1)$ is in the nice range, then C^0 stable maps are C^∞ stable (for example, see [2]), and consequently they are Thom maps. For example, this is the case if $n \leq 8$.

Furthermore, let $\rho_{n,n-1}(2)$ be the C^0 *equivalence relation modulo two regular fiber components* for fibers in $\tau(n, n-1)$: i.e., two fibers in $\tau(n, n-1)$ are $\rho_{n,n-1}(2)$-equivalent if they become C^0-equivalent after we add some regular fiber components to each of them with the numbers of added components having the same parity. Note that, under this equivalence, we do not distinguish the fibers of types $\widetilde{\mathrm{b}0}^0$ with $\widetilde{\mathrm{b}0}^1$.

For a C^0 equivalence class $\widetilde{\mathcal{F}}$ of singular fibers, denote by $\widetilde{\mathcal{F}}_o$ (or $\widetilde{\mathcal{F}}_e$) the equivalence class with respect to $\rho_{n,n-1}(2)$ which consists of singular fibers of type $\widetilde{\mathcal{F}}$ with an odd number (resp. even number) of regular fiber components.

Then, for the universal complex $\mathcal{C}(b\mathcal{S}_{\mathrm{pr}}(3,2),\rho_{3,2}(2))$, the coboundary homomorphism is given by the following formulae:

