TOPOLOGY OF SINGULAR FIBERS OF GENERIC MAPS

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ABSTRACT. This talk has two purposes.

(1) We classify singular fibers of proper C^∞ stable maps of orientable 4-manifolds into 3-manifolds up to *right-left equivalence*. We show that the signature of the source oriented 4-manifold of such a stable map coincides with the algebraic number of singular fibers of a certain type.

(2) For a generic map of negative codimension in general, we have a stratification of the target manifold according to the fibers. Using this, we define the *universal complexes of singular fibers* similar to Vassiliev's universal complexes of multi-singularities. We show that their cohomology groups give rise to cobordism invariants of smooth maps with a given set of local and global singularities.

1. INTRODUCTION

Let $f : M \to N$ be a proper smooth map of an n-dimensional manifold M into a p-dimensional manifold N. When its codimension $p - n$ is nonnegative, i.e. when $n \leq p$, for any point y in the target N, the inverse image $f^{-1}(y)$ consists of a finite number of points, provided that f is generic enough. Hence, in order to study the semi-local behavior of a generic map f around (the inverse image of) a point $y \in N$, we have only to consider the *multi-germ* $f : (M, f^{-1}(y)) \to (N, y)$. Therefore, we can use the well-developed theory of multi-jet spaces and their sections in order to study such semi-local behaviors of generic maps.

However, if the codimension $p - n$ is strictly negative, then the inverse image $f^{-1}(y)$ is no longer a discrete set. In general, $f^{-1}(y)$ forms a complex of positive dimension $n - p$. Hence, we have to study the map germ $f : (M, f^{-1}(y)) \to (N, y)$ along a set of positive dimension, namely along a *singular fiber*. Surprisingly enough, there has been no systematic study of such map germs in the literature, as long as the author knows.

In this talk, we consider the codimension -1 case, i.e. the case with $n - p = 1$, and classify the right-left equivalence classes of generic map germs $f : (M, f^{-1}(y)) \to (N, y)$ for $n = 2, 3, 4$ (Theorem 2.2). For the case $n = 3$, Kushner, Levine and Porto [6, 7] classified the singular fibers of C^∞ stable maps of 3-manifolds into surfaces up to *diffeomorphism*; however, they did not mention a classification up to *right-left equivalence* (for details, see Definition 2.1 in §2).

Given a generic map $f : M \to N$ of negative codimension, the target manifold N is naturally stratified according to the right-left equivalence classes of f-fibers. By carefully investigating how the strata are incident to each other, we get some information on the homology class represented by a set of the points in the target whose associated fibers are of certain types. This leads to some limitations on the co-existence of singular fibers. As an interesting and very important consequence of such co-existence results, we show that for a C^∞ stable map $f : M \to N$ of a closed orientable 4-manifold M into a 3-manifold N, the Euler characteristic of the source manifold M has the same parity as the number of singular fibers of type III^8 as depicted in Fig. 1 (Theorem 2.4). Furthermore, when the source 4-manifold is oriented, its signature coincides with the algebraic number of singular fibers of type III^8 (Theorem 2.5). Note that these kinds of results would be impossible if we used the multi-germs of a given map at the singular points contained in a fiber instead of considering the topology of the fibers. In other words, our idea of essentially using the topology of singular fibers leads to new information on the global structure of generic maps.

Furthermore, the natural stratification of the target manifold according to the fibers enables us to generalize Vassiliev's universal complex of multi-singularities [12] to our case. In this talk, we define such *universal complexes of singular fibers* and compute the corresponding cohomology groups in certain cases. It turns out that cohomology classes of such complexes give rise to cobordism invariants for maps with a given set of singularities in the sense of Rimányi and Szűcs [9].

For more details, refer to the preprint [10].

2. Classification of singular fibers and their topology

Definition 2.1. Let $f_i : M_i \to N_i$ be smooth maps, $i = 0, 1$. For $y_i \in N_i$, we say that the fibers over y_0 and y_1 are *diffeomorphic* (or *homeomorphic*) if $(f_0)^{-1}(y_0) \subset M_0$ and $(f_1)^{-1}(y_1) \subset M_1$ are diffeomorphic (resp. homeomorphic) as subsets of smooth manifolds. Furthermore, we say that the fibers over y_0 and y_1 are C^∞ *equivalent* (or C^0 *equivalent*), if for some open neighborhoods U_i of y_i in N_i, there exist diffeomorphisms (resp. homeomorphisms) $\widetilde{\varphi} : (f_0)^{-1}(U_0) \to (f_1)^{-1}(U_1)$ and $\varphi : U_0 \to U_1$ with $\varphi(y_0) = y_1$ which make the following diagram commutative:

$$\begin{array}{ccc} ((f_0)^{-1}(U_0), (f_0)^{-1}(y_0)) & \xrightarrow{\widetilde{\varphi}} & ((f_1)^{-1}(U_1), (f_1)^{-1}(y_1)) \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f_1} \\ (U_0, y_0) & \xrightarrow{\varphi} & (U_1, y_1). \end{array} \tag{2.1}$$

When the fibers over y_0 and y_1 are C^∞ (or C^0) equivalent, we also say that the map germs $f_0 : (M_0, (f_0)^{-1}(y_0)) \to (N_0, y_0)$ and $f_1 : (M_1, (f_1)^{-1}(y_1)) \to (N_1, y_1)$ are smoothly (or topologically) *right-left equivalent*. Note that then $(f_0)^{-1}(y_0)$ and $(f_1)^{-1}(y_1)$ are diffeomorphic (resp. homeomorphic) to each other in the above sense.

Theorem 2.2. *Let $f : M \to N$ be a proper C^∞ stable map of an orientable 4-manifold M into a 3-manifold N. Then, every singular fiber of f is equivalent to the disjoint union of one of the fibers as in Fig. 1 and a finite number of copies of a fiber of the trivial circle bundle.*

In Fig. 1, κ denotes the codimension of the set of points in N whose corresponding fibers are equivalent to the relevant one. Furthermore, $\mathrm{I}^*, \mathrm{II}^*$ and III^* mean the names of the corresponding singular fibers, and "/" is used only for separating the figures. Note that the list of singular fibers for proper stable maps of orientable 3-manifolds into surfaces consists of those fibers with $\kappa \leq 2$ of Fig. 1, and that the list for proper Morse functions on orientable surfaces consists of those with $\kappa \leq 1$.

Theorem 2.2 is proved as follows. We first list up all the possible 1-dimensional complexes which arise as a singular fiber, by a careful combinatorial argument. Then we use Ehresmann's fibration theorem together with a classification of multi-germs up to right equivalence [13, 2] to construct diffeomorphisms as in (2.1).

Theorem 2.2 has been generalized to the case of nonorientable source in [14].

As an immediate corollary to Theorem 2.2, we have the following. Compare this with a result of Damon [1] about stable map germs in nice dimensions.

Corollary 2.3. *Two fibers of proper C^∞ stable maps of orientable n-manifolds into $(n-1)$-manifolds ($n = 2, 3, 4$) are C^∞ equivalent if and only if they are C^0 equivalent.*