$$
\begin{aligned}
\delta_0(\widetilde{\mathrm{b0}}_o) &= \delta_0(\widetilde{\mathrm{b0}}_e) = \widetilde{\mathrm{bI}}^2 + \widetilde{\mathrm{bI}}^3 + \widetilde{\mathrm{bI}}^4 + \widetilde{\mathrm{bI}}^6 + \widetilde{\mathrm{bI}}^8,\\
\delta_1(\widetilde{\mathrm{bI}}_o^2) &= \widetilde{\mathrm{bII}}^{2,3} + \widetilde{\mathrm{bII}}^{2,4} + \widetilde{\mathrm{bII}}^{2,6} + \widetilde{\mathrm{bII}}^{2,8} + \widetilde{\mathrm{bII}}_e^{a} + \widetilde{\mathrm{bII}}_e^{b} + \widetilde{\mathrm{bII}}_o^{d},\\
\delta_1(\widetilde{\mathrm{bI}}_e^2) &= \widetilde{\mathrm{bII}}^{2,3} + \widetilde{\mathrm{bII}}^{2,4} + \widetilde{\mathrm{bII}}^{2,6} + \widetilde{\mathrm{bII}}^{2,8} + \widetilde{\mathrm{bII}}_o^{a} + \widetilde{\mathrm{bII}}_o^{b} + \widetilde{\mathrm{bII}}_e^{d},\\
\delta_1(\widetilde{\mathrm{bI}}_o^3) &= \widetilde{\mathrm{bII}}^{2,3} + \widetilde{\mathrm{bII}}^{3,4} + \widetilde{\mathrm{bII}}^{3,6} + \widetilde{\mathrm{bII}}^{3,8} + \widetilde{\mathrm{bII}}_e^{13} + \widetilde{\mathrm{bII}}_o^{22} + \widetilde{\mathrm{bII}}_o^{a},\\
\delta_1(\widetilde{\mathrm{bI}}_e^3) &= \widetilde{\mathrm{bII}}^{2,3} + \widetilde{\mathrm{bII}}^{3,4} + \widetilde{\mathrm{bII}}^{3,6} + \widetilde{\mathrm{bII}}^{3,8} + \widetilde{\mathrm{bII}}_o^{13} + \widetilde{\mathrm{bII}}_e^{22} + \widetilde{\mathrm{bII}}_e^{a},\\
\delta_1(\widetilde{\mathrm{bI}}_o^4) &= \widetilde{\mathrm{bII}}^{2,4} + \widetilde{\mathrm{bII}}^{3,4} + \widetilde{\mathrm{bII}}^{4,6} + \widetilde{\mathrm{bII}}^{4,8} + \widetilde{\mathrm{bII}}_e^{13} + \widetilde{\mathrm{bII}}_o^{22} + \widetilde{\mathrm{bII}}_o^{23} + \widetilde{\mathrm{bII}}^{24}\\
&\quad + \widetilde{\mathrm{bII}}_o^{b} + \widetilde{\mathrm{bII}}_o^{f},\\
\delta_1(\widetilde{\mathrm{bI}}_e^4) &= \widetilde{\mathrm{bII}}^{2,4} + \widetilde{\mathrm{bII}}^{3,4} + \widetilde{\mathrm{bII}}^{4,6} + \widetilde{\mathrm{bII}}^{4,8} + \widetilde{\mathrm{bII}}_o^{13} + \widetilde{\mathrm{bII}}_e^{22} + \widetilde{\mathrm{bII}}_e^{23} + \widetilde{\mathrm{bII}}^{24}\\
&\quad + \widetilde{\mathrm{bII}}_e^{b} + \widetilde{\mathrm{bII}}_e^{f},\\
\delta_1(\widetilde{\mathrm{bI}}_o^5) &= \widetilde{\mathrm{bII}}^{2,5} + \widetilde{\mathrm{bII}}^{3,5} + \widetilde{\mathrm{bII}}^{4,5} + \widetilde{\mathrm{bII}}^{5,6} + \widetilde{\mathrm{bII}}^{5,8} + \widetilde{\mathrm{bII}}^{15} + \widetilde{\mathrm{bII}}_o^{23} + \widetilde{\mathrm{bII}}^{25}\\
&\quad + \widetilde{\mathrm{bII}}_o^{30} + \widetilde{\mathrm{bII}}_o^{38} + \widetilde{\mathrm{bII}}_o^{e},\\
\delta_1(\widetilde{\mathrm{bI}}_e^5) &= \widetilde{\mathrm{bII}}^{2,5} + \widetilde{\mathrm{bII}}^{3,5} + \widetilde{\mathrm{bII}}^{4,5} + \widetilde{\mathrm{bII}}^{5,6} + \widetilde{\mathrm{bII}}^{5,8} + \widetilde{\mathrm{bII}}^{15} + \widetilde{\mathrm{bII}}_e^{23} + \widetilde{\mathrm{bII}}^{25}\\
&\quad + \widetilde{\mathrm{bII}}_e^{30} + \widetilde{\mathrm{bII}}_e^{38} + \widetilde{\mathrm{bII}}_e^{e},\\
\delta_1(\widetilde{\mathrm{bI}}_o^6) &= \widetilde{\mathrm{bII}}^{2,6} + \widetilde{\mathrm{bII}}^{3,6} + \widetilde{\mathrm{bII}}^{4,6} + \widetilde{\mathrm{bII}}^{6,8} + \widetilde{\mathrm{bII}}_e^{c} + \widetilde{\mathrm{bII}}_o^{d} + \widetilde{\mathrm{bII}}_e^{e} + \widetilde{\mathrm{bII}}_e^{f},\\
\delta_1(\widetilde{\mathrm{bI}}_e^6) &= \widetilde{\mathrm{bII}}^{2,6} + \widetilde{\mathrm{bII}}^{3,6} + \widetilde{\mathrm{bII}}^{4,6} + \widetilde{\mathrm{bII}}^{6,8} + \widetilde{\mathrm{bII}}_o^{c} + \widetilde{\mathrm{bII}}_e^{d} + \widetilde{\mathrm{bII}}_o^{e} + \widetilde{\mathrm{bII}}_o^{f},\\
\delta_1(\widetilde{\mathrm{bI}}_o^7) &= \widetilde{\mathrm{bII}}^{2,7} + \widetilde{\mathrm{bII}}^{3,7} + \widetilde{\mathrm{bII}}^{4,7} + \widetilde{\mathrm{bII}}^{6,7} + \widetilde{\mathrm{bII}}^{7,8} + \widetilde{\mathrm{bII}}^{22} + \widetilde{\mathrm{bII}}_e^{23} + \widetilde{\mathrm{bII}}_o^{d}\\
&\quad + \widetilde{\mathrm{bII}}_o^{f},\\
\delta_1(\widetilde{\mathrm{bI}}_e^7) &= \widetilde{\mathrm{bII}}^{2,7} + \widetilde{\mathrm{bII}}^{3,7} + \widetilde{\mathrm{bII}}^{4,7} + \widetilde{\mathrm{bII}}^{6,7} + \widetilde{\mathrm{bII}}^{7,8} + \widetilde{\mathrm{bII}}^{22} + \widetilde{\mathrm{bII}}_o^{23} + \widetilde{\mathrm{bII}}_e^{d}\\
&\quad + \widetilde{\mathrm{bII}}_e^{f},\\
\delta_1(\widetilde{\mathrm{bI}}_o^8) &= \widetilde{\mathrm{bII}}^{2,8} + \widetilde{\mathrm{bII}}^{3,8} + \widetilde{\mathrm{bII}}^{4,8} + \widetilde{\mathrm{bII}}^{6,8} + \widetilde{\mathrm{bII}}_o^{23} + \widetilde{\mathrm{bII}}^{24} + \widetilde{\mathrm{bII}}_o^{c} + \widetilde{\mathrm{bII}}_o^{e},\\
\delta_1(\widetilde{\mathrm{bI}}_e^8) &= \widetilde{\mathrm{bII}}^{2,8} + \widetilde{\mathrm{bII}}^{3,8} + \widetilde{\mathrm{bII}}^{4,8} + \widetilde{\mathrm{bII}}^{6,8} + \widetilde{\mathrm{bII}}_e^{23} + \widetilde{\mathrm{bII}}^{24} + \widetilde{\mathrm{bII}}_e^{c} + \widetilde{\mathrm{bII}}_e^{e},\\
\delta_1(\widetilde{\mathrm{bI}}_o^9) &= \widetilde{\mathrm{bII}}^{2,9} + \widetilde{\mathrm{bII}}^{3,9} + \widetilde{\mathrm{bII}}^{4,9} + \widetilde{\mathrm{bII}}^{6,9} + \widetilde{\mathrm{bII}}^{8,9} + \widetilde{\mathrm{bII}}^{27} + \widetilde{\mathrm{bII}}_e^{35} + \widetilde{\mathrm{bII}}_o^{37},\\
\delta_1(\widetilde{\mathrm{bI}}_e^9) &= \widetilde{\mathrm{bII}}^{2,9} + \widetilde{\mathrm{bII}}^{3,9} + \widetilde{\mathrm{bII}}^{4,9} + \widetilde{\mathrm{bII}}^{6,9} + \widetilde{\mathrm{bII}}^{8,9} + \widetilde{\mathrm{bII}}^{27} + \widetilde{\mathrm{bII}}_o^{35} + \widetilde{\mathrm{bII}}_e^{37},\\
\delta_1(\widetilde{\mathrm{bI}}_o^{10}) &= \widetilde{\mathrm{bII}}^{2,10} + \widetilde{\mathrm{bII}}^{3,10} + \widetilde{\mathrm{bII}}^{4,10} + \widetilde{\mathrm{bII}}^{6,10} + \widetilde{\mathrm{bII}}^{8,10} + \widetilde{\mathrm{bII}}_e^{30} + \widetilde{\mathrm{bII}}^{32}\\
&\quad + \widetilde{\mathrm{bII}}^{33} + \widetilde{\mathrm{bII}}_o^{35} + \widetilde{\mathrm{bII}}_o^{37} + \widetilde{\mathrm{bII}}_o^{38} + \widetilde{\mathrm{bII}}^{39},\\
\delta_1(\widetilde{\mathrm{bI}}_e^{10}) &= \widetilde{\mathrm{bII}}^{2,10} + \widetilde{\mathrm{bII}}^{3,10} + \widetilde{\mathrm{bII}}^{4,10} + \widetilde{\mathrm{bII}}^{6,10} + \widetilde{\mathrm{bII}}^{8,10} + \widetilde{\mathrm{bII}}_o^{30} + \widetilde{\mathrm{bII}}^{32}\\
&\quad + \widetilde{\mathrm{bII}}^{33} + \widetilde{\mathrm{bII}}_e^{35} + \widetilde{\mathrm{bII}}_e^{37} + \widetilde{\mathrm{bII}}_e^{38} + \widetilde{\mathrm{bII}}^{39},
\end{aligned}
$$

where $\widetilde{\mathcal{F}}$ denotes $\widetilde{\mathcal{F}}_o + \widetilde{\mathcal{F}}_e$.

Then, by a straightforward calculation, we obtain the following.

Proposition 4.7. *The cohomology groups of* $\mathcal{C}(b\mathcal{S}_{\mathrm{pr}}(3,2),\rho_{3,2}(2))$ *are described as follows:*

(1) $H^0(b\mathcal{S}_{\mathrm{pr}}(3,2),\rho_{3,2}(2)) \cong \mathbb{Z}_2$*, generated by* $[\widetilde{\mathrm{b0}}_o + \widetilde{\mathrm{b0}}_e]$,

(2) $H^1(b\mathcal{S}_{\rm pr}(3,2),\rho_{3,2}(2)) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$, *generated by*

$$\begin{aligned}\beta &= [\widetilde{\mathrm{bI}}^6 + \widetilde{\mathrm{bI}}^7 + \widetilde{\mathrm{bI}}^8] = [\widetilde{\mathrm{bI}}^2 + \widetilde{\mathrm{bI}}^3 + \widetilde{\mathrm{bI}}^4 + \widetilde{\mathrm{bI}}^7],\\ \gamma &= [\widetilde{\mathrm{bI}}^2_o + \widetilde{\mathrm{bI}}^3_e + \widetilde{\mathrm{bI}}^4_e + \widetilde{\mathrm{bI}}^6_o + \widetilde{\mathrm{bI}}^8_e] = [\widetilde{\mathrm{bI}}^2_e + \widetilde{\mathrm{bI}}^3_o + \widetilde{\mathrm{bI}}^4_o + \widetilde{\mathrm{bI}}^6_e + \widetilde{\mathrm{bI}}^8_o].\end{aligned}$$

Note that the ranks of $C^i(b\mathcal{S}_{\rm pr}(3,2),\rho_{3,2}(2))$, $i=0,1,2$, are equal to 2, 18 and 160, respectively.

Let

$$s_{\kappa *}\colon\ H^\kappa(b\mathcal{S}_{pr}(3,2),\rho_{3,2}(2)) \to H^\kappa(b\mathcal{S}_{pr}(2,1),\rho_{2,1}(2))$$

be the homomorphism induced by suspension s_κ. Then, Remark 3.5 shows that $s_{\kappa*}\beta$ induces a trivial $b\mathcal{S}_{\rm pr}(3,2)$-cobordism invariant for stable Morse functions $f\colon V\to W$ on compact surfaces V with boundary into $W=\mathbb{R}$ or S^1. Furthermore, by using the same method as in [7, Lemma 14.1], we can show that $s_{\kappa*}\gamma$ also induces a trivial $b\mathcal{S}_{\rm pr}(3,2)$-cobordism invariant. Therefore, unfortunately, the above cohomology information does not provide us of a new cobordism invariant.

Let us now consider a certain restricted class of stable maps. Let M be a 3-dimensional manifold with boundary and N a surface without boundary. A stable map $f\colon M\to N$ which is a submersion on a neighborhood of ∂M is called an *admissible stable map*. In other words, such a map can be characterized as a stable map without definite $\Sigma^{2,0}_{1,0}$ points nor indefinite $\Sigma^{2,0}_{1,0}$ points. Denote by $\mathcal{AS}_{\rm pr}(3,2)$ the set of fibers for proper C^∞ admissible stable maps of 3-manifolds with boundary into surfaces without boundary. Note that the suspension of a stable Morse function on a compact surface is always admissible.