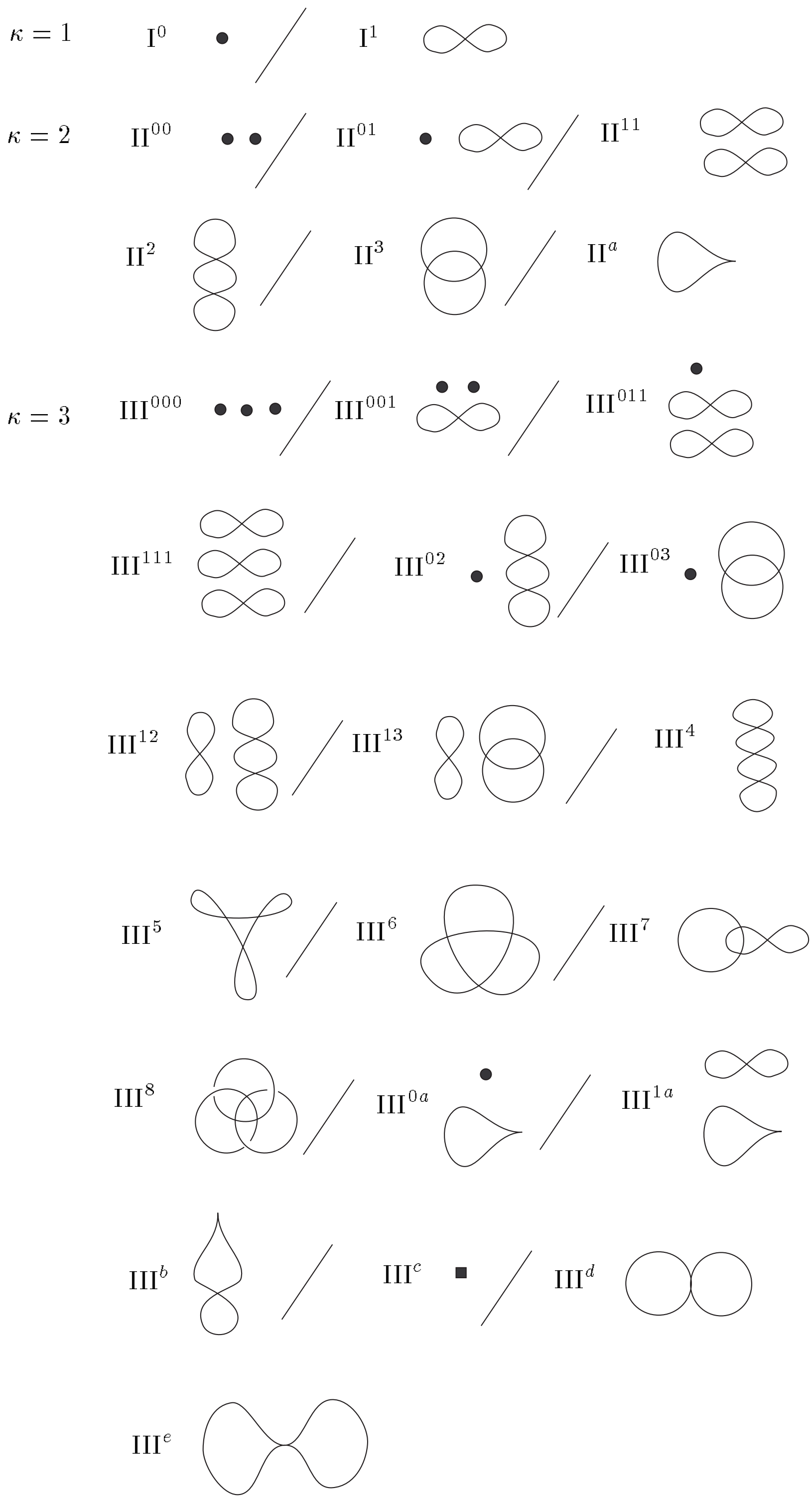

FIGURE 1. List of singular fibers of proper C^∞ stable maps of orientable 4-manifolds into 3-manifolds

By studying the adjacency of singular fibers, we obtain the following results.

Theorem 2.4. *Let $f : M \to N$ be a C^∞ stable map of a closed orientable 4-manifold into a 3-manifold. Then the number of singular fibers of f of type III^8 has the same parity as the Euler characteristic of M.*

When the 4-manifold M is oriented, one can assign a sign $(= \pm 1)$ to each fiber of type III^8. Then, we have the following.

Theorem 2.5 (T. Yamamoto and O. Saeki [11])**.** *Let $f : M \to N$ be a C^∞ stable map of a closed oriented 4-manifold into a 3-manifold. Then the algebraic number of singular fibers of f of type III^8 coincides with the signature of M.*

3. Universal complex for singular fibers

For $n, p \geq 0$, let $\mathcal{T}_{\mathrm{pr}}(n,p)$ denote the set of all proper Thom maps between manifolds of dimensions n and p. Recall that a Thom map is a stratified map with respect to Whitney regular stratifications of the source and the target such that it is a submersion on each stratum and satisfies a certain regularity condition (for example, see [3]).

Definition 3.1. We say that an equivalence relation $\rho = \rho_{n,p}$ among the fibers of elements of $\mathcal{T}_{\mathrm{pr}}(n,p)$ is *admissible* if the following holds.

(1) C^0 equivalent fibers are equivalent with respect to ρ.
(2) For any two proper Thom maps $f_i : M_i \to N_i$ and for any points $y_i \in N_i$, $i = 0, 1$, whose fibers are equivalent to each other with respect to ρ, there exist neighborhoods U_i of y_i in N_i, $i = 0, 1$, and a homeomorphism $\varphi : U_0 \to U_1$ such that $\varphi(y_0) = y_1$ and $\varphi(U_0 \cap \widetilde{\mathfrak{F}}(f_0)) = U_1 \cap \widetilde{\mathfrak{F}}(f_1)$ for every equivalence class $\widetilde{\mathfrak{F}}$ of fibers with respect to ρ, where $\widetilde{\mathfrak{F}}(f_i)$ is the set of points in N_i over which lies a fiber of f_i of type $\widetilde{\mathfrak{F}}$.