Then, a straightforward calculation shows the following.

Proposition 4.8. *The cohomology groups of the universal complex*

$$\mathcal{C}(\mathcal{AS}_{pr}(3,2),\rho_{3,2}(2))$$

for admissible stable maps of 3*-manifolds with boundary to surfaces without boundary with respect to the* C^0 *equivalence modulo two regular fiber components are described as follows:*

(1) $H^0(\mathcal{AS}_{\rm pr}(3,2),\rho_{3,2}(2)) \cong \mathbb{Z}_2$, *generated by* $[\widetilde{\mathrm{b0}}_o + \widetilde{\mathrm{b0}}_e]$,
(2) $H^1(\mathcal{AS}_{\rm pr}(3,2),\rho_{3,2}(2)) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$, *generated by*

$$\begin{aligned}\alpha &= [\widetilde{\mathrm{bI}}^2 + \widetilde{\mathrm{bI}}^3 + \widetilde{\mathrm{bI}}^4 + \widetilde{\mathrm{bI}}^5 + \widetilde{\mathrm{bI}}^9 + \widetilde{\mathrm{bI}}^{10}],\\ \beta &= [\widetilde{\mathrm{bI}}^2 + \widetilde{\mathrm{bI}}^3 + \widetilde{\mathrm{bI}}^4 + \widetilde{\mathrm{bI}}^7] = [\widetilde{\mathrm{bI}}^6 + \widetilde{\mathrm{bI}}^7 + \widetilde{\mathrm{bI}}^8],\\ \gamma &= [\widetilde{\mathrm{bI}}^2_o + \widetilde{\mathrm{bI}}^3_e + \widetilde{\mathrm{bI}}^4_e + \widetilde{\mathrm{bI}}^6_o + \widetilde{\mathrm{bI}}^8_e] = [\widetilde{\mathrm{bI}}^2_e + \widetilde{\mathrm{bI}}^3_o + \widetilde{\mathrm{bI}}^4_o + \widetilde{\mathrm{bI}}^6_e + \widetilde{\mathrm{bI}}^8_o].\end{aligned}$$

Note that the ranks of $C^i(\mathcal{AS}_{\rm pr}(3,2),\rho_{3,2}(2))$, $i=0,1,2$, are equal to 2, 18 and 154, respectively.

Let

$$s_{\kappa *}\colon\ H^\kappa(\mathcal{AS}_{\rm pr}(3,2),\rho_{3,2}(2)) \to H^\kappa(\mathcal{AS}_{\rm pr}(2,1),\rho_{2,1}(2))$$

be the homomorphism induced by suspension s_κ. Then, we obtain the following cobordism invariants.

Corollary 4.9. *Let* W *be the real line* $\mathbb{R}$ *or the circle* S^1.

(1) *The cohomology class* $s_{\kappa*}\alpha$ *induces a non-trivial* $\mathcal{AS}_{\rm pr}(3,2)$*-cobordism invariant for stable Morse functions* $f\colon V\to W$ *on compact surfaces* V *with boundary into* W.

(2) *The cohomology classes* $s_{\kappa*}\beta$ *and* $s_{\kappa*}\gamma$ *induce trivial* $\mathcal{AS}_{\rm pr}(3,2)$*-cobordism invariants for stable Morse functions* $f\colon V\to W$ *of compact surfaces* V *with boundary into* W.

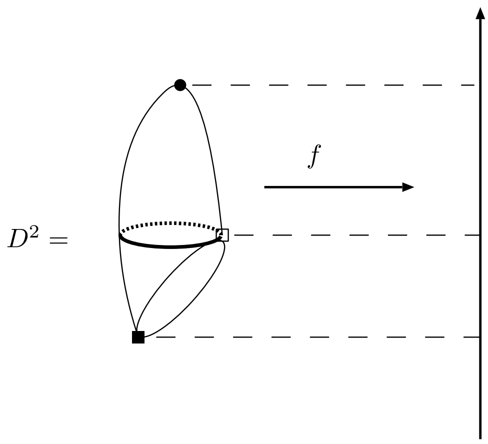

FIGURE 7. A stable Morse function on the disk

Proof. A stable Morse function $f\colon D^2 \to \mathbb{R}$ given by the height function as depicted in Figure 7 shows that the cobordism invariant $s_{\kappa *}\alpha$ is non-trivial, where D^2 denotes the 2-dimensional disk. The same construction works also for $W = S^1$.

On the other hand, Remark 3.5 shows that the invariant associated with $s_{\kappa *}\beta$ is trivial. The triviality of the invariant associated with $s_{\kappa *}\gamma$ is proved by using the same argument as in [7, Lemma 14.1], so we omit the details here. □

Remark 4.10. The above corollary shows that even if we take a non-trivial cohomology class of the universal complex, the corresponding cohomology class in the target manifold can be always trivial.

4.2. **Universal complex of co-orientable singular fibers of stable maps.** Let us consider the equivalence classes of singular fibers which are (strongly) co-orientable in the sense of [7, Definition 10.5]. An equivalence class $\widetilde{\mathcal{F}}$ of fibers of stable maps is *strongly co-orientable* if for a stable map $M \to N$ and a point $q \in N$ whose fiber belongs to $\widetilde{\mathcal{F}}$, every local homeomorphism around $q \in N$ preserving the adjacent equivalence classes necessarily preserves the orientation of the normal direction to the submanifold corresponding to $\widetilde{\mathcal{F}}$.

Then, we obtain the following.

Lemma 4.11. *Those equivalence classes with respect to $\rho_{3,2}(2)$ which are strongly co-orientable are* $\widetilde{\mathrm{b0}}_*$, $\widetilde{\mathrm{bI}}^2_*$, $\widetilde{\mathrm{bI}}^3_*$, $\widetilde{\mathrm{bI}}^4_*$, $\widetilde{\mathrm{bI}}^6_*$, $\widetilde{\mathrm{bI}}^8_*$, $\widetilde{\mathrm{bII}}^{2,3}_*$, $\widetilde{\mathrm{bII}}^{2,4}_*$, $\widetilde{\mathrm{bII}}^{2,6}_*$, $\widetilde{\mathrm{bII}}^{2,8}_*$, $\widetilde{\mathrm{bII}}^{3,4}_*$, $\widetilde{\mathrm{bII}}^{3,6}_*$, $\widetilde{\mathrm{bII}}^{3,8}_*$, $\widetilde{\mathrm{bII}}^{4,6}_*$, $\widetilde{\mathrm{bII}}^{4,8}_*$, $\widetilde{\mathrm{bII}}^{6,8}_*$, $\widetilde{\mathrm{bII}}^{13}_*$, $\widetilde{\mathrm{bII}}^{22}_*$, $\widetilde{\mathrm{bII}}^{23}_*$, $\widetilde{\mathrm{bII}}^{24}_*$, $\widetilde{\mathrm{bII}}^{a}_*$, $\widetilde{\mathrm{bII}}^{b}_*$, $\widetilde{\mathrm{bII}}^{c}_*$, $\widetilde{\mathrm{bII}}^{d}_*$, $\widetilde{\mathrm{bII}}^{e}_*$, $\widetilde{\mathrm{bII}}^{f}_*$, *where $*$ denotes o or e. The other equivalence classes are not strongly co-orientable.*

Let us fix a co-orientation for each co-orientable equivalence class of codimension one in such a way that the co-orientation points from $\widetilde{\mathrm{b0}}_e$ to $\widetilde{\mathrm{b0}}_o$. For each co-orientable equivalence class of codimension two, we fix a co-orientation as in Figures 8, 9 and 10. (For those equivalence classes which do not appear in the figures, we fix their co-orientations in a similar fashion.)

Then, for the universal complex

$$\mathcal{CO}(b\mathcal{S}_{\mathrm{pr}}(3,2), \rho_{3,2}(2))$$

of co-orientable fibers, the coboundary homomorphism is given by the following formulae. For the definition of the coboundary homomorphism, see [7, Chapters 7 and 8]. Note that the cochain complex is defined over $\mathbb{Z}$.