For an equivalence class $\widetilde{\mathfrak{F}}$ of fibers with respect to ρ, its codimension $\kappa = \kappa(\widetilde{\mathfrak{F}})$ is well-defined. For an equivalence class $\widetilde{\mathfrak{G}}$ of codimension $\kappa + 1$, we take a proper Thom map f with $\widetilde{\mathfrak{G}}(f) \neq \emptyset$. Let $\Sigma \subset \widetilde{\mathfrak{G}}(f)$ be a top dimensional stratum, and B_Σ a small disk which intersects Σ transversely exactly at its center and whose dimension coincides with the codimension of Σ. Then $B_\Sigma \cap \overline{\widetilde{\mathfrak{F}}(f)}$ consists of a finite number of arcs which have $B_\Sigma \cap \Sigma$ as a common end point. Let $[\widetilde{\mathfrak{F}} : \widetilde{\mathfrak{G}}] \in \mathbf{Z}_2$ denote the number of such arcs modulo two, which clearly does not depend on the choice of B_Σ, Σ or f by Definition 3.1 (2).

Let us construct a complex of fibers with coefficients in $\mathbf{Z}_2$ with respect to the admissible equivalence relation ρ as follows. For $\kappa \geq 0$, let $C^\kappa(\mathcal{T}_{\mathrm{pr}}(n,p), \rho)$ be the $\mathbf{Z}_2$-vector space consisting of all formal linear combinations,

$$\sum_{\kappa(\widetilde{\mathfrak{F}})=\kappa} m_{\widetilde{\mathfrak{F}}} \widetilde{\mathfrak{F}} \quad (m_{\widetilde{\mathfrak{F}}} \in \mathbf{Z}_2),$$

which may possibly contain infinitely many terms, of the equivalence classes $\widetilde{\mathfrak{F}}$ of fibers with codimension κ with respect to the equivalence relation ρ. For $\kappa < 0$, we put $C^\kappa(\mathcal{T}_{\mathrm{pr}}(n,p), \rho) = 0$. Define the $\mathbf{Z}_2$-linear map

$$\delta_\kappa : C^\kappa(\mathcal{T}_{\mathrm{pr}}(n,p), \rho) \to C^{\kappa+1}(\mathcal{T}_{\mathrm{pr}}(n,p), \rho)$$

by

$$\delta_\kappa(\widetilde{\mathfrak{F}}) = \sum_{\kappa(\widetilde{\mathfrak{G}})=\kappa+1} [\widetilde{\mathfrak{F}} : \widetilde{\mathfrak{G}}] \widetilde{\mathfrak{G}}, \tag{3.1}$$

for $\widetilde{\mathfrak{F}}$ with $\kappa(\widetilde{\mathfrak{F}}) = \kappa$. We warn the reader that the sum appearing in the right hand side of (3.1) may possibly contain infinitely many terms. Nevertheless, for a given equivalence class $\widetilde{\mathfrak{G}}$ of fibers, the number of equivalence classes $\widetilde{\mathfrak{F}}$ of fibers with codimension κ such that $[\widetilde{\mathfrak{F}} : \widetilde{\mathfrak{G}}] \neq 0$ is finite by virtue of the local finiteness of the Whitney regular stratifications and the definition of an admissible equivalence relation. Hence, the linear map δ_κ is well-defined.

It is not difficult to show that $\delta_{\kappa+1} \circ \delta_\kappa = 0$. Hence, $(C^\kappa(\mathcal{T}_{\rm pr}(n,p),\rho), \delta_\kappa)_\kappa$ constitutes a complex and its cohomology groups $H^\kappa(\mathcal{T}_{\rm pr}(n,p),\rho)$ are well-defined.

Remark 3.2. One can naturally define the *multi-singularity equivalence* by using the multi-germs at the singular points contained in a fiber. This is an admissible equivalence relation. Then the universal complex of singular fibers with respect to the multi-singularity equivalence corresponds to Vassiliev's universal complex of multi-singularities [12] (see also [5, 8]).

It turns out that cohomology classes of the above constructed universal complex of singular fibers give rise to cobordism invariants of singular maps in the sense of Rimányi and Szűcs [9]. More precisely, every $f \in \mathcal{T}_{\rm pr}(n,p)$ naturally induces a homomorphism $\varphi_f : H^\kappa(\mathcal{T}_{\rm pr}(n,p),\rho) \to H^\kappa(N; \mathbf{Z}_2)$, and φ_f restricted to a certain subgroup is a singular cobordism invariant of f. For example, we show that for the fold cobordism of Morse functions on oriented surfaces, a complete cobordism invariant constructed in [4] is also obtained in this way (in fact, we need a *co-oriented* version of the universal complex of singular fibers for this purpose).

We also show that the above homomorphism φ_f can be used to characterize cocycles of the universal complex of singular fibers.

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