$$
\begin{aligned}
\delta_0(\widetilde{\mathrm{b0}_o}) &= \widetilde{\mathrm{bI}}^2_o + \widetilde{\mathrm{bI}}^2_e + \widetilde{\mathrm{bI}}^3_o + \widetilde{\mathrm{bI}}^3_e + \widetilde{\mathrm{bI}}^4_o + \widetilde{\mathrm{bI}}^4_e + \widetilde{\mathrm{bI}}^6_o + \widetilde{\mathrm{bI}}^6_e + \widetilde{\mathrm{bI}}^8_o + \widetilde{\mathrm{bI}}^8_e,\\
\delta_0(\widetilde{\mathrm{b0}_e}) &= -\widetilde{\mathrm{bI}}^2_o - \widetilde{\mathrm{bI}}^2_e - \widetilde{\mathrm{bI}}^3_o - \widetilde{\mathrm{bI}}^3_e - \widetilde{\mathrm{bI}}^4_o - \widetilde{\mathrm{bI}}^4_e - \widetilde{\mathrm{bI}}^6_o - \widetilde{\mathrm{bI}}^6_e - \widetilde{\mathrm{bI}}^8_o - \widetilde{\mathrm{bI}}^8_e,\\
\delta_1(\widetilde{\mathrm{bI}}^2_o) &= \widetilde{\mathrm{bII}}^{2,3}_o - \widetilde{\mathrm{bII}}^{2,3}_e + \widetilde{\mathrm{bII}}^{2,4}_o - \widetilde{\mathrm{bII}}^{2,4}_e + \widetilde{\mathrm{bII}}^{2,6}_o - \widetilde{\mathrm{bII}}^{2,6}_e + \widetilde{\mathrm{bII}}^{2,8}_o - \widetilde{\mathrm{bII}}^{2,8}_e\\
&\quad - \widetilde{\mathrm{bII}}^{a}_e - \widetilde{\mathrm{bII}}^{b}_e + \widetilde{\mathrm{bII}}^{d}_o,\\
\delta_1(\widetilde{\mathrm{bI}}^2_e) &= \widetilde{\mathrm{bII}}^{2,3}_o - \widetilde{\mathrm{bII}}^{2,3}_e + \widetilde{\mathrm{bII}}^{2,4}_o - \widetilde{\mathrm{bII}}^{2,4}_e + \widetilde{\mathrm{bII}}^{2,6}_o - \widetilde{\mathrm{bII}}^{2,6}_e + \widetilde{\mathrm{bII}}^{2,8}_o - \widetilde{\mathrm{bII}}^{2,8}_e\\
&\quad + \widetilde{\mathrm{bII}}^{a}_o + \widetilde{\mathrm{bII}}^{b}_o - \widetilde{\mathrm{bII}}^{d}_e,\\
\delta_1(\widetilde{\mathrm{bI}}^3_o) &= -\widetilde{\mathrm{bII}}^{2,3}_o + \widetilde{\mathrm{bII}}^{2,3}_e + \widetilde{\mathrm{bII}}^{3,4}_o - \widetilde{\mathrm{bII}}^{3,4}_e + \widetilde{\mathrm{bII}}^{3,6}_o - \widetilde{\mathrm{bII}}^{3,6}_e + \widetilde{\mathrm{bII}}^{3,8}_o - \widetilde{\mathrm{bII}}^{3,8}_e\\
&\quad + \widetilde{\mathrm{bII}}^{13}_e + \widetilde{\mathrm{bII}}^{22}_o - \widetilde{\mathrm{bII}}^{a}_o,\\
\delta_1(\widetilde{\mathrm{bI}}^3_e) &= -\widetilde{\mathrm{bII}}^{2,3}_o + \widetilde{\mathrm{bII}}^{2,3}_e + \widetilde{\mathrm{bII}}^{3,4}_o - \widetilde{\mathrm{bII}}^{3,4}_e + \widetilde{\mathrm{bII}}^{3,6}_o - \widetilde{\mathrm{bII}}^{3,6}_e + \widetilde{\mathrm{bII}}^{3,8}_o - \widetilde{\mathrm{bII}}^{3,8}_e\\
&\quad - \widetilde{\mathrm{bII}}^{13}_o - \widetilde{\mathrm{bII}}^{22}_e + \widetilde{\mathrm{bII}}^{a}_e,\\
\delta_1(\widetilde{\mathrm{bI}}^4_o) &= -\widetilde{\mathrm{bII}}^{2,4}_o + \widetilde{\mathrm{bII}}^{2,4}_e - \widetilde{\mathrm{bII}}^{3,4}_o + \widetilde{\mathrm{bII}}^{3,4}_e + \widetilde{\mathrm{bII}}^{4,6}_o - \widetilde{\mathrm{bII}}^{4,6}_e + \widetilde{\mathrm{bII}}^{4,8}_o - \widetilde{\mathrm{bII}}^{4,8}_e\\
&\quad - \widetilde{\mathrm{bII}}^{13}_e - \widetilde{\mathrm{bII}}^{22}_o + \widetilde{\mathrm{bII}}^{23}_o + \widetilde{\mathrm{bII}}^{24}_o - \widetilde{\mathrm{bII}}^{24}_e - \widetilde{\mathrm{bII}}^{b}_o - \widetilde{\mathrm{bII}}^{f}_o\\
\delta_1(\widetilde{\mathrm{bI}}^4_e) &= -\widetilde{\mathrm{bII}}^{2,4}_o + \widetilde{\mathrm{bII}}^{2,4}_e - \widetilde{\mathrm{bII}}^{3,4}_o + \widetilde{\mathrm{bII}}^{3,4}_e + \widetilde{\mathrm{bII}}^{4,6}_o - \widetilde{\mathrm{bII}}^{4,6}_e + \widetilde{\mathrm{bII}}^{4,8}_o - \widetilde{\mathrm{bII}}^{4,8}_e\\
&\quad + \widetilde{\mathrm{bII}}^{13}_o + \widetilde{\mathrm{bII}}^{22}_e - \widetilde{\mathrm{bII}}^{23}_e + \widetilde{\mathrm{bII}}^{24}_o - \widetilde{\mathrm{bII}}^{24}_e + \widetilde{\mathrm{bII}}^{b}_e + \widetilde{\mathrm{bII}}^{f}_e,\\
\delta_1(\widetilde{\mathrm{bI}}^6_o) &= -\widetilde{\mathrm{bII}}^{2,6}_o + \widetilde{\mathrm{bII}}^{2,6}_e - \widetilde{\mathrm{bII}}^{3,6}_o + \widetilde{\mathrm{bII}}^{3,6}_e - \widetilde{\mathrm{bII}}^{4,6}_o + \widetilde{\mathrm{bII}}^{4,6}_e + \widetilde{\mathrm{bII}}^{6,8}_o - \widetilde{\mathrm{bII}}^{6,8}_e\\
&\quad - \widetilde{\mathrm{bII}}^{c}_e - \widetilde{\mathrm{bII}}^{d}_o - \widetilde{\mathrm{bII}}^{e}_e - \widetilde{\mathrm{bII}}^{f}_e,\\
\delta_1(\widetilde{\mathrm{bI}}^6_e) &= -\widetilde{\mathrm{bII}}^{2,6}_o + \widetilde{\mathrm{bII}}^{2,6}_e - \widetilde{\mathrm{bII}}^{3,6}_o + \widetilde{\mathrm{bII}}^{3,6}_e - \widetilde{\mathrm{bII}}^{4,6}_o + \widetilde{\mathrm{bII}}^{4,6}_e + \widetilde{\mathrm{bII}}^{6,8}_o - \widetilde{\mathrm{bII}}^{6,8}_e\\
&\quad + \widetilde{\mathrm{bII}}^{c}_o + \widetilde{\mathrm{bII}}^{d}_e + \widetilde{\mathrm{bII}}^{e}_o + \widetilde{\mathrm{bII}}^{f}_o,\\
\delta_1(\widetilde{\mathrm{bI}}^8_o) &= -\widetilde{\mathrm{bII}}^{2,8}_o + \widetilde{\mathrm{bII}}^{2,8}_e - \widetilde{\mathrm{bII}}^{3,8}_o + \widetilde{\mathrm{bII}}^{3,8}_e - \widetilde{\mathrm{bII}}^{4,8}_o + \widetilde{\mathrm{bII}}^{4,8}_e - \widetilde{\mathrm{bII}}^{6,8}_o + \widetilde{\mathrm{bII}}^{6,8}_e\\
&\quad - \widetilde{\mathrm{bII}}^{23}_o - \widetilde{\mathrm{bII}}^{24}_o + \widetilde{\mathrm{bII}}^{24}_e - \widetilde{\mathrm{bII}}^{c}_o - \widetilde{\mathrm{bII}}^{e}_o,\\
\delta_1(\widetilde{\mathrm{bI}}^8_e) &= -\widetilde{\mathrm{bII}}^{2,8}_o + \widetilde{\mathrm{bII}}^{2,8}_e - \widetilde{\mathrm{bII}}^{3,8}_o + \widetilde{\mathrm{bII}}^{3,8}_e - \widetilde{\mathrm{bII}}^{4,8}_o + \widetilde{\mathrm{bII}}^{4,8}_e - \widetilde{\mathrm{bII}}^{6,8}_o + \widetilde{\mathrm{bII}}^{6,8}_e\\
&\quad + \widetilde{\mathrm{bII}}^{23}_e - \widetilde{\mathrm{bII}}^{24}_o + \widetilde{\mathrm{bII}}^{24}_e + \widetilde{\mathrm{bII}}^{c}_e + \widetilde{\mathrm{bII}}^{e}_e.
\end{aligned}
$$

Note that in a particular case, similar formulas have been obtained in [8, §6].

Proposition 4.12. *The cohomology groups of the universal complex*

$$\mathcal{CO}(b\mathcal{S}_{\mathrm{pr}}(3,2), \rho_{3,2}(2))$$

of co-orientable fibers for proper stable maps of 3*-dimensional manifolds with boundary into surfaces without boundary, are described as follows:*

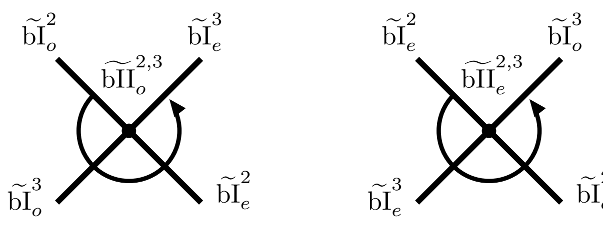

FIGURE 8. Co-orientations for $\widetilde{\mathrm{bII}}^{2,3}_*$

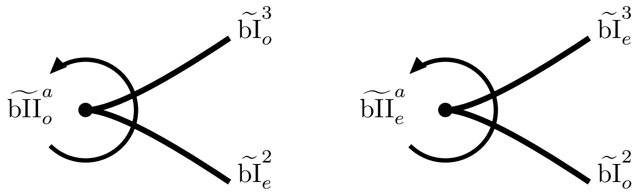

FIGURE 9. Co-orientations for $\widetilde{\mathrm{bII}}^a_*$

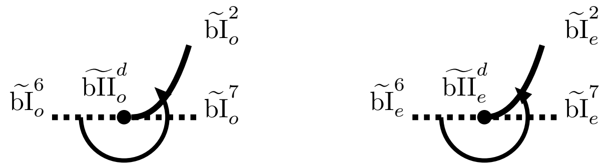

FIGURE 10. Co-orientations for $\widetilde{\mathrm{bII}}^d_*$

(1) $H^0(b\mathcal{S}_{\mathrm{pr}}(3,2)^{\mathrm{ori}}, \rho_{3,2}(2)) \cong \mathbb{Z}$, *generated by* $[\widetilde{\mathrm{b0}}_o + \widetilde{\mathrm{b0}}_e]$,
(2) $H^1(b\mathcal{S}_{\mathrm{pr}}(3,2)^{\mathrm{ori}}, \rho_{3,2}(2)) \cong \mathbb{Z}$, *generated by*

$$\gamma_1 = [\widetilde{\mathrm{bI}}^2_o + \widetilde{\mathrm{bI}}^3_e + \widetilde{\mathrm{bI}}^4_e + \widetilde{\mathrm{bI}}^6_o + \widetilde{\mathrm{bI}}^8_e] = [\widetilde{\mathrm{bI}}^2_e + \widetilde{\mathrm{bI}}^3_o + \widetilde{\mathrm{bI}}^4_o + \widetilde{\mathrm{bI}}^6_e + \widetilde{\mathrm{bI}}^8_o].$$

Note that the ranks of $C^i(b\mathcal{S}_{\mathrm{pr}}(3,2)^{\mathrm{ori}}, \rho_{3,2}(2))$, $i = 0,1,2$, are equal to 2, 10 and 40, respectively.

We can also prove that the $b\mathcal{S}_{\mathrm{pr}}(3,2)^{\mathrm{ori}}$-cobordism invariant $s_{\kappa *}\gamma_1$ is trivial by using the same argument as in [7, Lemma 14.1].

Proposition 4.13. *The cohomology groups of the universal complex*

$$\mathcal{CO}(\mathcal{AS}_{\mathrm{pr}}(3,2)^{\mathrm{ori}}, \rho_{3,2}(2))$$

of co-orientable fibers for proper admissible stable maps of 3*-dimensional manifolds with boundary into surfaces without boundary, are described as follows:*

(1) $H^0(\mathcal{AS}_{\mathrm{pr}}(3,2)^{\mathrm{ori}}, \rho_{3,2}(2)) \cong \mathbb{Z}$, *generated by* $[\widetilde{\mathrm{b0}}_o + \widetilde{\mathrm{b0}}_e]$,
(2) $H^1(\mathcal{AS}_{\mathrm{pr}}(3,2)^{\mathrm{ori}}, \rho_{3,2}(2)) \cong \mathbb{Z}$, *generated by*

$$\gamma_2 = [\widetilde{\mathrm{bI}}^2_o + \widetilde{\mathrm{bI}}^3_e + \widetilde{\mathrm{bI}}^4_e + \widetilde{\mathrm{bI}}^6_o + \widetilde{\mathrm{bI}}^8_e] = [\widetilde{\mathrm{bI}}^2_e + \widetilde{\mathrm{bI}}^3_o + \widetilde{\mathrm{bI}}^4_o + \widetilde{\mathrm{bI}}^6_e + \widetilde{\mathrm{bI}}^8_o]$$

Note that the ranks of $C^i(\mathcal{AS}_{\mathrm{pr}}(3,2)^{\mathrm{ori}}, \rho_{3,2}(2))$, $i = 0,1,2$, are 2, 10 and 34, respectively.

We can also prove that the $\mathcal{AS}_{\mathrm{pr}}(3,2)^{\mathrm{ori}}$-cobordism invariant $s_{\kappa *}\gamma_2$ is trivial by using the same argument as in [7, Lemma 14.1].

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REFERENCES

[1] J. Damon, *The relation between C^∞ and topological stability*, Bol. Soc. Brasil. Mat. **8** (1977), 1–38.

[2] A. du Plessis and T. Wall, *The geometry of topological stability*, London Math. Soc. Monographs, New Series 9, Oxford Science Publ., The Clarendon Press, Oxford Univ. Press, New York, 1995.

[3] C.G. Gibson, K. Wirthmüller, A.A. du Plessis and E.J.N. Looijenga, *Topological stability of smooth mappings*, Lecture Notes in Math., Vol. 552, Springer-Verlag, Berlin, New York, 1976.

[4] M. Golubitsky and V. Guillemin, *Stable mappings and their singularities*, Grad. Texts in Math., Vol. 14, Springer, New York-Heidelberg, 1973.

[5] L.F. Martins and A.C. Nabarro, *Projections of hypersurfaces in* $\mathbb{R}^4$ *with boundary to planes*, Glasgow Math. J. **56** (2014), 149–167.

[6] T. Ohmoto, *Vassiliev complex for contact classes of real smooth map-germs*, Rep. Fac. Sci. Kagoshima Univ. Math. Phys. Chem. **27** (1994), 1–12.

[7] O. Saeki, *Topology of singular fibers of differentiable maps*, Lecture Notes in Math., Vol. 1854, Springer, Berlin, 2004.

[8] O. Saeki, *Cobordism of Morse functions on surfaces, the universal complex of singular fibers and their application to map germs*, Algebr. Geom. Topol. **6** (2006), 539–572.

[9] O. Saeki and S. Takahashi, *Visual data mining based on differential topology: a survey*, to appear in Pacific Journal of Mathematics for Industry.

[10] O. Saeki, S. Takahashi, D. Sakurai, Hsiang-Yun Wu, K. Kikuchi, H. Carr, D. Duke and T. Yamamoto, *Visualizing multivariate data using singularity theory*, The Impact of Applications on Mathematics, Proceedings of Forum "Math-for-Industry" 2013, Springer, to appear.

[11] O. Saeki and T. Yamamoto, *Singular fibers of stable maps and signatures of* 4*-manifolds*, Geom. Topol. **10** (2006), 359–399.

[12] O. Saeki and T. Yamamoto, *Singular fibers and characteristic classes*, Topology Appl. **155** (2007), 112–120.

[13] N. Shibata, *On non-singular stable maps of* 3*-manifolds with boundary into the plane*, Hiroshima Math. J. **39** (2000), 415–435.

[14] V.A. Vassilyev, *Lagrange and Legendre characteristic classes*, Translated from the Russian, Advanced Studies in Contemporary Mathematics, Vol. 3, Gordon and Breach Science Publishers, New York, 1988.

[15] T. Yamamoto, *Classification of singular fibres of stable maps of* 4*-manifolds into* 3*-manifolds and its applications*, J. Math. Soc. Japan **58** (2006), 721–742.

[16] T. Yamamoto, *Euler number formulas in terms of singular fibers of stable maps*, in "Real and complex singularities", pp. 427–457, World Sci. Publ., Hackensack, NJ, 2007.

[17] T. Yamamoto, *Singular fibers of two-colored maps and cobordism invariants*, Pacific J. Math. **234** (2008), 379–398.

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